Thai Journal of Mathematics
Volume 20 Number 1 (2022)
Pages 203–209

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

On the Reversible Geodesics for a Finsler Space with Randers Change of Quartic Metric

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Abstract In this paper, we consider a Finsler space with a Randers change of Quartic metric $F = \sqrt[4]{\alpha^4 + \beta^4} + \beta$. The conditions for this space to be with reversible geodesics are obtained. Further, we study some geometrical properties of F with reversible geodesics and prove that the Finsler metric F induces a generalized weighted quasi-distance d_F on M.

MSC: 53C60; 53B40; 58B20

Keywords: Riemannian spaces; reversible geodesics; weighted quasi metric; randers change; quartic metric

Submission date: 10.01.2019 / Acceptance date: 02.11.2021

1. Introduction

An interesting topic in Finsler geometry is to study the reversible geodesics of a Finsler metric. Recall that, a Finsler space is said to have reversible geodesics if for every one of its oriented geodesic paths, the same path traversed in the opposite sense is also a geodesic. In the last decade many interesting and applicable results have been obtained on the theory of Finsler spaces with reversible geodesics. In [1], Crampin discussed Randers space with reversible geodesics. In ([2, 3]), Masca, Sabau and Shimada have discussed reversible geodesics with (α, β)-metric and two dimensional Finsler space with (α, β)-metric to be with reversible geodesic, respectively. In [4], Sabau and Shimada have given some important results on reversible geodesics. In [5], Shanker and Baby have discussed reversible geodesics for generalized (α, β)-metric. Recently, Shanker and Rani [6] have studied weighted quasi metric associated with Finsler spaces with reversible geodesics.
In this paper, we find conditions for a Finsler space (M, F) with Randers change of Quartic metric $F = \sqrt[4]{\alpha^4 + \beta^4} + \beta$ to be with reversible geodesics. The main results of this paper lies in theorem 3.1, 4.1, 4.2, 5.2 and 5.6.

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2. Preliminaries

Let $F^n = (M, F)$ be a connected n-dimensional Finsler manifold and let TM $= \bigcup_{x\in M} T_xM$ denotes the tangent bundle of M with local coordinates u = (x, y) = $(x^i, y^i) \in TM$, where $i = 1, ..., n$, $y = y^i\dfrac{\partial}{\partial x^i}$.

If γ : [0, 1] $\longrightarrow$ M is a piecewise C^∞ curve on M, then its Finslerian length is defined as

$$L_F(\gamma) = \int_0^1 F(\gamma(t), \dot{\gamma}(t))dt, \tag{2.1}$$

and the Finslerian distance function d_F : M $\times$ M $\longrightarrow$ [0, ∞) is defined by $d_F(p, q) = inf_\gamma L$, where infimum is taken over all piecewise C^∞ curves γ on M joining the points p, q $\in$ M. In general, this is not symmetric.

A curve γ : [0, 1] $\longrightarrow$ M is called a geodesic of (M, F) if it minimizes the Finslerian length for all piecewise C^∞ curves that keep their endpoints fixed. We denote the reverse Finsler metric of F as $\tilde{F}$: TM $\longrightarrow$ (0, ∞), given by $\tilde{F}(x, y) = F(x, -y)$. One can easily see that $\tilde{F}$ is also a Finsler metric.

Lemma 2.1. *A Finsler metric is with a reversible geodesic if and only if for any geodesic γ(t) of F, the reverse curve $\tilde{\gamma}(t)$ = γ(1-t) is also a geodesic of F.*

Lemma 2.2. *Let (M, F) be a connected, complete Finsler manifold with associated distance function d_F : $M \times M \longrightarrow$ [0, ∞). Then, d_F is a symmetric distance function on $M \times M$ if and only if F is a reversible Finsler metric, i.e., $F(x, y) = F(x, -y)$.*

Lemma 2.3. *A smooth curve γ : [0, 1] $\longrightarrow$ M is a constant Finslerian speed geodesic of (M, F) if and only if it satisfies $\ddot{\gamma}$ + 2 $G^i(\gamma(t), \dot{\gamma}(t))$ = 0, i = 1, ..., n, where the functions G^i : TM $\longrightarrow \mathbb{R}$ are given by*

$$G^i(x, y) = \Gamma^i_{jk}(x, y)y^iy^j, \tag{2.2}$$

with $\Gamma^i_{jk}(x, y) = \dfrac{g^{is}}{2}\Big(\dfrac{\partial g_{sj}}{\partial x^k} + \dfrac{\partial g_{sk}}{\partial x^j} - \dfrac{\partial g_{jk}}{\partial x^s}\Big)$.

Remark 2.4. It is well known [7] that the vector field $\Gamma = y^i\dfrac{\partial}{\partial x^i} - 2G^i\dfrac{\partial}{\partial y^i}$, is a vector field on TM, whose integral lines are the canonical lifts $\tilde{\gamma}(t)$ = (γ(t), $\dot{\gamma}(t)$) of the geodesics of γ. This vector field Γ is called the canonical geodesics spray of the Finsler space (M, F) and G^i are called the coefficients of the geodesics spray Γ.

Definition 2.5. If F and $\tilde{F}$ are two different fundamental Finsler functions on the same manifold M, then they are said to be projectively equivalent if their geodesics coincide as set points.

Lemma 2.6. *A Finsler structure (M, F) is with a reversible geodesic if and only if F and its reverse function $\tilde{F}$ are projectively equivalent.*

3. Reversible Geodesics for a Finsler Space with Randers Change of Quartic metric.

Consider a Finsler space (M, F) with a special (α, β)-metric $F = \sqrt[4]{\alpha^4 + \beta^4} + \beta$. Here, F can be treated as the Randers change of Quartic-metric $\tilde{F} = \sqrt[4]{\alpha^4 + \beta^4}$. One can easily

see that $\tilde{F}(x,-y)=\tilde{F}(x,y)$.
As we know that [4] if (M,F) is a non-Riemannian $n(n\geq 2)$-dimensional Finsler space with (α,β)-metric, which is not absolute homogeneous, then F is with reversible geodesics if and only if $F(\alpha,\beta)=F_0(\alpha,\beta)+\epsilon\beta$, where F_0 is absolute homogeneous (α,β)-metric, ϵ is a non-zero constant and β is a closed 1-form on the Manifold M.

In our case, $F_0=\tilde{F}$, which is absolute homogeneous. If β is a closed 1-form, then F is with reversible geodesics. Further, a necessary and sufficient condition for F to have reversible geodesics is that [4]

$$\tilde{\Gamma}\Big(\frac{\partial F}{\partial y^i}\Big)-\frac{\partial F}{\partial x^i}=0, \tag{3.1}$$

where $\tilde{\Gamma}$ is the reverse of Γ, the geodesic spray of F; moreover $\tilde{\Gamma}$ is geodesic spray of $\tilde{F}$. We have F $=\tilde{F}+\beta$, where, $\tilde{F}=\sqrt[4]{\alpha^4+\beta^4}$.

Therefore,

$$\begin{aligned}
&\tilde{\Gamma}\Big(\frac{\partial F}{\partial y^i}\Big)-\frac{\partial F}{\partial x^i}\\
&=\tilde{\Gamma}\Big(F_\alpha\frac{\partial\alpha}{\partial y^i}+F_\beta\frac{\partial\beta}{\partial y^i}\Big)-F_\alpha\frac{\partial\alpha}{\partial x^i}-F_\beta\frac{\partial\beta}{\partial x^i}\\
&=\tilde{\Gamma}(F_\alpha)\frac{\partial\alpha}{\partial y^i}+F_\alpha\tilde{\Gamma}\Big(\frac{\partial\alpha}{\partial y^i}\Big)+\tilde{\Gamma}(F_\beta)\Big(\frac{\partial\beta}{\partial y^i}\Big)+F_\beta\tilde{\Gamma}\Big(\frac{\partial\beta}{\partial y^i}\Big)-F_\alpha\frac{\partial\alpha}{\partial x^i}-F_\beta\frac{\partial\beta}{\partial x^i}\\
&=\tilde{\Gamma}(F_\alpha)\frac{\partial\alpha}{\partial y^i}+F_\alpha\Big[\tilde{\Gamma}\Big(\frac{\partial\alpha}{\partial y^i}\Big)-\frac{\partial\alpha}{\partial x^i}\Big]+\tilde{\Gamma}(F_\beta)\frac{\partial\beta}{\partial y^i}+F_\beta\Big[\tilde{\Gamma}\Big(\frac{\partial\beta}{\partial y^i}\Big)-\frac{\partial\beta}{\partial x^i}\Big].
\end{aligned} \tag{3.2}$$

For the Riemannian metric α, the Euler-Lagrange equation gives $\tilde{\Gamma}\Big(\frac{\partial\alpha}{\partial y^i}\Big)-\frac{\partial\alpha}{\partial x^i}=0$. Also, one knows [4] that if $(M,\ F(\alpha,\beta))$ is a Finsler space with $(\alpha,\ \beta)$-metric, then $f(x,y)\frac{\partial\alpha}{\partial y^i}+g(x,y)b_i=0,\ \forall\ i=1,\ 2,\ \ldots,\ n$, implies that $f=g=0$, for any smooth functions f and g on TM. It is known that, if β is closed and F is projectively equivalent to the Riemannian metric α, then $\tilde{\Gamma}(F_\alpha)\frac{\partial\alpha}{\partial y^i}+\tilde{\Gamma}(F_\beta)b_i=0$ and hence by using lemma 2.7 of [4], we find that $\tilde{\Gamma}(F_\alpha)=0,\tilde{\Gamma}(F_\beta)=0$.

Again, since $F=\sqrt[4]{\alpha^4+\beta^4}+\beta$, therefore $F_\beta=1+\dfrac{\beta^3}{(\alpha^4+\beta^4)^{\frac{3}{4}}}$.
Now, using the above results, the equation (3.2) reduces to the form

$$\begin{aligned}
\tilde{\Gamma}\Big(\frac{\partial F}{\partial y^i}\Big)-\frac{\partial F}{\partial x^i}&=F_\beta\Big[\tilde{\Gamma}\frac{\partial\beta}{\partial y^i}-\frac{\partial\beta}{\partial x^i}\Big]\\
&=\Big(1+\frac{\beta^3}{(\alpha^4+\beta^4)^{\frac{3}{4}}}\Big)\Big[\bar{\Gamma}(b_i)-\frac{\partial b_j}{\partial x^i}y^j\Big]\\
&=\Big(1+\frac{\beta^3}{(\alpha^4+\beta^4)^{\frac{3}{4}}}\Big)\Big[\frac{\partial b_i}{\partial y^j}y^j-\frac{\partial b_j}{\partial x^i}y^j\Big]\\
&=\Big(1+\frac{\beta^3}{(\alpha^4+\beta^4)^{\frac{3}{4}}}\Big)\Big[\frac{\partial b_i}{\partial x^j}-\frac{\partial b_j}{\partial x^i}\Big]y^j.
\end{aligned} \tag{3.3}$$

Now, $1+\frac{\beta^3}{(\alpha^4+\beta^4)^{\frac{3}{4}}}$ can not be zero. Therefore, from equation (3.1) and (3.3) we conclude that F is with reversible geodesics if and only if $\Big(\frac{\partial b_i}{\partial x^j}-\frac{\partial b_j}{\partial x^i}\Big)y^j=0.$
i.e., $\tilde{F}$ is with reversible geodesic if and only if β is closed 1-form. Hence, we have the following theorem:

Theorem 3.1. *A Finsler space (M,F) with Randers change of Quartic metric $F=\tilde{F}+\beta$, where, $\tilde{F}=\sqrt[4]{(\alpha^4+\beta^4)}$, is with reversible geodesics if and only if the differential 1-form β is closed on M.*

4. Projective Flatness of Randers Change of Quartic Metric

A Finsler space (M,F) is called (locally) projectively flat if all its geodesics are straight lines [8]. An equivalent condition is that the spray coefficients G^i of F can be expressed as $G^i=P(x,y)y^i$, where $P(x,y)=\frac{1}{2F}\frac{\partial F}{\partial x^k}y^k$.

An equivalent characterization of projective flatness is the Hamel's relation [9]

$$\frac{\partial^2 F}{\partial x^m\partial y^k}y^m-\frac{\partial F}{\partial x^k}=0.$$

Recall that ([5], [6]) if F $=F_0+\epsilon\,\beta$ is a Finsler metric, where F_0 is an absolute homogeneous $(\alpha,\,\beta)$-metric, then any two of the following properties imply the third one:

(1) F is projectively flat;
(2) F_0 is projectively flat;
(2) β is closed.

In our case, F $=\tilde{F}+\beta$, where $\tilde{F}=\sqrt[4]{(\alpha^4+\beta^4)}$, which is absolute homogeneous. Hence, we have the following theorem:

Theorem 4.1. *Let (M,F) be a Finsler space with Randers change of Quartic metric $F=\sqrt[4]{(\alpha^4+\beta^4)}+\beta$. Then, F is projectively flat if and only if $\tilde{F}$ is projectively flat.*

Proof. Let (M,F) be projectively flat, then by Hamel's relation for projective flatness, we have

$$\frac{\partial^2 F}{\partial x^m\partial y^k}y^m-\frac{\partial F}{\partial x^k}=0.$$

The proof directly follows from it. ∎

Theorem 4.2. *Let (M,F) be a Finsler space with Randers change of Quartic metric. If F is projectively flat, then it is with reversible geodesics.*

Proof. Applying Hammel's equation, one can easily see that F is projectively flat if and only if $\tilde{F}$ is projectively flat, which implies that both F and $\tilde{F}$ are projectively equivalent to the standard Euclidean metric and therefore F must be projective to $\tilde{F}$. Thus, F must be with a reversible geodesic. ∎

5. Weighted Quasi Metric Associated with Randers Change of Quartic Metric

It is well known that the Riemannian spaces can be represented as metric spaces. Indeed, for a Riemannian space (M, α), one can define the induced metric space $(\mathbf{M}, d_\alpha)$ with the metric

$$d_\alpha : M \times M \longrightarrow [0, \infty), d_\alpha(x, y) = \inf_{\gamma \in \Gamma_{xy}} \int_a^b \alpha(\gamma(t), \dot{\gamma}(t))dt, \tag{5.1}$$

where $\Gamma_{xy} = \{\gamma : [a, b] \longrightarrow M \mid \gamma \text{ is piecewise}, \gamma(\text{a}) = x, \gamma(\text{b}) = y\}$ is the set of curves joining x and y, $\dot{\gamma}(t)$ is the tangent vector to γ at γ(t). Then d_α is a metric on M satisfying the following conditions:

1. Positiveness : $d_\alpha(x, y) > 0$, if x $\neq$ y, $d_\alpha(x, x) = 0$, $x, y \in$ X.
2. Symmetry : $d_\alpha(x, y) = d_\alpha(y, x)$, $\forall\ x, y \in M$.
3. Triangle inequality: $d_\alpha(x, y) \leq d_\alpha(\text{x, z}) + d_\alpha(z, y)$, $\forall\ x, y, z \in M$.

Similar to the Riemannian space, one can induce the metric d_F to a Finsler space (M, F), given by

$$d_F : M \times M \longrightarrow [0, \infty), d_F(x, y) = \inf_{\gamma \in \Gamma_{xy}} \int_a^b F(\gamma(t), \dot{\gamma}(t))dt. \tag{5.2}$$

But in this case unlike the Riemannian case, d_F lacks the symmetric condition. In fact, d_F is a special case of quasi metric defined below:

Definition 5.1. A quasi metric d on a set X is a function $d : X \times X \longrightarrow [0, \infty)$ that satisfies the following axioms:

1. Positiveness : $d(x, y) > 0$, if $x \neq y$, $d(x, x) = 0$, $x, y \in X$.
2. Triangle inequality : $d(x, y) \leq d(x, z) + d(z, y)$, $\forall\ x, y, z \in X$.
3. Separation axiom : $d(x, y) = d(y, x) = 0 \Rightarrow x = y$, $\forall\ x, y \in X$.

One special class of quasi metric spaces are the so called weighted quasi metric spaces (M, d, w), where d is a quasi-metric on M and for each d, there exist a function $w : M \longrightarrow [0, \infty)$, called the *weight* of d that satisfies

4. *Weightability :* $d(x, y) + w(x) = d(y, x) + w(y)$, $\forall\ x, y \in M$.

In this case, the weight function w is $\mathbb{R}$-valued, and is called generalized weight.

Theorem 5.2. *Let (M, F) be an n-dimensional simply connected smooth Finsler manifold with F as Randers change of Quartic metric. Then, F induces generalized weighted quasi metric d_F on M.*

Proof. We consider that (M, F) is a Finsler space with F $= \beta + \sqrt[4]{(\alpha^4 + \beta^4)}$, which can be written as $F = \tilde{F} + \beta$, where $\tilde{F} = \sqrt[4]{(\alpha^4 + \beta^4)}$ is an absolute homogeneous Finsler metric on M and β an exact 1-form.

Let $\gamma_{xy} \in \Gamma_{xy}$ be an F-geodesic, which is in the same time $\tilde{F}$-geodesic, then from equation (5.2), we get

$$\begin{aligned} d_F(x, y) &= \int_a^b F(\gamma(t), \dot{\gamma}(t))dt \\ &= \int_a^b \left(\beta + \sqrt[4]{(\alpha^4 + \beta^4)}\right)dt \end{aligned}$$

$$= \int_a^b \left(\sqrt[4]{(\alpha^4+\beta^4)}\right)dt + \int_a^b \beta dt$$
$$= \int_{\gamma_{xy}} \left(\sqrt[4]{(\alpha^4+\beta^4)}\right) + \int_{\gamma_{xy}} \beta. \tag{5.3}$$

Consider a fixed point $a \in M$ and define the function $w_a : M \longrightarrow \mathbb{R}$ by $w_a(x) := d_F(a,x) - d_F(x,a)$.
From the equation (5.2) it follows that

$$w_a(x) = \int_{\gamma_{ax}} \beta - \int_{\gamma_{xa}} \beta = -2\int_{\gamma_{xa}} \beta, \tag{5.4}$$

where we have used the Stokes theorem for the 1-form β on the closed domain D with boundary $\partial \mathrm{D} := \gamma_{ax} \bigcup \gamma_{xa}$.
It can be easily seen that w_a is an anti-derivative of β. This is well defined if and only if the integral in the R.H.S. of equation (5.4) is path independent, i.e., β must be exact.

Then d_F is a weighted quasi-metric with generalized weight w_a. Next we have

$$d_F(x,y) + w_a(x) = \int_{\gamma_{xy}} \left(\sqrt[4]{(\alpha^4+\beta^4)}\right) - \int_{\gamma_{xa}} \beta - \int_{\gamma_{ya}} \beta, \tag{5.5}$$

where we have again used the Stokes theorem for the one form β on the closed domain with boundary $\gamma_{ax} \bigcup \gamma_{xy} \bigcup \gamma_{ya}$.

Similarly,

$$d_F(y,x) + w_a(y) = \int_{\gamma_{yx}} \left(\sqrt[4]{(\alpha^4+\beta^4)}\right) - \int_{\gamma_{ya}} \beta - \int_{\gamma_{xa}} \beta. \tag{5.6}$$

From the equations (5.5) and (5.6) we conclude that d_F is weighted quasi-metric with generalized weight w_a. This completes the proof. ∎

Next, recall the following lemma:

Lemma 5.3. ([10], [11]) *Let (M, d) be any quasi-metric space. Then d is weightable if and only if there exists $w : M \longrightarrow [0, \infty)$ such that*

$$d(x,y) = \rho(x,y) + \frac{1}{2}[w(x) - w(y)], \forall x,y \in M, \tag{5.7}$$

where ρ is the symmetrized distance function of d. Moreover, we have

$$\frac{1}{2}[w(x) - w(y)] \le \rho(x,y), \forall x,y \in M. \tag{5.8}$$

The proof is trivial from the definition of weighted quasi-metric.

Remark 5.4. If (M, F) is a Finsler space with a special (α, β)-metric $F = \sqrt[4]{(\alpha^4+\beta^4)} + \beta$, then the induced quasi-metric d_F and the symmetrized metric ρ induced the same topology on M. This follow immediately from ([7], [8]).

Remark 5.5. From lemma 5.3, It can be seen that the assumption of w to be smooth is not essential.

Next, we discuss an interesting geometric property concerning the geodesic triangles.

Theorem 5.6. *Let (M, F) be a Finsler space with the Randers change of Quartic-metric* $F = \sqrt[4]{(\alpha^4 + \beta^4)} + \beta$*. Then the parameteric length of any geodesic triangle on M does not depend on the orientation, that is,*

$$d_F(x,y) + d_F(y,z) + d_F(z,x) = d_F(x,z) + d_F(z,y) + d_F(y,x), \forall x,y,z \in M.$$

Proof. Since the Randers change of Quartic metric F = $\sqrt[4]{(\alpha^4 + \beta^4)} + \beta$ can be treated as the Randers change of absolute homogeneous Finsler metric $\tilde{F} = \sqrt[4]{(\alpha^4 + \beta^4)}$, i.e., F = $\tilde{F} + \beta$ with $\mathrm{d}\beta = 0$, from theorem 5.2 it follows that the quasi-metric is weightable and therefore equation (5.7) holds good. By using the formula (5.7), a simple calculation gives the required result. ∎

Acknowledgements

We would like to thank the referees for their comments and suggestions on the manuscript.

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