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An Iterative Approach for Obtaining a Closed-form Expansion for the Conditional Expectations of the Extended Cox-Ingersoll-Ross Process

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Abstract In this paper, we develop an iterative approach for obtaining a closed-form expansion for the conditional expectation of the valuation process, defined by

$$V_{t,T} := e^{-\int_t^T g(v_s)ds} f(v_T) + \int_t^T h(v_s) e^{-\int_t^s g(v_u)du} ds$$

for $0 \le t \le T$, where v_t is assumed to follow the extended Cox-Ingersoll-Ross process, for any smooth real-valued functions f, g, and h. The novel analytical approach presented here at least serves for two major purposes: (i) to avoid the requirement of numerical integration or Monte Carlo (MC) simulations to compute the conditional expectation, which can substantially reduce the computational burden; (ii) to provide a simple closed-form expansion for the conditional expectation, which can be easily used by market practitioners. Furthermore, a multi-step closed-form expansion is constructed in order to improve the accuracy of our approach. The performance of the current approach is demonstrated by comparing our numerical results with some exact solutions and MC simulations from several examples.

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Keywords: ECIR process; conditional expectation; closed-form expansion

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1. Introduction

The extended Cox-Ingersoll-Ross (ECIR) process has a form of

$$dv_t = \kappa(t)(\theta(t) - v_t)dt + \sigma(t)\sqrt{v_t}dW_t, \tag{1.1}$$

where v_t is an instantaneous variance and W_t is a standard Brownian motion under a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $(\mathcal{F}_t)_{t\geqslant 0}$. All of the parameters i.e., $\theta(t)$, $\kappa(t)$, and $\sigma(t)$, are set to be smooth and bounded time dependent functions.

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A special class of the ECIR process is that of the class of Cox-Ingersoll-Ross (CIR) process which has a form of

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW_t \tag{1.2}$$

where the parameters θ, κ, and σ are set to be constant.

Even though the CIR process is the most common model used to describe the dynamics of the instantaneous variance or interest rates in the Heston model of stochastic volatility or in stochastic interest rate models [1], there are many empirical evidences supporting the theory that the data generating process governing the dynamics of many economics variables might vary over time, because of economic climate changes or time effects. Following that case, the ECIR process is more suitable for describing the data than the corresponding CIR process, because the ECIR process uses time dependent parameter functions to represent possible time varying expected trends and volatilities of the market and the economy. Consequently, many researchers in commodity markets such as Schneider and Tavin [2], and Arismendi et al.[3], described seasonal stochastic volatility by using the ECIR process in which $\theta(t)$ describes the long-term mean variance level of commodity prices which is assumed to be a function of time.

Considering option pricing when the underlying process is assumed to follow the ECIR process (1.1), we define the *valuation process* of a *contingent claim* (f, g, h) by

$$V_{t,T} := e^{-\int_t^T g(v_s)ds} f(v_T) + \int_t^T h(v_s) e^{-\int_t^s g(v_u)du} ds \tag{1.3}$$

for real-valued functions f, g, and h. In this context, the processes $f(v_t)$, $g(v_t)$, and $h(v_t)$ for $t \in [0, T]$ represent, respectively, a terminal payoff, an interest rate process, and a payoff rate process. According to the theorem for option pricing proposed by Karatzas and Shreve [4] (see page 378), the fair price of the contingent claim (f, g, h) at a current time t is the conditional expectation of the valuation process (1.3) with respect to the risk-neutral probability measure P and current σ- field $\mathcal{F}_t$, such as

$$E^P[V_{t,T}|\mathcal{F}_t] = E^P[V_{t,T}|v_t = v] \tag{1.4}$$

for $t \in [0, T]$ and $v > 0$, where we denote by $E^P[X|\mathcal{F}_t]$, the conditional expectation of a random variable X with respect to the probability measure P and σ- field $\mathcal{F}_t$.

We define

$$X_{t,T} := e^{-\int_t^T g(v_s)ds} \tag{1.5}$$

$$Y_{t,T} := \int_t^T h(v_s) e^{-\int_t^s g(v_u)du} ds \tag{1.6}$$

for $t \in [0, T]$. Hence, the valuation process (1.3) can be expressed as

$$V_{t,T} = X_{t,T} f(v_T) + Y_{t,T} \tag{1.7}$$

and the conditional expectation (1.4) can be explicitly written in terms of a triple integral as

$$E^P[V_{t,T}|v_t = v] = \int_{D_Y} \int_{D_X} \int_{D_V} (xf(v) + y) p_{vxy}(v, x, y, t + \tau|v, t) dv dx dy \tag{1.8}$$

for $\tau = T - t \geq 0$ where $p_{vxy}(v, x, y, t+\tau|v, t)$ denotes the joint-transition density of the processes $v_t, X_{t,T}$, and $Y_{t,T}$ defined on the domains $D_V \subseteq \mathbb{R}^+, D_X \subseteq \mathbb{R}^+$, and $D_Y \subseteq \mathbb{R}$, respectively.

Focusing on computation, various analytical or numerical methods can be adopted to obtain exact or numerical solutions for the triple integral on the RHS of (1.8) providing that the joint-transition density p_{vxy} is available in closed-form. However, to derive p_{vxy} in closed-form, we need to solve the forward Kolmogorov equation, associated with the processes $v_t, X_{t,T}$, and $Y_{t,T}$ (see Karatzas and Shreve [4] on page 282) and this is a difficult and complicated task in general for arbitrary real-valued functions f, g, and h. According to literature, the conditional expectation (1.4) has closed-form formulas for some special cases. For example, Dufresne [5] proposed a closed-form formula for the case $f(v) = v^\gamma$ for any $\gamma > \frac{-2\kappa\theta}{\sigma^2}$ and $g = h = 0$ in which v_t is assumed to follow the CIR process (1.2). Rujivan [6] extended Dufresne's [5] work to the ECIR processes (1.1) for any $\gamma \in \mathbb{R}$. Sutthimat et al. [7] extended Rujivan's [6] work to product of polynomial and exponential function of affine transform. Thamrongrat and Rujivan [8] derived a closed-form formula for the conditional expectation of the valuation process for $f(v) = v^{\gamma_1}$and $h(v) = v^{\gamma_2}$ for any $\gamma_1, \gamma_2 \in \mathbb{R}$, and any integrable function g. Very recently, Rujivan [9] and Rujivan and Rakwongwan [10] used the results proposed in [6, 8] to price variance swaps and volatility swaps. Moreover, Thamrongrat and Rujivan [11] used the results proposed in [7] to determine the fair prices of interest rate swaps in terms of bond prices under the ECIR model (1.1), whereas Prathom and Rujivan [12] applied the results proposed in [6] to derive the conditional moments of quadratic variance diffusion processes.

In terms of numerical methods based on simulations, as alternative to the methods previously introduced, the Monte Carlo (MC) method is the most influential one which can be directly adopted to obtain approximates for the conditional expectation (1.4). Nevertheless, this approach consumes much computational time and effort to generate sample paths of $V_{t,T}$ which is a path dependence process, depending on the underlying processes $v_t, X_{t,T}$, and $Y_{t,T}$. In this study, we present an iterative approach to derive a closed-form expansion for the conditional expectation (1.4). Very interestingly, the derivation of our approach has completely avoided the utilization of the joint-transition density p_{vxy}.

There are two major contributions of this paper. First, our closed-form expansion produces approximates for the conditional expectation (1.4) without employing numerical integration or MC simulations. Clearly, this can substantially reduce the computational burden which is a major drawback of numerical integration and MC method. Second, our closed-form expansion has a simple form, which can be easily used by market practitioners. With these contributions, our closed-form expansion should be valuable in both theoretical and practical senses.

The approach presented here in this paper has a major difference from Rujivan's [6] approach as follows. Rujivan's [6] work aimed to derive a closed-form formula for the conditional moments of the ECIR processes. In other words, the arbitrary real-valued functions were set to be $f(v) = v^\gamma$ and $g(v) = h(v) = 0$ for any $\gamma \in \mathbb{R}$ and $v > 0$. Since f which is an initial condition for solving the associated PDE has a form as a polynomial function in v, the solution of the associated PDE was assumed as a power series in v, in which the coefficient functions depend only on time and can be determined by solving a system of ordinary differential equations (ODE), recursively. On the other hand, the current work aims to derive a closed-form expansion for the conditional expectation of the valuation process (1.3) based on the ECIR process (1.1) and for smooth real-valued functions f, g, and h. It is clear that Rujivan's [6] approach cannot be adopted to derive the closed-form expansion when f, g, and h are not polynomial functions in v. Furthermore, we apply the tower property to the conditional expectation (1.4) and construct a multi-step closed-form expansion in order to improve the accuracy of our approach.

The following two assumptions proposed by Maghsoodi [13] are needed, in order to ensure that the stochastic differential equation (SDE) (1.1) has a pathwise unique strong solution, in which v_t avoids zero a.s. P for all $t \in (0, T]$.

Assumption 1.1. The parameter functions $\theta(t)$, $\kappa(t)$, and $\sigma(t)$ are strictly positive and continuous on $[0, T]$ such that the dimension of the ECIR process (1.1), defined by $\delta(t) := \frac{4\theta(t)\kappa(t)}{\sigma^2(t)}$, is bounded.

Assumption 1.2. The inequality $\delta(t) \geq 2$ holds for all $t \in [0, T]$.

Moreover, the following assumption ensures that the conditional expectation (1.8) exists for all $t \in (0, T]$ and $v > 0$.

Assumption 1.3. The functions f, g, and h are smooth functions and satisfy the polynomial growth condition $|f(v)| + |g(v)| + |h(v)| \leq Cv^N$ for some constant $C > 0$, positive integer N, and for all $v > 0$.

The organization of the paper is as follows. In Section 2, we prove the main theorem, which shall be adopted to derive a closed-form expansion for the conditional expectation (1.4), based on the CIR process (1.2) and ECIR process (1.1), respectively. We further construct a multi-step closed-form expansion for the conditional expectation (1.4) in order to improve the accuracy of the current approach. Under the special case of contingent claims as previously introduced, we finish the section by deriving some closed-form formulas for the conditional expectation (1.4), based on the ECIR process (1.1) and CIR process (1.2), respectively. In Section 3, we study the accuracy of our current approach by comparing our numerical results with some exact solutions and MC simulations from several examples. We conclude the paper in the last section.

2. Main Results

In this section, we first present an iterative approach for obtaining a closed-form expansion for the conditional expectation (1.4), based on the CIR process (1.2).

2.1. Our closed-form expansion for the CIR process

Theorem 2.1. *Suppose v_t follows the CIR process (1.2). By assuming that Assumptions 1-3 hold, we denote*

$$U_C(v, \tau) := E^P[V_{t,T} | v_t = v] \tag{2.1}$$

for $v > 0$ and $\tau = T - t \geq 0$. Then, $U_C(v, \tau)$ can be expressed as

$$U_C(v, \tau) = \sum_{k=0}^{\infty} A_k(v) \frac{\tau^k}{k!} \tag{2.2}$$

for all $(v, \tau) \in D_C$ where D_C is a subset of $(0, \infty) \times [0, \infty)$,

$$A_0(v) = f(v) \tag{2.3}$$

$$A_1(v) = \frac{1}{2}\sigma^2 v A_0''(v) + \kappa(\theta - v)A_0'(v) - g(v)A_0(v) + h(v) \tag{2.4}$$

$$A_k(v) = \frac{1}{2}\sigma^2 v A_{k-1}''(v) + \kappa(\theta - v)A_{k-1}'(v) - g(v)A_{k-1}(v) \tag{2.5}$$

for $k = 2, 3, \ldots$, and the derivatives are computed with respect to v.

Proof. Applying the Feynman-Kac theorem, U_C satisfies the PDE:

$$-\frac{\partial U_C}{\partial \tau} + \frac{1}{2}\sigma^2 v \frac{\partial^2 U_C}{\partial v^2} + \kappa(\theta - v)\frac{\partial U_C}{\partial v} - g(v)U_C(v,\tau) + h(v) = 0 \tag{2.6}$$

for all $v > 0$ and $0 < \tau \leq T$, subject to the initial condition

$$U_C(v, 0) = f(v) \tag{2.7}$$

for all $v > 0$.

Next, we compute the partial derivatives of U_C in (2.6) using the solution form (2.2). Inserting the partial derivatives obtained into (2.6) gives

$$\sum_{k=1}^{\infty} \frac{-A_k(v)\tau^{k-1}}{(k-1)!} + \frac{1}{2}\sigma^2 v \left\{\sum_{k=0}^{\infty} \frac{A_k{}''(v)\tau^k}{k!}\right\} + \kappa(\theta - v)\left\{\sum_{k=0}^{\infty} \frac{A_k{}'(v)\tau^k}{k!}\right\} - g(v)\left\{\sum_{k=0}^{\infty} \frac{A_k(v)\tau^k}{k!}\right\} + h(v) = 0. \tag{2.8}$$

It should be noted here that we need to impose an assumption on the infinite series on the RHS of (2.2) and LHS of (2.8) in order to obtain (2.8) by interchanging the partial derivative signs and infinite-sum signs in (2.6). The assumption is that there is a subset D_C of $(0,\infty) \times [0,\infty)$, such that the infinite series on the RHS of (2.2) and LHS of (2.8) converge uniformly on D_C with respect to (v, τ).

Under the assumption and initial condition (2.7), by collecting the coefficients of τ^k in (2.8) for $k = 0, 1, \ldots$, and equating them to zero, the coefficient functions $A_k(v), k = 0, 1, \ldots$, must satisfy (2.3), (2.4), and (2.5), recursively. ∎

2.2. Our closed-form expansion for the ECIR process

We need the following assumption in order to derive a closed-form expansion for the conditional expectation (1.4), based on the ECIR process (1.1).

Assumption 2.2. For any $T > 0$, the Taylor expansions of the parameter functions $\theta(\cdot)$, $\kappa(\cdot)$, and $\sigma(\cdot)$ centered at T converge to $\theta(t)$, $\kappa(t)$, and $\sigma(t)$, respectively, for all $t \in [0, T + \epsilon)$ for some $\epsilon > 0$.

Theorem 2.3. *Suppose v_t follows the ECIR process (1.1). By assuming that Assumptions 1-4 hold, we denote*

$$U_E(v, \tau) := E^P[V_{t,T}|v_t = v] \tag{2.9}$$

for $v > 0$ and $\tau = T - t \geq 0$ and set $\alpha_1(t) = \sigma^2(t)$, $\alpha_2(t) = \kappa(t)\theta(t)$, and $\alpha_3(t) = \kappa(t)$ for $t \geq 0$. Then, $U_E(v,\tau)$ can be expressed as

$$U_E(v,\tau) = \sum_{k=0}^{\infty} B_k(v)\frac{\tau^k}{k!} \tag{2.10}$$

for all $(v,\tau) \in D_E$ where D_E is a subset of $(0,\infty) \times [0,\infty)$,

$$B_0(v) = f(v) \tag{2.11}$$

$$B_1(v) = \frac{1}{2}\alpha_1(T)vB_0{}''(v) + \{\alpha_2(T) - \alpha_3(T)v\}\, B_0{}'(v) - g(v)B_0(v) + h(v) \tag{2.12}$$

$$\begin{aligned} B_k(v) = &\sum_{j=0}^{k-1} c_{jk} \left\{\frac{1}{2}\alpha_1^{(k-j-1)}(T)vB_j{}''(v) + \{\alpha_2^{(k-j-1)}(T) - \alpha_3^{(k-j-1)}(T)v\}B_j{}'(v)\right\} \\ &- g(v)B_{k-1}(v), \end{aligned} \tag{2.13}$$

where $c_{jk} = \frac{(-1)^{k-j-1}(k-1)!}{j!(k-j-1)!}$ for $k = 2,3,\ldots$. The $(k-j-1)^{th}$ derivatives of $\alpha_i(t), i = 1,2,3,$ are computed with respect to t while the derivatives $B_j{}''(v)$ and $B_j{}'(v)$ are computed with respect to v.

Proof. Similar to the proof of Theorem 2.1, we first apply the Feynman-Kac theorem and obtain that U_E satisfies the PDE:

$$-\frac{\partial U_E}{\partial \tau} + \frac{1}{2}\alpha_1(t)v\frac{\partial^2 U_E}{\partial v^2} + (\alpha_2(t) - \alpha_3(t)v)\frac{\partial U_E}{\partial v} - g(v)U_E(v,\tau) + h(v) = 0 \tag{2.14}$$

for all $v > 0$, $0 < \tau \leq T$ and $0 \leq t < T$, subject to the initial condition

$$U_E(v,0) = f(v) \tag{2.15}$$

for all $v > 0$.

Following the procedure as presented in the proof of Theorem 2.1 gives

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{-B_k(v)\tau^{k-1}}{(k-1)!} + \frac{\alpha_1(t)}{2}v\left\{\sum_{k=0}^{\infty}\frac{B_k{}''(v)\tau^k}{k!}\right\} + (\alpha_2(t) - \alpha_3(t)v)\left\{\sum_{k=0}^{\infty}\frac{B_k{}'(v)\tau^k}{k!}\right\} \\ - g(v)\left\{\sum_{k=0}^{\infty}\frac{B_k(v)\tau^k}{k!}\right\} + h(v) = 0, \end{aligned} \tag{2.16}$$

where we assume that there is a subset D_E of $(0,\infty)\times[0,\infty)$ such that the infinite series on the RHS of (2.10) and LHS of (2.16) converge uniformly on D_E with respect to (v,τ).

Under Assumption 4, we have

$$\alpha_i(t) = \sum_{k=0}^{\infty} \alpha_i^{(k)}(T)\frac{(-1)^k\tau^k}{k!} \tag{2.17}$$

for $i = 1,2,3$ for all $t \in [0,T]$. Replacing these Taylor expansions into (2.16) and collecting the coefficients of τ^k for $k = 0,1,\ldots$, the coefficient function $B_0(v)$ must satisfy (2.11) derived by using the initial condition (2.15). Consequently, the remaining coefficient functions $B_k(v), k = 1, ...,$ must follow (2.12) and (2.13), recursively. ∎

It should be noticed that (2.13) reduces to (2.5) when the parameter functions $\theta(t)$, $\kappa(t)$, and $\sigma(t)$ are constants.

2.3. A multi-step closed-form expansion

This subsection constructs a multi-step closed-form expansion for the conditional expectation (1.4) by applying the tower property. We shall demonstrate later in Section 3 that the multi-step closed-form expansion produces more accurate results than the closed-form expansion (2.10).

Let $\wp_n(t,T) := \{t_0 = t, t_1, ..., t_n = T\}$ be a partition on $[t,T]$ for any positive integer $n \geq 2$ and set $\tau(n) := \{\tau_i = t_i - t_{i-1} | i = 1, ..., n\}$. First, we define

$$x_{t_j,t_k} := e^{-\int_{t_j}^{t_k} g(v_s)ds} f(v_{t_k}) \tag{2.18}$$

for $t_j \leq t_k$ in $\wp_n(t,T)$. By applying the tower property, we have

$$E_{t_0}^P[x_{t_0,t_n}] = E_{t_0}^P[e^{-\int_{t_0}^{t_{n-1}} g(v_s)ds} E_{t_{n-1}}^P[x_{t_{n-1},t_n}]], \tag{2.19}$$

where we denote $E_{t_j}^P[x_{t_k,t_l}] := E^P[x_{t_k,t_l} | v_{t_j}]$ for $t_j \leq t_k \leq t_l$ in $\wp_n(t,T)$. Utilizing Theorem 2.3, we have

$$E_{t_{n-1}}^P[x_{t_{n-1},t_n}] = \sum_{k_n=0}^{\infty} B_{k_n}^{(1)}(v_{t_{n-1}}) \frac{\tau_n^{k_n}}{k_n!} \tag{2.20}$$

where $B_0^{(1)}(v) = f(v)$ and $B_{k_n}^{(1)}(v), k_n = 1, 2, ...,$ are computed using (2.12) with $h(v) = 0$ and (2.13), respectively. Inserting (2.20) into (2.19) and applying the tower property again give us

$$E_{t_0}^P[x_{t_0,t_n}] = E_{t_0}^P[e^{-\int_{t_0}^{t_{n-2}} g(v_s)ds} \sum_{k_n=0}^{\infty} E_{t_{n-2}}^P[e^{-\int_{t_{n-2}}^{t_{n-1}} g(v_s)ds} B_{k_n}^{(1)}(v_{t_{n-1}})] \frac{\tau_n^{k_n}}{k_n!}]. \tag{2.21}$$

By applying Theorem 2.3 to the conditional expectation with respect to t_{n-2} on the RHS of (2.21), we obtain

$$E_{t_{n-2}}^P[e^{-\int_{t_{n-2}}^{t_{n-1}} g(v_s)ds} B_{k_n}^{(1)}(v_{t_{n-1}})] = \sum_{k_{n-1}=0}^{\infty} B_{k_{n-1},k_n}^{(1)}(v_{t_{n-2}}) \frac{\tau_{n-1}^{k_{n-1}}}{k_{n-1}!}, \tag{2.22}$$

for all $k_n = 1, 2, ...,$ where $B_{0,k_n}^{(1)}(v) = B_{k_n}^{(1)}(v)$ and $B_{k_{n-1},k_n}^{(1)}(v), k_{n-1} = 1, 2, ...,$ are computed using (2.12) with $h(v) = 0$ and (2.13), respectively. Inserting (2.22) into (2.21) gives

$$E_{t_0}^P[x_{t_0,t_n}] = E_{t_0}^P[e^{-\int_{t_0}^{t_{n-2}} g(v_s)ds} \sum_{k_{n-1},k_n=0}^{\infty} B_{k_{n-1},k_n}^{(1)}(v_{t_{n-2}}) \frac{\tau_{n-1}^{k_{n-1}}}{k_{n-1}!} \frac{\tau_n^{k_n}}{k_n!}]. \tag{2.23}$$

Repeating the procedure for n times, we thus obtain

$$E_{t_0}^P[x_{t_0,t_n}] = \sum_{k_1,...,k_n=0}^{\infty} \left\{ B_{k_1,...,k_n}^{(1)}(v) \prod_{j=1}^{n} \frac{\tau_j^{k_j}}{k_j!} \right\}, \tag{2.24}$$

where $B^{(1)}_{k_1,...,k_n}(v)$ for $k_1,...,k_n = 0,1,...,$ are computed using (2.12) with $h(v) = 0$ and (2.13), respectively, as previously demonstrated.

Second, we define

$$y_{t_j,t_k} := \int_{t_j}^{t_k} h(v_s)e^{-\int_{t_0}^{s} g(v_u)du}ds \tag{2.25}$$

for $t_j \le t_k$ in $\wp_n(t,T)$. By applying the tower property, we have

$$E^P_{t_0}[y_{t_0,t_n}] = E^P_{t_0}[y_{t_0,t_{n-1}}] + E^P_{t_0}[E^P_{t_{n-1}}[y_{t_{n-1},t_n}]], \tag{2.26}$$

where we denote $E^P_{t_j}[y_{t_k,t_l}] := E^P[y_{t_k,t_l}|v_{t_j}]$ for $t_j \le t_k \le t_l$ in $\wp_n(t,T)$. Utilizing Theorem 2.3, we have

$$\begin{aligned} E^P_{t_{n-1}}[y_{t_{n-1},t_n}] &= E^P_{t_{n-1}}\left[\int_{t_{n-1}}^{t_n} h(v_s)e^{-\int_{t_0}^{s} g(v_u)du}ds\right] \\ &= e^{-\int_{t_0}^{t_{n-1}} g(v_u)du}E^P_{t_{n-1}}\left[\int_{t_{n-1}}^{t_n} h(v_s)e^{-\int_{t_{n-1}}^{s} g(v_u)du}ds\right] \\ &= e^{-\int_{t_0}^{t_{n-2}} g(v_u)du}\sum_{k_n=0}^{\infty} e^{-\int_{t_{n-2}}^{t_{n-1}} g(v_u)du}B^{(2)}_{k_n}(v_{t_{n-1}})\frac{\tau_n^{k_n}}{k_n!}, \end{aligned} \tag{2.27}$$

where $B^{(2)}_0(v) = 0, B^{(2)}_1(v) = h(v)$ and $B^{(2)}_{k_n}(v), k_n = 2,3,...,$ are computed using (2.13). Inserting (2.27) into the second term on the RHS of (2.26) and applying the tower property, we obtain

$$\begin{aligned} &E^P_{t_0}[E^P_{t_{n-1}}[y_{t_{n-1},t_n}]] \\ &= E^P_{t_0}\left[e^{-\int_{t_0}^{t_{n-2}} g(v_u)du}\sum_{k_n=0}^{\infty} E^P_{t_{n-2}}\left[e^{-\int_{t_{n-2}}^{t_{n-1}} g(v_u)du}B^{(2)}_{k_n}(v_{t_{n-1}})\right]\frac{\tau_n^{k_n}}{k_n!}\right]. \end{aligned} \tag{2.28}$$

Next, we apply Theorem 2.3 to the conditional expectation with respect to t_{n-2} on the RHS of (2.28)

$$E^P_{t_{n-2}}\left[e^{-\int_{t_{n-2}}^{t_{n-1}} g(v_u)du}B^{(2)}_{k_n}(v_{t_{n-1}})\right] = \sum_{k_{n-1}=0}^{\infty} B^{(2)}_{k_{n-1},k_n}(v_{t_{n-2}})\frac{\tau_{n-1}^{k_{n-1}}}{k_{n-1}!} \tag{2.29}$$

for all $k_n = 1,2,...,$ where $B^{(2)}_{0,k_n}(v) = B^{(2)}_{k_n}(v)$ and $B^{(2)}_{k_{n-1},k_n}(v), k_{n-1} = 1,2,...,$ are computed using (2.12) and (2.13), respectively, with $h(v) = 0$. Inserting (2.29) into (2.28), we obtain

$$E^P_{t_0}[E^P_{t_{n-1}}[y_{t_{n-1},t_n}]] = E^P_{t_0}\left[e^{-\int_{t_0}^{t_{n-2}} g(v_u)du}\sum_{k_{n-1},k_n=0}^{\infty} B^{(2)}_{k_{n-1},k_n}(v_{t_{n-2}})\frac{\tau_{n-1}^{k_{n-1}}}{k_{n-1}!}\frac{\tau_n^{k_n}}{k_n!}\right]. \tag{2.30}$$

Repeating the procedure for n times, we thus obtain

$$E^P_{t_0}[E^P_{t_{n-1}}[y_{t_{n-1},t_n}]] = \sum_{k_1,...,k_n=0}^{\infty}\left\{B^{(2)}_{k_1,...,k_n}(v)\prod_{j=1}^{n}\frac{\tau_j^{k_j}}{k_j!}\right\} \tag{2.31}$$

where $B^{(2)}_{k_1,...,k_n}(v)$ for $k_1, ..., k_n = 1, 2, ...$, are computed using (2.12) and (2.13), respectively, with $h(v) = 0$, as previously demonstrated. Furthermore, we can show by following the procedure for obtaining (2.31) that

$$E^P_{t_0}[E^P_{t_{l-1}}[y_{t_{l-1},t_l}]] = \sum_{k_1,...,k_l=0}^{\infty}\left\{B^{(2)}_{k_1,...,k_l}(v)\prod_{j=1}^{l}\frac{\tau_j^{k_j}}{k_j!}\right\}, \tag{2.32}$$

for all $l = 1, ..., n$, where $B^{(2)}_{k_1,...,k_l}(v)$ for $k_1, ..., k_l = 1, 2, ...$, are computed using (2.12) and (2.13), respectively, with $h(v) = 0$.

Using (2.32) and the tower property for n times, the conditional expectation on the LHS of (2.26) can be written as

$$E^P_{t_0}[y_{t_0,t_n}] = \sum_{l=1}^{n}\left\{\sum_{k_1,...,k_l=0}^{\infty}\left\{B^{(2)}_{k_1,...,k_l}(v)\prod_{j=1}^{l}\frac{\tau_j^{k_j}}{k_j!}\right\}\right\}. \tag{2.33}$$

Theorem 2.4. *Suppose v_t follows the ECIR process (1.1) and $n \geq 2$ is a positive integer. By assuming that Assumptions 1-4 hold, a multi-step closed-form expansion for the conditional expectation (1.4) with respect to a partition $\wp_n(t,T) = \{t_0 = t, t_1, ..., t_n = T\}$ can be written as*

$$\begin{aligned} U_E(v,\tau(n);\wp_n) := &\sum_{k_1,...,k_n=0}^{\infty}\left\{B^{(1)}_{k_1,...,k_n}(v;\wp_n)\prod_{j=1}^{n}\frac{\tau_j^{k_j}}{k_j!}\right\} \\ &+\sum_{l=1}^{n}\left\{\sum_{k_1,...,k_l=0}^{\infty}\left\{B^{(2)}_{k_1,...,k_l}(v;\wp_n)\prod_{j=1}^{l}\frac{\tau_j^{k_j}}{k_j!}\right\}\right\} \end{aligned} \tag{2.34}$$

for $v > 0$, where $\tau(n) = \{\tau_i = t_i - t_{i-1} | i = 1, ..., n\}$. Moreover, the formula (2.34) reduces to (2.10) when $n = 1$.

Proof. Since $V_{t,T} = x_{t_0,t_n} + y_{t_0,t_n}$. From (2.24) and (2.33), we immediately obtain (2.34). According to (2.12) and (2.13), the functions $B^{(1)}_{k_1,...,k_n}(v)$ and $B^{(2)}_{k_1,...,k_n}(v)$ for $k_1, ..., k_n = 0, 1, ...$, written in (2.24) and (2.33), respectively, must depend on all points in $\wp_n(t,T)$. Hence, we write $B^{(1)}_{k_1,...,k_n}(v;\wp_n)$ and $B^{(2)}_{k_1,...,k_n}(v;\wp_n)$ in (2.34) instead of $B^{(1)}_{k_1,...,k_n}(v)$ and $B^{(2)}_{k_1,...,k_n}(v)$, respectively. ∎

The calculation of (2.34) can easily be done with the aid of a symbolic package, such as Maple, Matlab, or Mathematica. For the reader's convenience, all Mathematica codes used in this paper are available from the authors on request.

2.4. Some closed-form solutions under a special case of contingent claims

This subsection presents some closed-form formulas for the conditional expectation (1.4), based on the ECIR process (1.1) and CIR process (1.2) derived by Thamrongrat and Rujivan [8] for $f(v) = v^{\gamma_1}$ and $h(v) = v^{\gamma_2}$ for any $\gamma_1, \gamma_2 \in \mathbb{R}$, and any integrable function g. They adopted the analytical approach presented by Rujivan [6] using the Feynman-Kac theorem to obtain the simple closed-form formulas written in the following theorems in which we shall use their formulas to demonstrate the accuracy of our current approach in the next section.

Theorem 2.5. *Suppose v_t follows the ECIR process (1.1) and a contingent claim (f, g, h) satisfies $f(v) = v^{\gamma_1}, h(v) = v^{\gamma_2}$, and $g(v) = r$ for some $\gamma_1, \gamma_2, r \in \mathbb{R}$. By assuming that Assumptions 1-4 hold, the conditional expectation (1.4) can be expressed as*

$$U_E(v,\tau) = e^{-r\tau}U_E^{(\gamma_1)}(v,\tau) + \sum_{k=0}^{\infty}\left(\int_t^T A_{\gamma_2-k}(s-t)e^{-r(s-t)}ds\right)v^{\gamma_2-k} \tag{2.35}$$

for $v > 0$ and $\tau = T - t \geq 0$ where the functions $U_E^{(\gamma_1)}(v,\tau)$ and $A_{\gamma_2-k}(s-t), k = 0, 1, ...,$ are given in Theorem 2.1 by Thamrongrat and Rujivan [8].

In particular, if $\gamma_1 = m$ and $\gamma_2 = n$ are nonnegative integers, then

$$U_E(v,\tau) = e^{-r\tau}U_E^{(m)}(v,\tau) + \sum_{j=0}^{n}\left(\int_t^T A_j(s-t;n)e^{-r(s-t)}ds\right)v^{j} \tag{2.36}$$

for $v > 0$ and $\tau = T-t \geq 0$, where the functions $U_E^{(m)}(v,\tau)$ and $A_j(s-t;n), j = 0, 1, ..., n,$ are given in Theorem 2.1 by Thamrongrat and Rujivan [8].

Proof. See Theorem 2.1 in Thamrongrat and Rujivan [8]. ∎

The integral terms on the RHS of (2.35)-(2.36) can be worked out when v_t follows the CIR process (1.2), as shown in the following theorem.

Theorem 2.6. *According to Theorem 2.5, if v_t follows the CIR process (1.2) then*

$$\begin{aligned} U_C(v,\tau) = &\sum_{k=0}^{\infty}\left\{c_k^{(\gamma_1)}\frac{e^{-(r+\gamma_1\kappa)\tau}}{k!}\left(\frac{e^{\kappa\tau}-1}{\kappa}\right)\right\}v^{\gamma_1-k} \\ &+\left\{c_k^{(\gamma_2)}\frac{1}{\kappa^k}\sum_{i=0}^{k}\frac{(-1)^{k-i+1}}{(k-i)!i!}\left(\frac{e^{-(r+(\gamma_2-i)\kappa)\tau}-1}{r+(\gamma_2-i)\kappa}\right)\right\}v^{\gamma_2-k} \end{aligned} \tag{2.37}$$

for $v > 0$ and $\tau = T - t > 0$, where we define $c_0^{(\gamma)} = 1$ and

$$c_k^{(\gamma)} = \prod_{l=1}^{k}(\gamma - l + 1)(\tfrac{1}{2}(\gamma - l)\sigma^2 + \kappa\theta)$$

for $k = 1, 2, ...,$ and $\gamma \in \mathbb{R}$.

In particular, if $\gamma_1 = m$ and $\gamma_2 = n$ are nonnegative integers, then

$$\begin{aligned} U_C(v,\tau) = &\sum_{j=0}^{\max(m,n)}\left\{d_j^{(m)}\frac{e^{-(r+m\kappa)\tau}}{(m-j)!}\left(\frac{e^{\kappa\tau}-1}{\kappa}\right)^{m-j}\right\}v^{j} \\ &+\sum_{j=0}^{\max(m,n)}\left\{d_j^{(n)}\frac{1}{\kappa^{n-j}}\sum_{i=0}^{n-j}\frac{(-1)^{n-j-i+1}}{(n-j-i)!i!}\left(\frac{e^{-(r+(n-i)\kappa)\tau}-1}{r+(n-i)\kappa}\right)\right\}v^{j} \end{aligned} \tag{2.38}$$

for $v > 0$ and $\tau = T - t > 0$, where for any nonnegative integer N, we define $d_j^{(N)} = 0$ for $j > N$, $d_N^{(N)} = 1$ and

$$d_j^{(N)} = \prod_{l=1}^{N-j}(N - l + 1)(\tfrac{1}{2}(N - l)\sigma^2 + \kappa\theta)$$

for $j < N$.

Proof. See Theorem 2.2 in Thamrongrat and Rujivan [8]. ∎

3. Numerical Results and Discussions

Here, we study the accuracy of the closed-form expansion (2.34) through numerical simulations. In our numerical study, we consider the ECIR process:

$$dv_t = \kappa\left(\frac{\sigma_0^2 de^{2\sigma_1 t}}{4\kappa} - v_t\right)dt + \sigma_0 e^{\sigma_1 t}\sqrt{v_t}dW_t \tag{3.1}$$

where $\kappa(t) = \kappa$, $\theta(t) = \frac{1}{4\kappa}\sigma_0^2 de^{2\sigma_1 t}$, and $\sigma(t) = \sigma_0 e^{\sigma_1 t}$, with κ, d, σ_0, and σ_1 are positive constants. The transition density of v_t, denoted by $p_V(w, t+\tau|v, t)$ for all $w > 0$, was found by Egorov et al. [14] (see page 121 of their paper).

For a special case, when $g = h = 0$, we can compute $U_E(v, \tau)$ by using the formula:

$$U_E(v,\tau) = E^P[f(v_T)|v_t = v] = \int_0^\infty f(w)p_V(w, t+\tau|v,t)dw \tag{3.2}$$

for all $v > 0$ and $\tau = T - t \geq 0$. As discussed in Rujivan [6], the semi-infinite integral (3.2) may not be available in closed-form, and numerical integration may be employed to approximate $U_E(v,\tau)$. However, it must be remarked that when τ is very small, i.e., $\tau = 0.01$ (year), the usual numerical integration methods cannot be performed well, because p_V behaves like a Dirac delta function, which has an infinitely thin spike at the initial value v. To avoid this problem, one can employ Monte Carlo simulations in order to approximate the value of $U_E(v,\tau)$.

In general f, g, and h may not vanish; the calculation of the conditional expectation (1.4) by integrating the joint-transition density of the two stochastic processes, $x_{t,T}$ and $y_{t,T}$ defined in (2.18) and (2.25), is clearly not an easy task. In fact, the joint-transition density of the two stochastic processes is a function of two state variables. This leads to the computation of a semi-infinite double integral that is more time-consuming than the simple case, when $g = h = 0$. Although MC simulations can also be employed under the latter case, it would dramatically increase computational time from the simple case, since a large number of sample points $x_{t_i,T}$ and $y_{t_i,T}$ for $t_i \in [t, T]$ are needed to approximate the integrals contained in (2.18) and (2.25), and a large number of sample paths of $x_{t,T}$ and $y_{t,T}$ are required in order to achieve a prescribed accuracy. The present section demonstrates one of the two major contributions of the paper such that the computational time of using the closed-form formula (2.34) to compute the conditional expectation (1.4) would be much less than that of the MC simulations.

Our MC simulations are based on a simple simulation of the ECIR processes by utilizing the Euler-Maruyama discretization. Higham and Mao [15] proved that the Euler-Maruyama discretization is an attractive method, providing qualitatively correct approximations. Since our aim is primarily to obtain some benchmark values for the closed-form expansion (2.34), we will not focus our attention on the use of other variance reduction techniques that could further enhance the computational efficiency. In our MC simulations, we have employed the simple Euler-Maruyama discretization for the ECIR process (1.1)

$$v_{t_i} = v_{t_{i-1}} + \kappa(t_{i-1})(\theta(t_{i-1}) - v_{t_{i-1}})\Delta t + \sigma(t_{i-1})\sqrt{v_{t_{i-1}}}\sqrt{\Delta t}W_{t_i} \tag{3.3}$$

where W_t is a standard normal random variable. The parameters are set to follow $\kappa = 0.03$, $d = 3.40$, $\sigma_0 = 0.01$, and $\sigma_1 = 0.02$ (unless otherwise stated). Our numerical experiments are performed under Microsoft Windows 7 64-bit on a processor Intel(R) Core(TM) i5-3210M 2.5 GHz machine with 8GB main memory.

3.1. Notations

In what follows, we let $U_E^{(N,K_1,K_2)}$ denote an $N-$step closed-form expansion of U_E in the form of (2.34), in which the partial sum of the infinite series (2.10) is used up to order K_1, with the Taylor expansion (2.17) up to order K_2. Moreover, $U_E^{(M,N_p)}$ denotes an approximate of U_E, obtained by MC simulations with a number of sample paths is N_p.

In the case that U_E has a closed-form formula, we measure the level of accuracy of our approach and of the MC simulations by using the percentage absolute relative errors (R_E) defined, respectively, by

$$R_E^{(N,K_1,K_2)}(v,\tau) := \left| \frac{U_E(v,\tau) - U_E^{(N,K_1,K_2)}(v,\tau)}{U_E(v,\tau)} \right| \times 100 \tag{3.4}$$

and

$$R_E^{(M,N_p)}(v,\tau) := \left| \frac{U_E(v,\tau) - U_E^{(M,N_p)}(v,\tau)}{U_E(v,\tau)} \right| \times 100 \tag{3.5}$$

for any $(v,\tau) \in (0,\infty) \times (0,\infty)$, providing that $U_E(v,\tau) \neq 0$. In addition, we use $U_C^{(N,K_1,K_2)}$, $U_C^{(M,N_p)}$, $R_C^{(N,K_1,K_2)}$ and $R_C^{(M,N_p)}$ instead of $U_E^{(N,K_1,K_2)}$, $U_E^{(M,N_p)}$, $R_E^{(N,K_1,K_2)}$ and $R_E^{(M,N_p)}$, respectively, when the ECIR process (3.1) becomes a CIR process under some cases of parameter setting.

On the other hand, when U_E has not been found in closed-form, for a fixed N, we define a sequence of absolute relative errors as

$$\epsilon^{(N,k)}(v,\tau) := \left| \frac{U_E^{(N,k+1,k+1)}(v,\tau) - U_E^{(N,k,k)}(v,\tau)}{U_E^{(N,k,k)}(v,\tau)} \right| \tag{3.6}$$

for any $(v,\tau) \in (0,\infty) \times (0,\infty)$, and $k = 1,2,...$ providing that $U_E^{(N,k,k)}(v,\tau) \neq 0$. Furthermore, we claim that if $\epsilon^{(N,k)}(v,\tau) < 0.5 \times 10^{-n}$, then the current approximate $U_E^{(N,k,k)}(v,\tau)$ of $U_E(v,\tau)$ is correct to at least n significant digits, where n is a positive integer.

The definite integrals contained in (2.18) and (2.25) are computed using the trapezoidal rule, with the number of sample points being 1,000. In our numerical tests, we compute the values of $U_E^{(N,K_1,K_2)}(v,\tau)$ and $U_E^{(M,N_p)}(v,\tau)$ at the initial values $v \in \{0.1, 0.2, ..., 2\}$. In order to demonstrate the performance of our current approach, we divide our numerical study into three parts as follows.

3.2. Comparison with some exact solutions

We first compare our numerical results with the exact solutions provided in Theorem 2.5.

Example 3.1. This example cosiders the closed-form formula (2.36) for ECIR processes with $m = 2$ and $n = 3$. Using (2.34) with the parameter functions $\theta(t) = \frac{\sigma_0^2 d e^{2\sigma_1 t}}{4\kappa}$, $\kappa(t) = \kappa$, and $\sigma(t) = \sigma_0 e^{\sigma_1 t}$, we obtain the $N-$step closed-form expansions for $N = 1,2$ and $K_1 = K_2 = 1$ as

$$U_E^{(1,1,1)}(v,\tau) = v^2 + \frac{1}{2} v \left((d+2)\sigma_0^2 e^{2\sigma_1 T} + 2v(-2\kappa - r + v) \right) \tau \tag{3.7}$$

and

$$U_E^{(2,1,1)}(v,\tau) = v^2 + \tfrac{1}{4}v\left((d+2)\sigma_0^2\left(e^{\sigma_1(t+T)} + e^{2\sigma_1 T}\right) + 4v(-2\kappa - r + v)\right)\tau$$
$$+ \quad \tfrac{1}{32}(-2\sigma_0^2 v e^{\sigma_1 T}\left\{e^{\sigma_1 t}(4(d+2)\kappa + 2(d+2)r - 3(d+4)v) + 2(d+2)(\kappa+r)e^{\sigma_1 T}\right\}$$
$$+ \quad d(d+2)\sigma_0^4 e^{\sigma_1(t+3T)} + 8v^2\left(4\kappa^2 + r^2 + 4\kappa r - rv - 3\kappa v\right))\tau^2 \tag{3.8}$$

where $\tau = T - t$. The $N-$step closed-form expansions for $N = 3, 4, 5$, and $K_1 = K_2 = 1$ can further be obtained by using (2.34). The exact solution $U_E(v,\tau)$ can be derived by using (2.36) with $m = 2$ and $n = 3$.

By setting $t = 0$, $T = 1$, and $r = 0.1$, Figures 1-2 display the convergence of the multi-step closed-form expansion (2.34) towards the exact solution U_E when N increases for all $v \in (0, 2)$. We can clearly see that R_E is reduced from 6.2%, with $N = 1$ being less than 1.2% with $N = 5$. Moreover, as displayed in Figure 3, when the final time is smaller, such as $T = 0.5$, $U_E^{(5,1,1)}(v, 0.5)$ perfectly matches the exact solution with $R_E^{(5,1,1)}(v, 0.5)$ being less than 0.6% for all $v \in (0, 2)$.

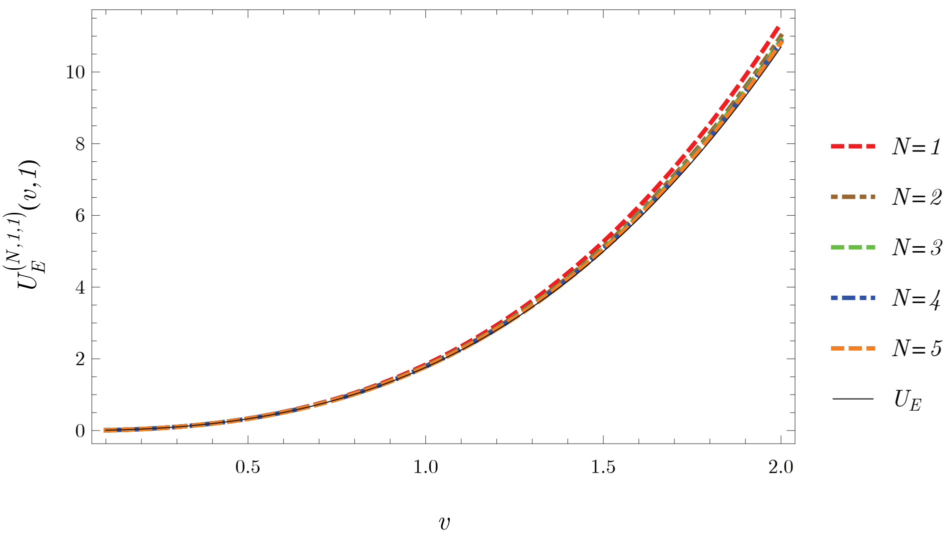

FIGURE 1. Convergence of the multi-step closed-form expansion (2.34) towards the exact solution when the number of steps is increased where $\tau = 1$ and the contingent claim is set in Example 1.

Example 3.2. In this example, we consider the closed-form formula (2.35) for ECIR processes with $\gamma_1 = \frac{1}{2}$ and $\gamma_2 = -\frac{1}{2}$. Using (2.34), with the same setting for the parameter functions and times t and T used in the previous example, we obtain the one-step closed-form expansions for $K_1 = K_2 = K = 1, 2$ as

$$U_E^{(1,1,1)}(v,\tau) = \sqrt{v} + \frac{\left((d-1)\sigma_0^2 e^{2\sigma_1 T} - 4v(\kappa + 2r) + 8\right)}{8\sqrt{v}}\tau \tag{3.9}$$

and

$$U_E^{(1,2,2)}(v,\tau) = U_E^{(1,1,1)}(v,\tau) + \lambda\tau^2 \tag{3.10}$$

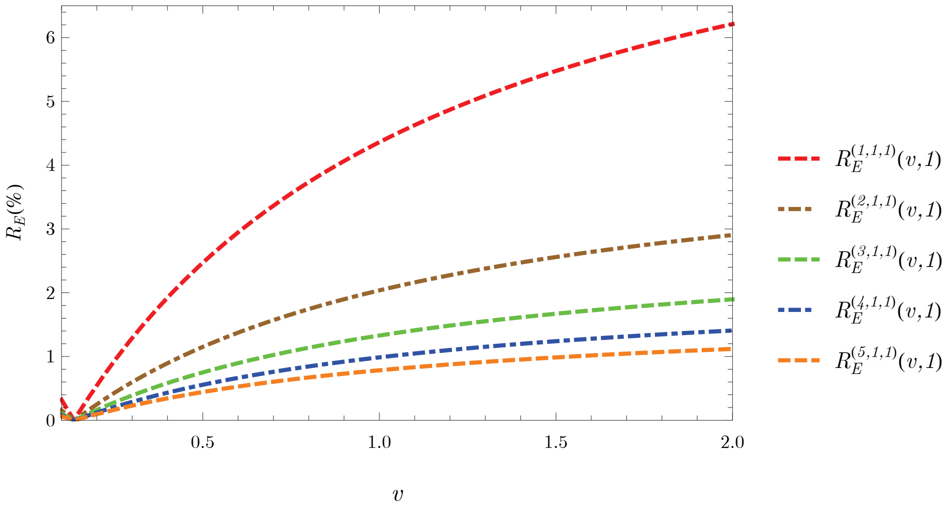

FIGURE 2. Convergence of the percentage absolute errors towards zero when the number of steps in the multi-step closed-form expansion (2.34) is increased where $\tau = 1$ and the contingent claim is set in Example 1.

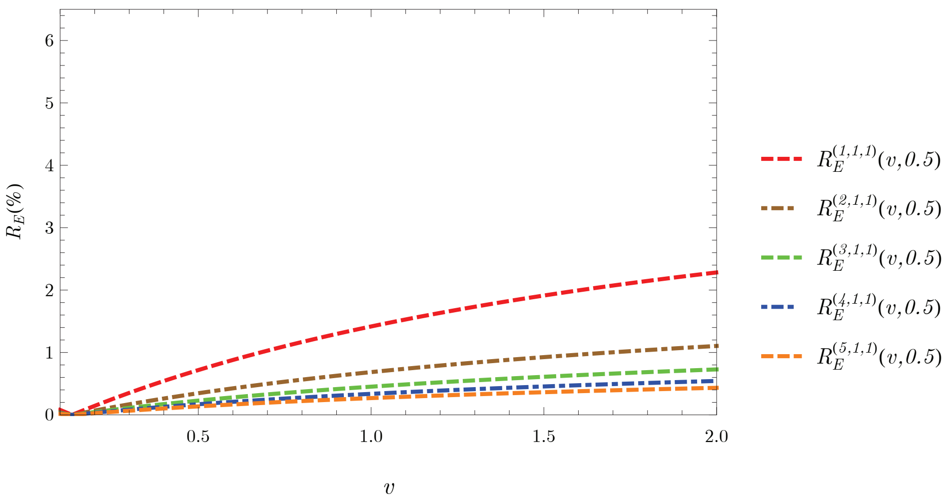

FIGURE 3. Convergence of the percentage absolute errors towards zero when the number of steps in the multi-step closed-form expansion (2.34) is increased where τ is reduced to 0.5 and the contingent claim is set in Example 1.

where $\lambda = \frac{\left(16v\left(-(d-1)\sigma_0^2(r+\sigma_1)e^{2\sigma_1 T}+2\kappa-4r\right)-(d-3)\sigma_0^2 e^{2\sigma_1 T}\left((d-1)\sigma_0^2 e^{2\sigma_1 T}+8\right)+16v^2(\kappa+2r)^2\right)}{128v^{3/2}}$ and $\tau = T - t$. The one-step closed-form expansions for $K_1 = K_2 = K = 3, 4, 5$, can also be

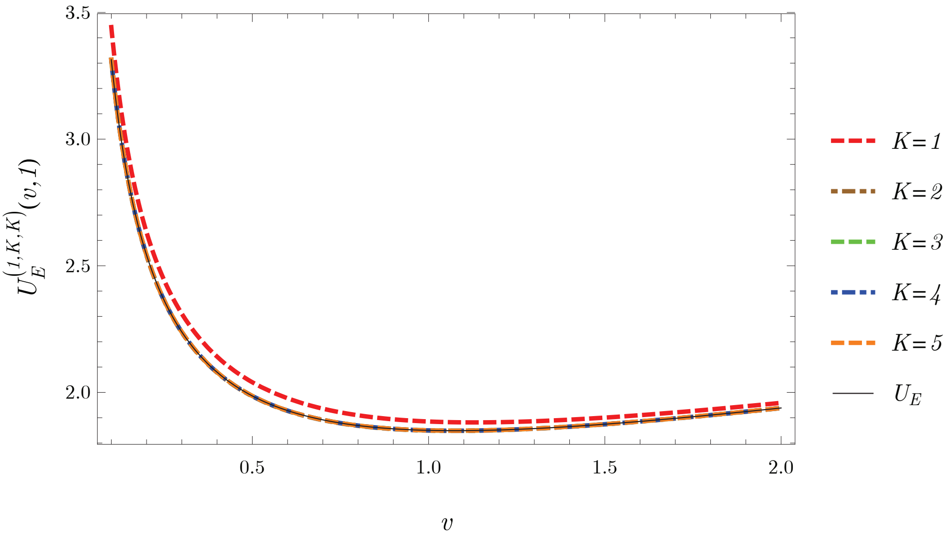

FIGURE 4. Convergence of the one-step closed-form expansion (2.10) towards the exact solution when the number of terms in the infinite series (2.10) and (2.17) used up to order K, is increased where $\tau = 1$ and the contingent claim is set in Example 2.

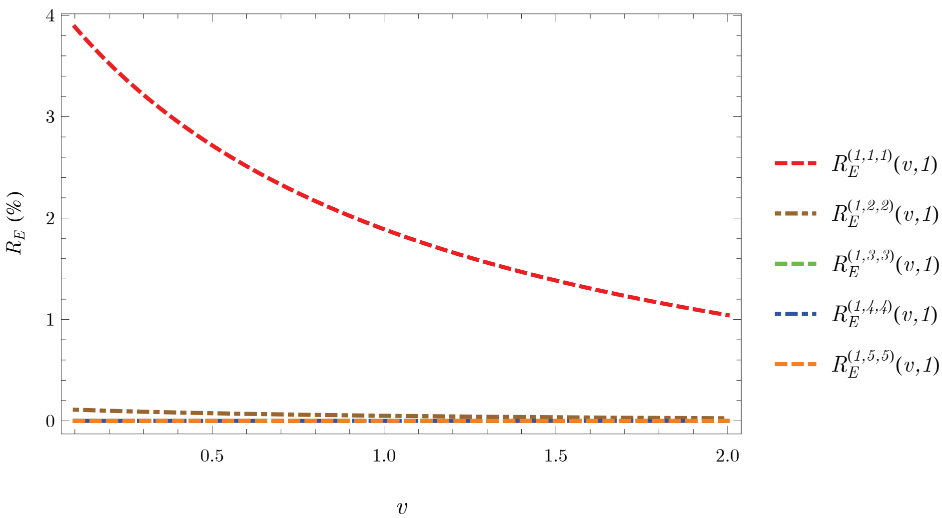

FIGURE 5. Convergence of the percentage absolute errors towards zero when the number of terms in the infinite series (2.10) and (2.17) used up to order K, is increased where $\tau = 1$ and the contingent claim is set in Example 2.

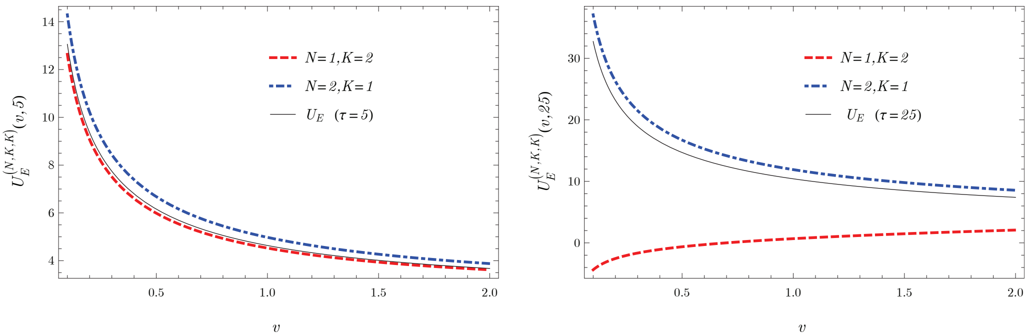

FIGURE 6. The efficiencies of the one-step closed-form expansion $U_E^{(1,2,2)}$ (Red) with $K = 2$ and the two-step closed-form expansion $U_E^{(2,1,1)}$ (Blue) with $K = 1$ compared with two different time-step sizes $\tau = 5$ and $\tau = 25$ where the contingent claim is set in Example 2.

obtained by using (2.34). The exact solution $U_E(v, \tau)$ can be deduced by using (2.35) with $\gamma_1 = \frac{1}{2}$ and $\gamma_2 = -\frac{1}{2}$.

As displayed in Figures 4-5, the current example demonstrates the convergence of the one-step closed-form expansion (2.10) towards the exact solution U_E. It can be easily seen from Figure 5 that R_E converges rapidly to zero when K is increased to $2, ..., 5$, respectively, for all $v \in (0, 2)$. The numerical results illustrate that the one-step closed-form expansion (2.10) is feasible and effective for approximating the exact solution U_E.

From our numerical results presented in Examples 1 and 2, an interesting question could be addressed, namely of what we should increase between N and K to improve the accuracy of our current approach when τ is large. Figure 6 shows that the two-step closed-form expansion $U_E^{(2,1,1)}$ gives better approximates for the exact solution than the one-step closed-form expansion $U_E^{(1,2,2)}$ when τ is large. Although one can increase both N and K to obtain much better approximates for the exact solution, it might take a very long time for symbolic computation to obtain a closed-form expansion when N and K are too large.

In the next example, we compare our current approach with MC simulations in terms of computational time by using the exact solutions provided in Theorem 2.6.

Example 3.3. Let us consider the closed-form formula (2.38) for CIR processes with $m = 3$ and $n = 4$. By setting $\sigma_1 = 0$, the ECIR process (3.1) becomes a CIR process with the parameter functions $\theta(t) = \frac{\sigma_0^2 d}{4\kappa}$, $\kappa(t) = \kappa$, and $\sigma(t) = \sigma_0$. Therefore, we set $K_2 = 0$. Using (2.34), one can obtain the $N-$step closed-form expansions for $N = 3, 4, 5$ and $K_1 = 2$. The exact solution $U_C(v, \tau)$ under this case can be obtained by using the closed-form formula (2.38) with the replacing rule $(\kappa \to \kappa, \theta \to \frac{\sigma_0^2 d}{4\kappa}, \sigma \to \sigma_0)$.

By setting $t = 0$, $T = 1$, and $r = 0.1$, we plot $R_E^{(N,2,0)}(v, \tau)$ against $R_E^{(M,N_p)}(v, \tau)$ for $v \in \{0.1, 0.2, ..., 2\}$, as shown in Figure 7 with $N = 3, 4, 5$ and $N_p = 10^3, 10^4, 10^5$. Figure 7 also shows the computational times written in the parentheses, obtained by using our current approach and MC simulations.

From Figure 7, when N_p reaches $1,000$, $R_C^{(M,N_p)}(v,\tau)$ is less than 0.30% already for all $v \in \{0.1, 0.2, ..., 2\}$. Such a relative error is further reduced when the number of paths is increased, demonstrating the convergence of the MC simulation towards the exact solution U_C. This is similar to our approach when the number of step in the multi-step closed-form expansion (2.34) is increased, as shown in Example 1.

On the other hand, in terms of computational time, the MC simulation takes a much longer time than the multi-step closed-form expansion (2.34). For example, in order to obtain R_C being less than 0.06% for all sample points $v \in \{0.1, 0.2, ..., 2\}$, $U_C^{(3,2,0)}$ consumed roughly 23.4 million folds of reduction in computational time (seconds), compared with $U_C^{(M,10^5)}$. Furthermore, it is clear from Figure 7 that our current approach is more efficient and robust than the MC simulation.

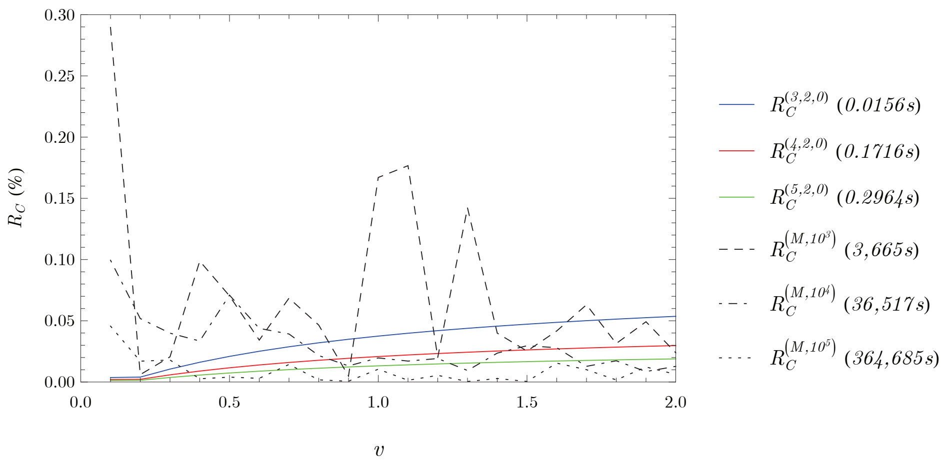

Figure 7. Convergences of the percentage absolute errors towards zero obtained from the multi-step closed-form expansion (2.34) and MC simulation with the computational times (seconds) written in the parentheses where the contingent claim is set in Example 3.

3.3. Comparison with Monte Carlo simulations

In this subsection, MC simulations are conducted to illustrate the accuracy of our current approach for some cases of contingent claims in which the corresponding conditional expectations, written in the form of (1.4), have not been found in closed-form.

Example 3.4. We choose $f(v) = sin(v), g(v) = \frac{v}{v+1}$ and $h(v) = cos(e^v)$. Using (2.10) with the contingent claim (f, g, h) and the same setting for the parameter functions used Example 1, we obtain the one-step closed-form expansion for $K_1 = K_2 = 1$ as

$$U_E^{(1,1,1)}(v,\tau) = \sin(v) + \frac{\left(\sigma_0^2(v+1)e^{2\sigma_1 T}(d\cos(v)-2v\sin(v))-4v(\kappa(v+1)\cos(v)+\sin(v))+4(v+1)\cos(e^v)\right)}{4(v+1)}\tau \tag{3.11}$$

where $\tau = T - t$. Utilizing (2.34), we can further obtain $U_E^{(1,k,k)}(v,\tau)$ for $k = 2, ..., 5$, which shall also be used to investigate the accuracy of our current approach in this example.

As displayed in Figure 8, we can clearly observe that our numerical results obtained by using our one-step closed-form expansion (3.11) perfectly match the results from the MC simulation with $N_p = 10,000$ at the initial values $v \in \{0.1, 0.2, , 2\}$ and $\tau = 0.1$. According to the convergence of the MC simulation towards the exact solution when N_p approaches infinity as presented in Example 3, this guarantees that our one-step closed-form expansion (3.11) also produces accurate approximates for the exact solution for $v \in \{0.1, 0.2, , 2\}$ and $\tau = 0.1$, but consumes a very much shorter time than the MC simulation.

Since the exact solution U_E has not been found in closed-form under this setting of contingent claim (f, g, h), it would also be interesting to investigate the level of accuracy of our one-step closed-form expansion (3.11) through the sequence of absolute relative errors $\epsilon^{(1,k)}(v,\tau)$ introduced in (3.6). Therefore, we have tabulated the absolute relative errors for $\tau = 0.01, 0.1$ and $k = 1, ..., 5$, in Table 1, where we set the initial value v to be varied from 0.0001 to 2.0.

By fixing τ and v, Table 1 illustrates that the accuracy of our one-step closed-form expansion (3.11) improves when k increases. Furthermore, we can conclude from our numerical results shown in Table 1 that using $k = 1$ is sufficient for obtaining an accurate approximation for the exact solution when τ is small.

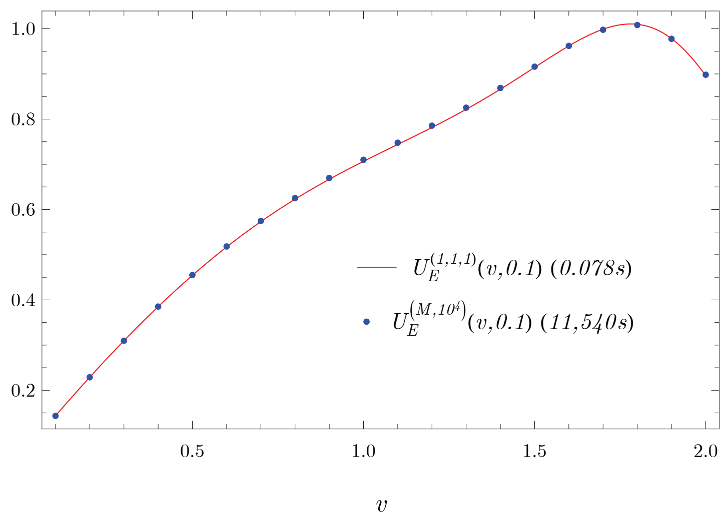

FIGURE 8. Comparison between the approximates for the exact solution $U_E(v, 0.1)$ obtained by using the one-step closed-form expansion (3.11) and MC simulation, where the computational times (seconds) are written in the parentheses and the contingent claim is set in Example 4.

Example 3.5. Let $f(v) = |v - 1|$, $g(v) = sin(v)$ and $h(v) = v$. This example presents how to adopt the multi-step closed-form expansion (2.34) when Assumption 3 is violated, such that the function $f(v) = |v-1|$ of the contingent claim (f, g, h) is not differentiable at $v = 1$. According to the study of PDE of parabolic type, the existence and uniqueness of

	$\epsilon^{(1,k)}(v,\tau)$, $\tau = 0.01$						
k	$v = 0.0001$	$v = 0.001$	$v = 0.01$	$v = 0.1$	$v = 1.0$	$v = 1.5$	$v = 2.0$
1	0.12 E-05	0.46 E-05	0.17 E-04	0.17 E-04	0.44 E-04	0.16 E-04	0.26 E-04
2	0.11 E-08	0.55 E-09	0.18 E-08	0.63 E-08	0.79 E-07	0.91 E-08	0.13 E-06
3	0.47 E-12	0.38 E-12	0.54 E-13	0.20 E-11	0.11 E-09	0.25 E-10	0.35 E-09
4	0	0	0	0.53 E-15	0.12 E-12	0.80 E-13	0.60 E-12
	$\epsilon^{(1,k)}(v,\tau)$, $\tau = 0.10$						
k	$v = 0.0001$	$v = 0.001$	$v = 0.01$	$v = 0.1$	$v = 1.0$	$v = 1.5$	$v = 2.0$
1	0.12 E-04	0.54 E-04	0.40 E-03	0.13 E-02	0.51 E-02	0.17 E-02	0.26 E-02
2	0.11 E-06	0.65 E-07	0.43 E-06	0.46 E-05	0.92 E-04	0.98 E-05	0.13 E-03
3	0.48 E-09	0.44 E-09	0.13 E-09	0.14 E-07	0.13 E-05	0.27 E-06	0.35 E-05
4	0.10 E-11	0.10 E-11	0.66 E-12	0.41 E-10	0.14 E-07	0.87 E-08	0.61 E-07

Remark: Zeros in the table signify $\epsilon^{(1,k)}(v,\tau) < 0.5 \times 10^{-16}$.

Table 1. Sequences of absolute relative errors with different time-step sizes $\tau = 0.01$ and $\tau = 0.10$ obtained by using the one-step closed-form expansion (2.10) where the contingent claim is set in Example 4.

the solution of the PDE (2.14) subject to an initial condition, which is a piecewise smooth function, is guaranteed by Theorem 12 in Friedman [16]. Since the exact solution under this case of contingent claims has not been found in closed-form, MC simulations can be employed to obtain an approximate, denoted by $U^{(M,N_p)}_{E,f}(v,\tau)$, for the exact solution $U_{E,f}(v,\tau)$ for any $v > 0$ and $\tau > 0$.

Although the derivative of f at $v = 1$ does not exist for all $n = 1, 2, \ldots$, we shall demonstrate that the multi-step closed-form expansion (2.34) is applicable to obtain good approximates for the exact solution $U_{E,f}(v,\tau)$ for any $v > 0$ and $\tau > 0$.

In order to test our hypothesis, we define $f^-(v) := 1 - v$ and $f^+(v) := v - 1$ for $v > 0$. Since f^- and f^+ are smooth functions. We can use the multi-step closed-form expansion (2.34) with the contingent claims (f^-, g, h) and (f^+, g, h) to derive $U^{(N,K,K)}_{E,f^-}(v,\tau)$ and $U^{(N,K,K)}_{E,f^+}(v,\tau)$, respectively, for any $N, K = 1, 2, \ldots$. Nevertheless, for a fixed τ and sufficient large values of N and K, we shall justify that

$$U_{E,f}(v,\tau) \approx \begin{cases} U^{(N,K,K)}_{E,f^-}(v,\tau) & ; v < v_\tau \\ U^{(N,K,K)}_{E,f^+}(v,\tau) & ; v > v_\tau \end{cases} \tag{3.12}$$

for some $v_\tau > 0$ depending on τ.

In our numerical test, we first compute $U^{(1,2,2)}_{E,f^-}(v,\tau)$ and $U^{(1,2,2)}_{E,f^+}(v,\tau)$ for $\tau = \tau_1 = 0.1$ and 100 values of v, sampled uniformly in a neighborhood of the non-differentiable point $v = 1$, i.e. $(1 - \epsilon, 1 + \epsilon)$ for $\epsilon = 0.05$. Next, we compare our numerical results with $U^{(M,N_p)}_{E,f}(v,\tau)$ for all sampled points v where we set $N_p = 10{,}000$ in MC simulation. In other words, we claim that $U_{E,f}(v,\tau) \approx U^{(M,10^4)}_{E,f}(v,\tau)$ for any $v > 0$ and $\tau > 0$.

Figure 9 illustrates that $U^{(1,2,2)}_{E,f^-}(v,\tau_1)$ for $v < v_{\tau_1}$ and $U^{(1,2,2)}_{E,f^+}(v,\tau_1)$ for $v > v_{\tau_1}$ perfectly match $U^{(M,10^4)}_{E,f}(v,\tau_1)$ in the neighborhood where $v_{\tau_1} \approx 1.003$ is determined by the intersection point between the two curves. When we set $\tau = \tau_2 = 1$, however, $U^{(1,2,2)}_{E,f^-}(v,\tau_2)$ and $U^{(1,2,2)}_{E,f^+}(v,\tau_2)$ are away from $U^{(M,10^4)}_{E,f}(v,\tau_2)$, as shown in Figure 10.

To obtain better approximates for $U_{E,f}(v,\tau_2)$, we increase the values of N and K to 2 and 4, respectively. Next, we compute $U^{(2,4,4)}_{E,f^-}(v,\tau_2)$ and $U^{(2,4,4)}_{E,f^+}(v,\tau_2)$ for all sampled points v. As displayed in Figure 11, $U^{(2,4,4)}_{E,f^-}(v,\tau_2)$ for $v < v_{\tau_2}$ and $U^{(2,4,4)}_{E,f^+}(v,\tau_2)$ for $v > v_{\tau_2}$ perfectly match $U^{(M,10^4)}_{E,f}(v,\tau_2)$ in the neighborhood where $v_{\tau_2} \approx 1.0304$.

Furthermore, our numerical results lead to a conclusion that $U^{(N,K,K)}_{E,f^-}(v_\tau,\tau)$ and $U^{(N,K,K)}_{E,f^+}(v_\tau,\tau)$ converge to the exact solution $U_{E,f}(v_\tau,\tau)$ when τ approaches zero. On the other hand, when τ is large, as shown in Figures 10-11, we need to increase the values of N and K in order to improve the accuracy of the approximation (3.12).

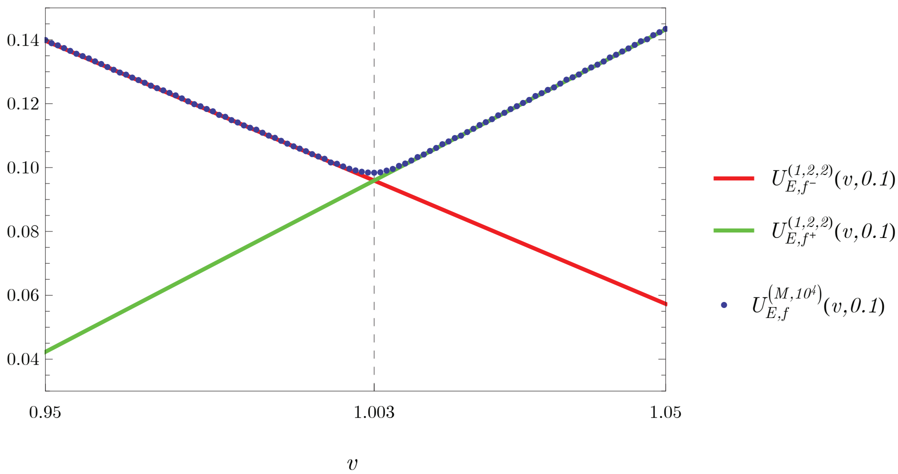

FIGURE 9. Comparison between the approximates for the exact solution $U_{E,f}(v,0.1)$ obtained by using the contingent claims (f^-,g,h), (f^+,g,h) with $N=1, K=2$ in (2.34) and MC simulation in Example 5.

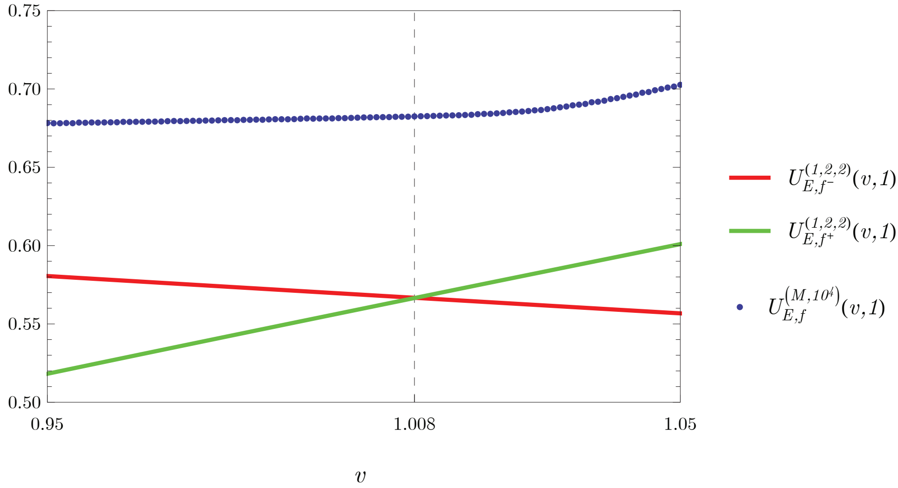

FIGURE 10. Comparison between the approximates for the exact solution $U_{E,f}(v,1)$ obtained by using the contingent claims (f^-, g, h), (f^+, g, h) with $N = 1, K = 2$ in (2.34) and MC simulation in Example 5.

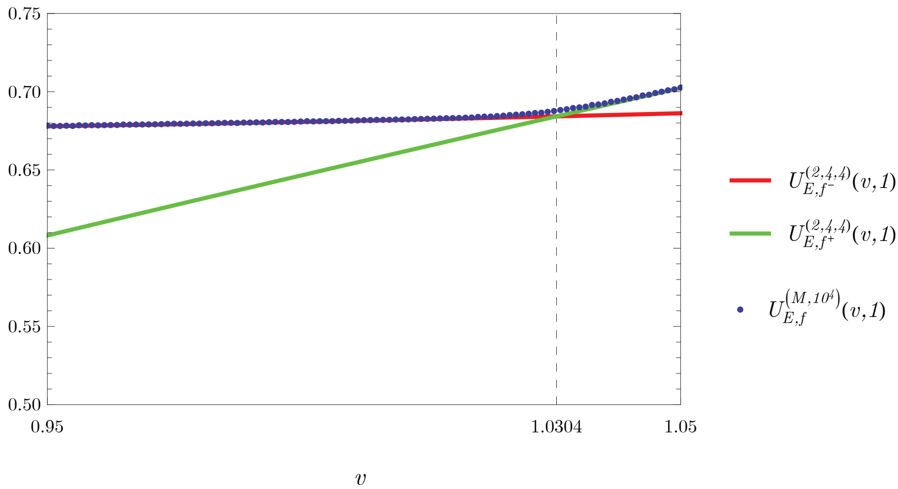

FIGURE 11. Comparison between the approximates for the exact solution $U_{E,f}(v,1)$ obtained by using the contingent claims (f^-, g, h), (f^+, g, h) with $N = 2, K = 4$ in (2.34) and MC simulation in Example 5.

4. Conclusions

This paper has proposed an iterative approach for obtaining a closed-form expansion for the conditional expectation of the valuation process, defined by $V_{t,T} := e^{-\int_t^T g(v_s)ds} f(v_T) + \int_t^T h(v_s) e^{-\int_t^s g(v_u)du} ds$ for $0 \le t \le T$, where v_t is assumed to follow the extended Cox-Ingersoll-Ross process, for any smooth real-valued functions f, g, and h. Our closed-form expansion has a simple form, assisting market practitioners to easily compute approximates for the conditional expectation. The accuracy and efficiency of the current approach have been tested by several examples, demonstrating its superiority over the MC method in terms of computational time and effort. Moreover, a multi-step closed-form expansion has been constructed in order to improve the accuracy of the current approach when the time-step size $T - t$ is large. Finally, we have presented how to adopt our approach for obtaining a closed-form expansion for the conditional expectation when f is a piecewise smooth function. Of course, how to resolve this problem when g and h so are piecewise smooth functions may be a limitation of this work. An extension to the present paper to solve the problem by utilizing the approximation technique as demonstrated in our last example may be a potential topic in the future.

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