Thai Journal of Mathematics
Special Issue (Annual Meeting in Mathematics, 2008) : 39-44
www.math.science.cmu.ac.th/thaijournal
Online ISSN 1686-0209

The Stability of the Pexiderized Cosine Functional Equation

C. Kusollerschariya and P. Nakmahachalasint

Abstract : In this paper, we study the stability of the following pexiderized cosine functional equation:

$$f_1(x+y)+f_2(x-y)=2g_1(x)g_2(y).$$

Keywords : Cosine functional equation, Stability, Superstability.
2000 Mathematics Subject Classification : 39B62, 39B82.

1 Introduction

In 1940, S.M. Ulam [8] proposed the stability problem of the additive functional equation: $f(x+y)=f(x)+f(y)$. One year later, D.H. Hyers [3] gave the positive answer to the problem as follows. "Let $f : E \to E'$ be a function from a Banach space to a Banach space which satisfies the inequality $\|f(x+y)-f(x)-f(y)\| \le \varepsilon$ for all $x, y \in E$. Then there exists a unique additive function φ satisfying the inequality $\|f(x)-\varphi(x)\| \le \varepsilon$." In 1978, a generalized version of Hyers' result was proven by Th. M. Rassias in [6] where $f : E \to E'$ satisfies the inequality $\|f(x+y)-f(x)-f(y)\| \le \theta\,(\|x\|^p+\|y\|^p)$ for all $x, y \in E$ and for some constants $\theta \ge 0$ and $0 \le p < 1$.

In 1979, J. Baker, J. Lawrence, and F. Zorzitto [2] introduced that if f satisfies the inequality $|E_1(f)-E_2(f)| \le \varepsilon$, then either f is bounded or $E_1(f)=E_2(f)$. This concept is now known as the *superstability*. In 1980, J. A. Baker [1] observed the superstability of the well-known cosine functional equation

$$f(x+y)+f(x-y)=2f(x)f(y). \tag{A}$$

The following functional equations are some generalized forms of the above functional equation:

$$f(x+y)+f(x-y)=2f(x)g(y), \tag{A_{fg}}$$
$$f(x+y)+f(x-y)=2g(x)f(y). \tag{A_{gf}}$$

The superstability of (A_{fg}) and (A_{gf}) were also studied in [4, 5, 7].

In this paper, we investigate the superstability of the pexiderized cosine functional equation

$$f_1(x+y)+f_2(x-y)=2g_1(x)g_2(y),$$

where f_1, f_2, g_1, and g_2 are functions from $\mathbb{R}$ to $\mathbb{C}$.

2 Main Results

We will study the superstability of the pexiderized cosine functional equation by starting with Theorem 1 where the unboundedness of g_1 is assumed.

Theorem 1. *Let $f_1, f_2, g_1, g_2 : \mathbb{R} \to \mathbb{C}$ be functions satisfying*

$$|f_1(x+y)+f_2(x-y)-2g_1(x)g_2(y)| \le \delta, \tag{2.1}$$

for all $x, y \in \mathbb{R}$. Then either g_1 is bounded or there exists an even function $h : \mathbb{R} \to \mathbb{C}$ with $h(0)=1$ such that

$$g_2(x+y)+g_2(x-y)=2g_2(x)h(y) \quad \forall x, y \in \mathbb{R}.$$

Proof. Suppose that g_1 is unbounded. Then we can choose a sequence $\{x_n\}$ such that $0 \ne |g_1(x_n)| \to \infty$ as $n \to \infty$. For each $n \in \mathbb{N}$, setting $x = x_n$ in (2.1) and dividing both sides of the resulting inequality by $|2g_1(x_n)|$, we have

$$\left| \frac{f_1(x_n+y)+f_2(x_n-y)}{2g_1(x_n)} - g_2(y) \right| \le \frac{\delta}{|2g_1(x_n)|}.$$

The right-hand side approaches to 0 as $n \to \infty$. Therefore,

$$g_2(y) = \lim_{n\to\infty} \frac{f_1(x_n+y)+f_2(x_n-y)}{2g_1(x_n)} \quad \forall y \in \mathbb{R}. \tag{2.2}$$

Substituting $(x,y)=(x_n+y,x)$ and then $(x,y)=(x_n-y,x)$ in (2.1), we obtained

$$|f_1((x_n+y)+x)+f_2((x_n+y)-x)-2g_1(x_n+y)g_2(x)| \le \delta$$

and

$$|f_1((x_n-y)+x)+f_2((x_n-y)-x)-2g_1(x_n-y)g_2(x)| \le \delta.$$

By the triangle inequality, the last two inequalities lead to

$$\begin{aligned} \Bigg| & \frac{f_1(x_n+(x+y))+f_2(x_n-(x+y))}{2g_1(x_n)} \\ & + \frac{f_1(x_n+(x-y))+f_2(x_n-(x-y))}{2g_1(x_n)} \\ & \qquad -2\left(\frac{g_1(x_n+y)+g_1(x_n-y)}{2g_1(x_n)}\right) g_2(x) \Bigg| \le \frac{2\delta}{|2g_1(x_n)|}. \end{aligned} \tag{2.3}$$

We notice that the right-hand side converges to zero as $n \to \infty$. So we define

$$h(y) = \lim_{n\to\infty} \frac{g_1(x_n + y) + g_1(x_n - y)}{2g_1(x_n)} \quad \text{for all} \quad y \in \mathbb{R}.$$

Notice that h is even and $h(0) = 1$.
Then, by letting $n \to \infty$ in (2.3), we see that

$$g_2(x + y) + g_2(x - y) = 2g_2(x)h(y) \quad \text{for all} \quad x, y \in \mathbb{R}$$

as desired. □

In the other way around, we will look at the case g_2 is unbounded.

Theorem 2. *Let $f_1, f_2, g_1, g_2 : \mathbb{R} \to \mathbb{C}$ be functions satisfying*

$$|f_1(x + y) + f_2(x - y) - 2g_1(x)g_2(y)| \le \delta, \tag{2.4}$$

for all $x, y \in \mathbb{R}$. Then either g_2 is bounded or there exists an even function $h : \mathbb{R} \to \mathbb{C}$ with $h(0) = 1$ such that

$$g_1(x + y) + g_1(x - y) = 2g_1(x)h(y) \quad \forall x, y \in \mathbb{R}.$$

Proof. Suppose that g_2 is unbounded. Then we choose a sequence $\{y_n\}$ such that $0 \ne |g_2(y_n)| \to \infty$ as $n \to \infty$. It can be shown similarly to the above theorem that

$$g_1(y) = \lim_{n\to\infty} \frac{f_1(x + y_n) + f_2(x - y_n)}{2g_2(y_n)} \quad \forall y \in \mathbb{R}. \tag{2.5}$$

We set $(x, y) = (x_n + y, x)$ and $(x, y) = (x_n - y, x)$, respectively, in (2.4) and proceed the same fashion of the previous proof. We are led to defining a function h as follows

$$h(y) = \lim_{n\to\infty} \frac{g_2(y_n + y) + g_2(y_n - y)}{2g_2(x_n)} \quad \text{for all} \quad y \in \mathbb{R}$$

and we then have

$$g_1(x + y) + g_1(x - y) = 2g_1(x)h(y) \quad \text{for all} \quad x, y \in \mathbb{R}$$

as desired. Also, note that h is even and $h(0) = 1$. □

Remark. The continuous solutions $f : \mathbb{R} \to \mathbb{R}$ of the functional equation

$$f(x + y) + f(x - y) = 2f(x)g(y)$$

have been thoroughly investigated [9] and they fall into 4 categories:

- $f(x) = c\cos\alpha x + d\sin\alpha x$, $g(x) = \cos\alpha x$;
- $f(x) = c\cosh\alpha x + d\sinh\alpha x$, $g(x) = \cosh\alpha x$;

- $f(x) = c + dx$, $g(x) \equiv 1$;
- $f(x) \equiv 0$, and g arbitrary, where $\alpha, c, d \in \mathbb{R}$.

Notice that if we further assume the evenness of f, then either $f \equiv 0$ or $\hat{f}(x) := \frac{f(x)}{f(0)}$ is equal to the cosine, the cosine hyperbolic, or the constant function 1 which satisfies the cosine functional equation. The following lemma which is easy to verify shows that the similar argument holds without assuming the continuity. To make it easy to write, we continue using this notation $\hat{f}$ and note that it is legel only when $f(0) \neq 0$.

Lemma 1. *Let $f, g : \mathbb{R} \to \mathbb{C}$ be functions satisfying*

$$f(x+y) + f(x-y) = 2f(x)g(y) \quad \textit{for all} \quad x, y \in \mathbb{R}.$$

If f is an even function, then either $f \equiv 0$ or $\hat{f}(x)$ satisfies (A).

We now apply the preceding lemma to obtain the following theorems.

Theorem 3. *Let $f_1, f_2, g_1, g_2 : \mathbb{R} \to \mathbb{C}$ be functions such that*

$$|f_1(x+y) + f_2(x-y) - 2g_1(x)g_2(y)| \leq \delta,$$

for all $x, y \in \mathbb{R}$. Suppose that g_2 is an even function and $g_2 \not\equiv 0$. Then either g_1 is bounded or $\hat{g_2}$ satisfies (A).

Proof. Assume that g_1 is unbounded. It follows from Theorem 1 that there is a function h such that, for every $x, y \in \mathbb{R}$,

$$g_2(x+y) + g_2(x-y) = 2g_2(x)h(y).$$

Then, by the use of Lemma 1, we conclude that $\hat{g_2}$ satisfies (A). □

Similarly, we come to the next theorem.

Theorem 4. *Let $f_1, f_2, g_1, g_2 : \mathbb{R} \to \mathbb{C}$ be functions such that*

$$|f_1(x+y) + f_2(x-y) - 2g_1(x)g_2(y)| \leq \delta,$$

for all $x, y \in \mathbb{R}$. Suppose that g_1 is an even function and $g_1 \not\equiv 0$. Then either g_2 is bounded or $\hat{g_1}$ satisfies (A).

From this point onwards, we observe the stability of the special cases of the proposed functional equation as corollaries. We also have to refer to the definitions of g_1 and g_2 in the proofs of the first two theorems.

Corollary 1. *Let $f, g_1, g_2 : \mathbb{R} \to \mathbb{C}$ be functions such that*

$$|f(x+y) + f(x-y) - 2g_1(x)g_2(y)| \leq \delta,$$

for all $x, y \in \mathbb{R}$. Suppose that $g_2 \not\equiv 0$. Then either g_1 is bounded or $\hat{g_2}$ satisfies (A).

Proof. Taking $f_1 = f_2 = f$ in Theorem 3, we infer the evenness of g_2 from its definition in (2.2) which completes the proof. ☐

Corollary 2. *Let $f, g_1, g_2 : \mathbb{R} \to \mathbb{C}$ be functions such that*

$$|f(x+y) + f(x-y) - 2g_1(x)g_2(y)| \leq \delta,$$

for all $x, y \in \mathbb{R}$. Suppose that f is even and $g_1 \not\equiv 0$. Then either g_2 is bounded or $\hat{g}_1$ satisfies (A).

Proof. Take $f_1 = f_2 = f$ in Theorem 3. The evenness of f and the definition of g_1 in (2.5) lead to the evenness of g_1 which completes the proof. ☐

Corollary 3. *Let $f, g : \mathbb{R} \to \mathbb{C}$ be functions such that*

$$|f(x+y) + f(x-y) - 2f(x)g(y)| \leq \delta,$$

for all $x, y \in \mathbb{R}$. Then either f is bounded or g satisfies (A).

Proof. Take $g_1 = f$ and $g_2 = g$ in Corollary 1 and recall (2.2), where f_1 and f_2 are substituted by f. We see that g is even and $g(0) = 1$; therefore, g, which is equal to $\hat{g}$, satisfies (A). ☐

Corollary 4. *Let $f, g : \mathbb{R} \to \mathbb{C}$ be functions such that*

$$|f(x+y) + f(x-y) - 2g(x)f(y)| \leq \delta,$$

for all $x, y \in \mathbb{R}$. Then, provided that f is even, either f is bounded or g satisfies (A).

Proof. Take $g_1 = g$ and $g_2 = f$ in Corollary 2 and recall (2.5), where f_1 and f_2 are substituted by f. The evenness of g follows from the evenness of f. Also, we see that $g(0) = 1$. Thus, g, which is equal to $\hat{g}$, satisfies (A). ☐

By taking $g = f$ in Corollary 3, we obtain the stability of the cosine functional equation.

Corollary 5. *Let $f : \mathbb{R} \to \mathbb{C}$ be functions such that*

$$|f(x+y) + f(x-y) - 2f(x)f(y)| \leq \delta, \tag{2.6}$$

for all $x, y \in \mathbb{R}$. Then either f is bounded or f satisfies (A).

References

[1] J. A. Baker, "The Stability of the Cosine Functional Equation," *Proceedings of the American Mathematical Society*, vol. 80, pp. 411-416, 1980.

[2] J. Baker, J. Lawrence, F. Zorzitto, "The Stability of the Equation $f(x+y) = f(x)f(y)$," *Proceedings of the American Mathematical Society*, vol. 74, pp. 242-246, 1979.

[3] D. H. Hyers, "On the Stability of the Linear Functional Equations," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, no. 4, pp. 222-224, 1941.

[4] Pl. Kannappan, G. H. Kim, "On the Stability of the Generalizad Cosine Functional Equations," *Anales Academiae Paedagogicae Cracoviensis; Studia Mathematica*, vol. 1, pp. 49-58, 2001.

[5] G. H. Kim, S. H. Lee, "Stability of the d'Alembert Type Functional Equations," *Nonlinear Functional Analysis & Applications*, vol. 9, no. 4, pp. 593-604, 2004.

[6] Th. M. Rassias, "On the Stability of the Linear Mapping in Banach Spaces," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 72, no. 2, pp. 297-300, 1978.

[7] L. Székelyhidi, "The Stability of d'Alembert-type Functional Equations," *Acta Scientiarum Mathematicarum*, vol. 44, no. 3-4, pp. 313-320, 1982.

[8] S. M. Ulam, *Problems in Modern Mathematics*, Chapter 6, John Wiley & Sons, New York, NY, USA, 1964.

[9] B. J. Venkatachala, *Functional Equations A Problem Solving Approach*, Chapter 5, Prism Books Pvt Ltd, Bangalore, 2002.

(Received 29 May 2008)

C. Kusollerschariya
Department of Mathematics
Faculty of Science
Chulalongkon University
Bangkok 10330, THAILAND.
e-mail : `dexchao@hotmail.com`

P. Nakmahachalasint
Department of Mathematics
Faculty of Science
Chulalongkon University
Bangkok 10330, THAILAND.
e-mail : `paisan.n@chula.ac.th`