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Generalized n-Derivations in Non-Archimedean Banach Algebras

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Abstract We have a verification on the behavior of almost generalized n-derivations from non-Archimedean Banach algebras into non-Archimedean Banach modules and give some applications of our results.

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1. Introduction and Preliminaries

A non-Archimedean valuation is a function $|\cdot|$ from a field $\mathcal{K}$ into $[0,\infty)$ such that 0 is the unique element having the 0 valuation, $|rs| = |r|\cdot|s|$ and the strong triangle inequality holds, i.e., $|r+s| \le \max\{|r|,|s|\}$ for all $r,s \in \mathcal{K}$. Any field endowed with a non-Archimedean valuation is said to be a non-Archimedean field. Note that every complete valued field is isomorphic to $\mathbb{R}$ or $\mathbb{C}$ or is non-Archimedean [1, Theorem 1].

In any such field we have $|\mathbf{1}| = |-\mathbf{1}| = 1$ and $|n\times\mathbf{1}| \le 1$ for all $n \in \mathbb{N}$, where $\mathbf{1}$ is the neutral element of the semigroup $(\mathcal{K},\cdot)$, $1\times\mathbf{1} = \mathbf{1}$ and $(n+1)\times\mathbf{1} = (n\times\mathbf{1}) + \mathbf{1}$ for $n \in \mathbb{N}$.

A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except 0 into 1 and $|0| = 0$. Another example of a non-Archimedean valuation is the function $|\cdot|_q$ from a field $\mathcal{K}$ into $[0,\infty)$ with $|0|_q = 0$, $|r|_q = \dfrac{1}{r}$ if $r > 0$ and $|r|_q = \dfrac{-1}{r}$ if $r < 0$, for any $r \in \mathcal{K}$. Standard examples of such fields are fields of p-adic numbers $\mathbb{Q}_p$. Let p be a prime, the set $\mathbb{Q}_p$ is defined as a completion of the rational numbers $\mathbb{Q}$ with respect to the norm $|\cdot|_p$ from $\mathbb{Q}$ into $\mathbb{R}$ given by $|0|_p = 0$ and $|a|_p = p^{-r}$ if $a \neq 0$, here $a = p^r\frac{m}{n}$ such that $r,m \in \mathbb{Z}$, $n \in \mathbb{N}$, and m and n are coprime to the prime number p. The absolute value $|\cdot|_p$ is non-Archimedean. There are also many other examples of non-Archimedean fields (see for example [2]).

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Definition 1.1. Let X be a linear space over a field $\mathcal{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$ that is non-trivial (i.e., we additionally assume that there is an $r_0 \in \mathcal{K}$ such that $|r_0| \neq 0, 1$). A function $\|\cdot\| : X \to [0, \infty)$ is said to be a non-Archimedean norm if it is a norm over $\mathcal{K}$ with the strong triangle inequality (ultrametric), i.e., $\|x+y\| \leq \max\{\|x\|, \|y\|\}$ for all $x, y \in X$. Then the pair $(X, \|\cdot\|)$ is called a non-Archimedean normed space.

By a complete non-Archimedean normed space or non-Archimedean Banach space we mean a non-Archimedean normed space in which every Cauchy sequence is convergent.

Remark 1.2. If $\|x\| \neq \|y\|$ then $\|x+y\| = \max\{\|x\|, \|y\|\}$. We may assume $\|x\| < \|y\|$. We need to show $\|x+y\| = \|y\|$. If not, we have $\|x+y\| < \|y\|$. But $\|y\| = \|y+x-x\| \leq \max\{\|y+x\|, \|x\|\} < \|y\|$, a contradiction. Note that this is a crucial property which is proper to the non-Archimedeanity of the norm.

Definition 1.3. Let $(\mathcal{A}, \|\cdot\|)$ be a non-Archimedean Banach algebra over $\mathcal{K}$. This means that the norm $\|\cdot\|$ of the Banach algebra satisfies the non-Archimedean property, i.e., $\|x+y\| \leq \max\{\|x\|, \|y\|\}$ for all $x, y \in \mathcal{A}$ (and if $\mathcal{A}$ is unital, $\|\mathbf{1}\| = 1$). A non-Archimedean Banach space $\mathcal{X}$ is a non-Archimedean Banach $\mathcal{A}$-bimodule if $\mathcal{X}$ is an $\mathcal{A}$-bimodule which satisfies $\max\{\|xa\|, \|ax\|\} \leq \|a\|\|x\|$ for all $a \in \mathcal{A}$ and $x \in \mathcal{X}$.

There are many examples of such kind of spaces, see [3–5]. Let us consider some basic examples of non-Archimedean Banach algebras.

Example 1.4. Let $\mathcal{K}$ be a non-Archimedean field and $\mathcal{K}^n := \{\mathrm{x} = (x_1, \ldots, x_n) : \ x_i \in \mathcal{K}, \ i = 1, \ldots, n\}$. Then $\mathcal{K}^n$ with a norm $\|\mathrm{x}\| = \max_i |x_i|$ and usual pointwise summation and multiplication operations, is a non-Archimedean Banach algebra over $\mathcal{K}$.

Example 1.5. Let $\mathcal{K}$ be as above and $c_0 := \{\mathrm{x} = \{x_n\}_{n\in\mathbb{N}} : \ x_n \in \mathcal{K}, \ \lim_{n\to\infty} x_n = 0\}$. Then c_0 with a norm $\|\mathrm{x}\| = \max_n |x_n|$ and usual pointwise summation and multiplication operations, is a non-Archimedean Banach algebra over $\mathcal{K}$.

In [6], the authors investigated the approximately additive mappings over p-adic fields. The stability of the Cauchy and monomial functional equations in normed spaces over fields with valuation was studied by Kaiser in [7, 8], see also [9, 10]. Interesting results concerning almost derivations of non-Archimedean Banach algebras have been obtained by many authors, see, e.g., [11, 12]). In the present paper we investigate the almost n-derivations of order m from non-Archimedean Banach algebras into non-Archimedean Banach modules and give some applications of our results.

2. Main Results

Let $\mathcal{A}$ be an algebra. An additive mapping $f : \mathcal{A} \longrightarrow \mathcal{A}$ is called an n-Jordan derivation if f satisfying

$$f(x^n) = \sum_{i=1}^{n} x^{i-1} f(x) x^{n-i}$$

for all $x \in \mathcal{A}$, where $n > 1$ is an integer. This is known as the nth power property (see, among others, [13, 14]). For more details of the nth power property, n-Jordan derivations and other applications, see, e.g., [14–18].

Remark 2.1. The monomial $f(x) = cx^m$ on real numbers is a solution of the functional equation

$$\sigma_y f(ax) = a^{m-2}\sigma_y f(x) + 2(a^2 - 1)\left(a^{m-2}f(x) - \kappa f(y)\right), \tag{2.1}$$

where a is an arbitrarily fixed nonzero integer different from -1 and 1, m is a positive integer less than 5 and $\kappa = 0$, if $m \neq 4$ and $\kappa = 1$, if $m = 4$. Here $\sigma_y f(x)$ denotes $\sigma_y f(x) = f(x+y) + f(x-y)$. Every solution of the functional equation (2.1) is called an m-mapping. The general solution of the functional equation (2.1) in vector spaces when $a = 2$ obtained in [19] and when $a \in \mathbb{Z} \setminus \{0, \pm 1\}$ obtained in [20].

Definition 2.2. We say that an m-mapping $f : \mathcal{A} \longrightarrow \mathcal{A}$ is an n-derivation of order m if f satisfying

$$f\left(\prod_{i=1}^{n} x_i\right) = \sum_{i=1}^{n}\prod_{\imath=1}^{i-1} x_\imath^m f(x_i) \prod_{\imath=i+1}^{n} x_\imath^m \tag{2.2}$$

for all $x_1, \ldots, x_n \in \mathcal{A}$, where $\prod_{\imath=l+1}^{l} x_\imath^m = 1 \in \mathbb{C}$ with $l \in \{0, n\}$.

Putting $m = 1$ and replacing each x_i by x in (2.2), we observe that f satisfies the nth power property; that is, f is an n-Jordan derivation. Note that 2-derivations of order 1 are a derivation, in the usual sense.

Example 2.3. Let us consider the algebra of 3×3 matrices

$$\mathcal{A} = \left\{ \begin{bmatrix} 0 & \alpha & \beta \\ 0 & 0 & \gamma \\ 0 & 0 & 0 \end{bmatrix} : \alpha, \beta, \gamma \in \mathcal{K} \right\}.$$

Then the mapping $f : \mathcal{A} \longrightarrow \mathcal{A}$, defined by

$$f\left(\begin{bmatrix} 0 & \alpha & \beta \\ 0 & 0 & \gamma \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & \alpha^2 & \beta^2 \\ 0 & 0 & \gamma^2 \\ 0 & 0 & 0 \end{bmatrix},$$

is an 3-derivation of order 2, while is not an 2-derivation of order 3 and is not a derivation.

Proposition 2.4. [4] *A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a non-Archimedean normed space $(X, \|\cdot\|)$ is Cauchy sequence if and only if* $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

Note that, any non-Archimedean norm is a continuous function from its domain to real numbers.

Proposition 2.5. *Let E be a normed space and X be a non-Archimedean normed space. Suppose $f : E \longrightarrow X$ is a mapping and continuous at $0 \in E$ such that $f(ax) = a^m f(x)$ for all $x \in E$, where $a \neq 1$ and m are arbitrarily fixed positive integers. Then, $f = 0$.*

Proof. Since f is continuous at $0 \in E$ and $f(0) = 0$, for all $\varepsilon > 0$, there exists $\delta > 0$ that, for all $x \in E$ with $\|x\| \leq \delta$,

$$\|f(x) - f(0)\| = \|f(x)\| \leq \varepsilon.$$

Also for any $x \in E$, there exists $n \in \mathbb{N}$ that $\left\|\dfrac{x}{a^n}\right\| \leq \delta$ and hence

$$\|f(x)\| = \left\|a^{mn} f\left(\frac{x}{a^n}\right)\right\| \leq \left\|f\left(\frac{x}{a^n}\right)\right\| \leq \varepsilon$$

for all $\varepsilon > 0$ and all $x \in E$. Therefore, $f = 0$. ■

From Remark 2.1 and Proposition 2.5, we deduce the following result.

Corollary 2.6. *Let E be a normed space and X be a non-Archimedean normed space. Suppose $f : E \longrightarrow X$ is an m-mapping and continuous at $0 \in E$. Then, $f = 0$.*

Notice that the argument above is a special case of a general result for non-Archimedean normed spaces, that is, every continuous function from a connected space to a non-Archimedean normed space is constant. This is a consequence of totally disconnectedness of every non-Archimedean normed space (see [4]).

In the rest of this paper, unless otherwise explicitly stated, we will assume that $\mathbb{R}^+$ is the set of nonnegative real numbers, n is an integer greater than 1, m is a positive integer less than 5, $f : \mathcal{A} \to \mathcal{X}$ is a mapping with $f(0) = 0$ whenever $m = 4$, $\kappa = 0$, if $m \neq 4$ and $\kappa = 1$, if $m = 4$, $a \neq 0, \pm 1$ is an arbitrarily fixed integer, $\mathcal{A}$ is a non-Archimedean Banach algebra and $\mathcal{X}$ is a non-Archimedean Banach $\mathcal{A}$-bimodule over a non-Archimedean field of characteristic different from 2 and a.

Definition 2.7. A function $\zeta : \mathbb{R} \to \mathbb{R}$ satisfying the equation $\zeta(xy) = \zeta(x)\zeta(y)$ is called a multiplicative function, and $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the inequality $\xi(xy) \leq \xi(x)\xi(y)$ is called a submultiplicative function.

Definition 2.8. A mapping $f : \mathcal{A} \to \mathcal{X}$ is called an almost n-derivation of order m if there exist functions $\omega : \mathcal{A} \times \mathcal{A} \longrightarrow \mathbb{R}^+$ and $\upsilon : \underbrace{\mathcal{A} \times \cdots \times \mathcal{A}}_{n \text{ times}} \longrightarrow \mathbb{R}^+$ such that $\lim_{k\to\infty} \frac{\omega(a^k x, a^k y)}{|a|^{mk}} = 0 = \lim_{k\to\infty} \frac{\upsilon(a^k x_1,\ldots,a^k x_n)}{(|a|^{mn})^k}$ $\Big($or $\lim_{k\to\infty} |a|^{mk}\omega\left(\frac{x}{a^k}, \frac{y}{a^k}\right) = 0 = \lim_{k\to\infty}(|a|^{mn})^k \upsilon\left(\frac{x_1}{a^k}, \ldots, \frac{x_n}{a^k}\right)\Big)$ and

$$\left\|\sigma_y f(ax) - a^{m-2}\sigma_y f(x) - 2(a^2-1)\left(a^{m-2}f(x) - \kappa f(y)\right)\right\| \leq \omega(x,y) \tag{2.3}$$

$$\left\| f\left(\prod_{i=1}^n x_i\right) - \sum_{i=1}^n \prod_{\imath=1}^{i-1} x_\imath^m f(x_i) \prod_{\imath=i+1}^n x_\imath^m \right\| \leq \upsilon(x_1,\ldots,x_n) \tag{2.4}$$

for all $x, y, x_1, \ldots, x_n \in \mathcal{A}$, where $n \geq 2$. Also, $f : \mathcal{A} \to \mathcal{X}$ is called an (ε,ξ)-n-derivation of order m if there exist a non-negative real number ε and a submultiplicative function ξ such that (2.3) and (2.4) hold for $\omega(x,y) = \varepsilon\left(\xi(\|x\|) + \xi(\|y\|)\right) := \varepsilon\left(\xi_{\|x\|} + \xi_{\|y\|}\right)$ and $\upsilon(x_1,\ldots,x_n) = \varepsilon \prod_{i=1}^n \xi(\|x_i\|) := \varepsilon \prod_{i=1}^n \xi_{\|x_i\|}$ for all $x, y, x_1, \ldots, x_n \in \mathcal{A}$.

We here present the following notion of n-derivations of order m on unital non-Archimedean algebras.

Proposition 2.9. *Suppose $\mathcal{A}$ is a unital non-Archimedean algebra, $\mathcal{X}$ is a unital non-Archimedean $\mathcal{A}$-bimodule and $f : \mathcal{A} \to \mathcal{X}$ is an n-derivation of order m. Then f is a derivation of order m.*

Proof. Since f satisfies (2.2), by setting each x_i in (2.2) with $\mathbf{1}$ we have $f\left(\mathbf{1}^n\right) = nf(\mathbf{1})$. Thus, $f(\mathbf{1}) = 0$. Substituting $x_i = \mathbf{1}$ for all $i = 3, 4, \ldots, n$ in (2.2), we get

$$\begin{aligned} f\left(x_1 x_2\right) &= f\left(x_1 x_2 \prod_{i=3}^n \mathbf{1}\right) = f(x_1)x_2^m \mathbf{1}^m \cdots \mathbf{1}^m + x_1^m f(x_2)\mathbf{1}^m \cdots \mathbf{1}^m + 0 + \cdots + 0 \\ &= f(x_1)x_2^m + x_1^m f(x_2) \end{aligned}$$

for all $x_1, x_2 \in \mathcal{A}$, so that f is a derivation of order m. ∎

Example 2.3 shows that Proposition 2.9 does not hold in general.

Theorem 2.10. *Let $f : \mathcal{A} \to \mathcal{X}$ be an almost n-derivation of order m and set*

$$\Omega(x) := \sup\left\{\frac{\omega(a^j x, 0)}{|a|^{mj}} : j \in \mathbb{N} \cup \{0\}\right\}, \ (x \in \mathcal{A}).$$

Then there exists a unique n-derivation of order m, $\Theta_{m,n} : \mathcal{A} \to \mathcal{X}$ such that $\|f(x) - \Theta_{m,n}(x)\| \leq \dfrac{\Omega(x)}{|2a^m|}$ for all $x \in \mathcal{A}$.

Proof. Setting $y = 0$ in (2.3) yields $\|f(ax) - a^m f(x)\| \leq \frac{\omega(x,0)}{|2|}$ for all $x, \in \mathcal{A}$. Replacing x by $a^k x$ in the inequality above and then dividing by $|a|^{mk+m}$ gives $\left\|\frac{f(a^{k+1}x)}{a^{mk+m}} - \frac{f(a^k x)}{a^{mk}}\right\| \leq \frac{\omega(a^k x,0)}{|2a^{mk+m}|}$ for all $x, \in \mathcal{A}$. Combining the last inequality and $\lim_{k\to\infty} \frac{\omega(a^k x,0)}{|a|^{mk}} = 0$, we obtain that $\left\{\frac{f(a^k x)}{a^{mk}}\right\}$ is a Cauchy sequence. Since the space $\mathcal{X}$ is complete, this sequence is convergent, and we define $\Theta_{m,n}(x) := \lim_{k\to\infty} \frac{f(a^k x)}{a^{mk}}$.

Using induction, it is easy to prove that

$$\left\|f(x) - \frac{f(a^k x)}{a^{mk}}\right\| \leq \frac{1}{|2a^m|} \max\left\{\frac{\omega(a^j x, 0)}{|a|^{mj}} : 0 \leq j < k\right\}$$

for all $k \in \mathbb{N}$ and all $x \in \mathcal{A}$. Letting $k \to \infty$ in this inequality, and using the fact that

$$\lim_{k\to\infty} \max\left\{\frac{\omega(a^j x, 0)}{|a|^{mj}} : 0 \leq j < k\right\} = \sup\left\{\frac{\omega(a^j x, 0)}{|a|^{mj}} : j \in \mathbb{N} \cup \{0\}\right\},$$

we see that $\|f(x) - \Theta_{m,n}(x)\| \leq \dfrac{\Omega(x)}{|2a^m|}$ for all $x \in \mathcal{A}$.

Substituting $x = a^k x$ and $y = a^k y$ in (2.3), dividing by $|a|^{mk}$, taking k to approach infinity in the resultant inequality and utilizing $\lim_{k\to\infty} \frac{\omega(a^k x, a^k y)}{|a|^{mk}} = 0$, we find that $\Theta_{m,n}$ satisfies (2.1). So, by Remark 2.1, $\Theta_{m,n}$ is an m-mapping. Also, it follows from the definition of $\Theta_{m,n}$ and (2.4) that

$$\begin{aligned}
&\left\|\Theta_{m,n}\left(\prod_{i=1}^n x_i\right) - \sum_{i=1}^n \prod_{\imath=1}^{i-1} x_\imath^m \Theta_{m,n}(x_i) \prod_{\imath=i+1}^n x_\imath^m\right\| \\
&= \lim_{k\to\infty} \frac{1}{(|a|^{mn})^k}\left\|f\left(\prod_{i=1}^n (a^k x_i)\right) - \sum_{i=1}^n \prod_{\imath=1}^{i-1} (a^k x_\imath)^m f(a^k x_i) \prod_{\imath=i+1}^n (a^k x_\imath)^m\right\| \\
&\leq \lim_{k\to\infty} \frac{\upsilon(a^k x_1, \ldots, a^k x_n)}{(|a|^{mn})^k} = 0,
\end{aligned}$$

and so $\Theta_{m,n}$ satisfies (2.2). Therefore, $\Theta_{m,n}$ is an n-derivation of order m.

Let us finally assume that $\Theta^*_{m,n} : \mathcal{A} \to \mathcal{X}$ is another n-derivation of order m such that $\|f(x) - \Theta^*_{m,n}(x)\| \leq \frac{\Omega(x)}{|2a^m|}$ for all $x \in \mathcal{A}$. Then for all $x \in \mathcal{A}$, we have

$$\begin{aligned}
\left\|\Theta_{m,n}(x) - \Theta^*_{m,n}(x)\right\| &= \lim_{\jmath\to\infty} \frac{1}{|a|^{m\jmath}} \left\|\Theta_{m,n}\left(a^\jmath x\right) - \Theta^*_{m,n}\left(a^\jmath x\right)\right\| \\
&\leq \lim_{\jmath\to\infty} \frac{1}{|a|^{m\jmath}} \max\left\{\left\|\Theta_{m,n}\left(a^\jmath x\right) - f\left(a^\jmath x\right)\right\|, \left\|f\left(a^\jmath x\right) - \Theta^*_{m,n}\left(a^\jmath x\right)\right\|\right\} \\
&\leq \frac{1}{|2a^m|} \lim_{\jmath\to\infty} \lim_{k\to\infty} \max\left\{\frac{\omega\left(a^j x, 0\right)}{|a|^{mj}} : \jmath \leq j < k + \jmath\right\} \\
&= \frac{1}{|2a^m|} \lim_{\jmath\to\infty} \frac{1}{|a|^{m\jmath}} \sup\left\{\frac{\omega\left(a^j x, 0\right)}{|a|^{mj}} : j \in \mathbb{N} \cup \{0\}\right\} \\
&= \frac{1}{|2a^m|} \lim_{\jmath\to\infty} \sup\left\{\frac{\omega\left(a^j x, 0\right)}{|a|^{mj}} : \jmath \leq j < \infty\right\} = 0,
\end{aligned}$$

and thus $\Theta_{m,n}(x) = \Theta^*_{m,n}(x)$. ∎

From Proposition 2.9 and Theorem 2.10, we deduce the following result.

Corollary 2.11. *If, under the conditions of Theorem 2.10, we assume in addition $\mathcal{A}$ is a unital non-Archimedean algebra and $\mathcal{X}$ is a unital non-Archimedean $\mathcal{A}$-bimodule, then there exists a unique derivation of order m, $\Upsilon_m : \mathcal{A} \to \mathcal{X}$ such that $\|f(x) - \Upsilon_m(x)\| \leq \dfrac{\Omega(x)}{|2a^m|}$ for all $x \in \mathcal{A}$.*

Corollary 2.12. *Let $f : \mathcal{A} \to \mathcal{X}$ be an (ε, ξ)-n-derivation of order m, $a > 1$ be a constant natural number and ξ be a submultiplicative function satisfying $\xi_{|a|} < |a|^\alpha$, where α is a fixed real number in (m, ∞). Then there exists a unique n-derivation of order m, $\Theta_{m,n} : \mathcal{A} \to \mathcal{X}$ such that $\|f(x) - \Theta_{m,n}(x)\| \leq \dfrac{\varepsilon\xi_{\|x\|}}{|2a^m|}$ for all $x \in \mathcal{A}$.*

Proof. Since $\frac{1}{|a|^m}\xi_{|a|} < \frac{|a|^\alpha}{|a|^m} = |a|^{\alpha-m} < 1$, taking $\omega(x, y) = \varepsilon\left(\xi_{\|x\|} + \xi_{\|y\|}\right)$ and $\upsilon(x_1, \ldots, x_n) = \varepsilon\prod_{i=1}^n \xi_{\|x_i\|}$ for all $x, y, x_1, \ldots, x_n \in \mathcal{A}$, we have

$$\begin{aligned}
\lim_{k\to\infty} \frac{\upsilon\left(a^k x_1, \ldots, a^k x_n\right)}{\left(|a|^{mn}\right)^k} &\leq \lim_{k\to\infty} \frac{\xi^{nk}_{|a|}}{\left(|a|^m\right)^{nk}} \upsilon\left(x_1, \ldots, x_n\right) \\
&\leq \lim_{k\to\infty} |a|^{(\alpha-m)nk} \upsilon\left(x_1, \ldots, x_n\right) = 0,
\end{aligned}$$

and $\lim_{k\to\infty} \frac{\omega\left(a^k x, a^k y\right)}{|a|^{mk}} \leq \lim_{k\to\infty} \left(\frac{\xi_{|a|}}{|a|^m}\right)^k \omega(x, y) = 0$. Also,

$$\Omega(x) = \sup\left\{\frac{\omega\left(a^j x, 0\right)}{|a|^{mj}} : j \in \mathbb{N} \cup \{0\}\right\} = \omega(x, 0) = \varepsilon\xi_{\|x\|},$$

and

$$\lim_{\jmath\to\infty}\lim_{k\to\infty}\max\left\{\frac{\omega\left(a^{j}x,0\right)}{|a|^{mj}}\ :\ \jmath\le j<k+\jmath\right\}=\lim_{\jmath\to\infty}\sup\left\{\frac{\omega\left(a^{j}x,0\right)}{|a|^{mj}}\ :\ \jmath\le j<\infty\right\}$$
$$=\lim_{\jmath\to\infty}\frac{\omega\left(a^{\jmath}x,0\right)}{|a|^{m\jmath}}$$
$$\le\lim_{\jmath\to\infty}\left(\frac{\xi_{|a|}}{|a|^{m}}\right)^{\jmath}\omega\left(x,0\right)=0$$

for all $x\in\mathcal{A}$. Hence, the result follows by Theorem 2.10. ∎

Remark 2.13. Let $f:\mathcal{A}\to\mathcal{X}$ be an almost n-derivation of order m and set

$$\mho\left(x\right):=\sup\left\{|a|^{mj}\omega\left(\frac{x}{a^{j+1}},0\right)\ :\ j\in\mathbb{N}\cup\{0\}\right\},\ (x\in\mathcal{A}).$$

By $\lim_{k\to\infty}|a|^{mk}\omega\left(\frac{x}{a^k},\frac{y}{a^k}\right)=0=\lim_{k\to\infty}(|a|^{mn})^{k}\upsilon\left(\frac{x_1}{a^k},\dots,\frac{x_n}{a^k}\right)$ and a similar method to the proof of Theorem 2.10, one can show that there exists a unique n-derivation of order m, $\Phi_{m,n}:=\lim_{k\to\infty}a^{mk}f\left(\frac{x}{a^k}\right)$ from $\mathcal{A}$ to $\mathcal{X}$ such that $\|f(x)-\Phi_{m,n}(x)\|\le\dfrac{\mho\left(x\right)}{|2|}$ for all $x\in\mathcal{A}$.

For the case $\omega(x,y):=\varepsilon\left(\xi_{\|x\|}+\xi_{\|y\|}\right)$ and $\upsilon(x_1,\dots,x_n):=\varepsilon\prod_{i=1}^{n}\xi_{\|x_i\|}$ (where $a>1$ is a constant natural number and ξ is a submultiplicative function satisfying $\xi_{\frac{1}{|a|}}<|a|^{-\alpha}$ and α is a fixed real number in $(-\infty,m)$), there exists a unique n-derivation of order m, $\Phi_{m,n}$ such that $\|f(x)-\Phi_{m,n}(x)\|\le\dfrac{\varepsilon\xi_{\|x\|}}{|2a^{\alpha}|}$ for all $x\in\mathcal{A}$.

Example 2.14. The classical example of the function ξ in Corollary 2.12 (Remark 2.13) is the mapping $\xi(t)=t^{p}$, $t\in[0,\infty)$, where $p>\alpha$ ($p<\alpha$) with the further assumption that $|a|<1$.

Here we present some conditions for an almost n-derivation of order m to be a derivation of order m.

Theorem 2.15. *If $f:\mathcal{A}\to\mathcal{X}$ is an almost n-derivation of order m, $\omega(x,y)$ is replaced by $\omega(0,y)$ and $|a|<1$, then f is an n-derivation of order m.*

Proof. Letting $x=y=0$ in (2.3), we obtain $\left\|2\left(a^{m}+\kappa(1-a^{2})-1\right)f\left(0\right)\right\|\le\omega(0,0)$. But since $\lim_{k\to\infty}|a|^{-mk}\omega\left(0,0\right)=0$, it follows that $\omega(0,0)=0$. Thus, $f(0)=0$. Setting $y=0$ in (2.3) and using $f(0)=0$, we get $f(ax)=a^{m}f(x)$ for all $x\in A$. So we will prove by induction that $f(a^{k}x)=a^{mk}f(x)$; that is,

$$f(x)=\frac{1}{a^{mk}}f(a^{k}x)\tag{2.5}$$

for all $x\in\mathcal{A}$ and $k\in\mathbb{N}$. On the other hand, by Theorem 2.10, the mapping $\Theta_{m,n}:\mathcal{A}\to\mathcal{X}$ defined by $\Theta_{m,n}(x):=\lim_{k\to\infty}\frac{f\left(a^{k}x\right)}{a^{mk}}$. is a unique n-derivation of order m. Then it follows from (2.5) that $f=\Theta_{m,n}$. Therefore, the mapping f is an n-derivation of order m. ∎

The following result is due to Proposition 2.9 and Theorem 2.15.

Corollary 2.16. *If $\mathcal{A}$ is a unital non-Archimedean algebra, $\mathcal{X}$ is a unital non-Archimedean $\mathcal{A}$-bimodule, $f:\mathcal{A}\to\mathcal{X}$ is an almost n-derivation of order m, $\omega(x,y)$ is replaced by $\omega(0,y)$ and $|a|<1$, then f is a derivation of order m.*

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