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Banach Contraction Theorem on Extended Fuzzy Cone b-metric Space

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Abstract In this paper, a generalization of b-metric space is introduced named as extended fuzzy cone b-metric space. A new contractive mapping, extended fuzzy cone b-contractive mapping is defined and a modified form of Banach Contraction Theorem is proved for single and pair of mappings which is termed as extended fuzzy cone b-Banach Contraction Theorem. The derived results are applied on Fredholm Integral Equations to obtain a unique solution.

MSC: 47H10; 47H09; 54H25; 54E15; 54E35; 54A40

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1. Introduction

Maurice Frechet initiated the concept of metric space [1] in 1906. A generalization of Metric Space i.e. b-metric space was introduced by I. A. Bakhtin [2] which was further utilised by Czerwick [3, 4] for establishing many results and examples [5–8]. This space moves to metric space by taking b=1. Huang and Zhang [9] introduced the concept of cone metric space in 2007 by replacing real numbers by ordered Banach space. A lot of research has been done in the field of cone metric spaces that can be seen in [10–19]. Further, a generalisation of b-metric space was introduced by Tayyab Kamran et.al, which is termed as extended b-metric space in 2017 [20].

The concept of fuzzy set theory was laid by Zadeh [21] in 1965. Using fuzzy sets, a new metric space i.e. fuzzy metric space was originated by Kramosil and Michalek [22] in 1975 which was redefined in a stronger version by George and Veeramani in 1994 [23].

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Later on, a combination of cone metric and b-metric was introduced by Hussain and Shah [24] in 2011 which is termed as cone b-metric space. By introducing fuzziness to cone metric space, Oner et al. [25] in 2015, defined fuzzy cone metric space and proved some fixed point results under this space that can be viewed in [19, 26–30]. Then, using the concept of b-metric space in fuzzy cone metric space, a new space is developed named as fuzzy cone b-metric space by T. Bag [31] and then modified by Posul et. al [32] in 2019 and some basic fixed point results are proved in [33, 34].

2. Preliminaries

Some basic definitions and terminology that helps in creating the new generalised metric space i.e. extended fuzzy cone b-metric space, are mentioned below:

Definition 2.1. [3] Let M be a non empty set and given $\lambda \geq 1$ be a real number. A function d: $M \times M \longrightarrow [0,\infty)$ is a b-metric on M if for all x,y,z $\in$ M, the following conditions hold:

- (B_1) d(x,y) = 0 $\Leftrightarrow$ x = y;
- (B_2) d(x,y) = d(y,x);
- (B_3) d(x,z) $\leq \lambda$ [d(x,y)+d(y,z)].

Then, the triplet (M,d,λ) is denoted as b-metric space.

Definition 2.2. [20] Let X be a non empty set and $\theta : X\times X \longrightarrow [1,\infty)$. A function d_θ : $X\times X \longrightarrow [0,\infty)$ is called an extended b-metric if for all x,y,z $\in$ X, it satisfies;

1. d_θ (x,y) = 0 $\Leftrightarrow$ x = y;
2. d_θ (x,y) = d_θ (y,x);
3. d_θ (x,z) $\leq \theta$(x,z) [d_θ (x,y) + d_θ (y,z)].

The pair (X,d_θ) is called an extended b-metric space.

Definition 2.3. [35] The binary operation $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is triangular norm (t-norm) if it satisfies the following conditions, for all a,b,c,d $\in$ [0,1];

1. $a * b = b * a$;
2. $a * 1 = a$;
3. $a * (b * c) = (a * b) * c$;
4. If $a \leq c$ and $b \leq d$, then $a * b \leq c * d$.

Example 2.4. Some commonly used t-norms are defined as follows;

1. $a \wedge b = \min\{a, b\}$;
2. $a \cdot b = ab$ i.e. usual multiplication in [0,1] ;
3. $a * b = \max\{a + b - 1, 0\}$.

Definition 2.5. [23] The triplet (X, N, $*$) is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and N is a fuzzy set in $X \times X \times [0,\infty)$ such that for all x,y,z $\in$ X,

1. N(x,y,0) = 0;
2. N(x,y,t) = 1 $\Leftrightarrow$ x = y, for all t > 0;
3. N(x,y,t) = N(y,x,t), for all $t \geq 0$;
4. N(x,z,t+s) $\geq$ N(x,y,t) $*$ N(y,z,s), for all t,s ≥ 0;
5. N(x,y,$\cdot$) : $[0,\infty) \longrightarrow [0,1]$ is left continuous and $\lim_{t\longrightarrow\infty}$ N(x,y,t) = 1.

Definition 2.6. [36] Let X be a non empty set. Let b $\geq$ 1 is a given real number and $*$ be a continuous t-norm. Then, N, a fuzzy set on X $\times$ X $\times$ [0,∞) is said to be fuzzy b-metric if for all x,y,z $\in$ X, the following conditions hold;

(1) N(x,y,0) = 0;
(2) N(x,y,t) = 1 $\Leftrightarrow$ x = y, for all t > 0;
(3) N(x,y,t) = N(y,x,t), for all t $\geq$ 0;
(4) N(x,z,b(t+s)) $\geq$ N(x,y,t) $*$ N(y,z,s), for all t,s $\geq$ 0;
(5) N(x,y,$\cdot$) : [0,∞) $\longrightarrow$ [0,1] is left continuous and $\lim_{t\longrightarrow\infty}$ N(x,y,t) = 1.

Definition 2.7. [37] Let X be a non empty set. Let θ : X $\times$ X $\longrightarrow$ [1,∞) and $*$ be a continuous t-norm. Then, N_θ, a fuzzy set on X $\times$ X $\times$ [0,∞) is said to be extended fuzzy b-metric if for all x,y,z $\in$ X, the following conditions hold;

(1) N_θ(x,y,0) = 0;
(2) N_θ(x,y,t) = 1 $\Leftrightarrow$ x = y, for all t > 0;
(3) N_θ(x,y,t) = N_θ(y,x,t), for all t $\geq$ 0;
(4) N_θ(x,z,θ(x,z)(t+s)) $\geq$ N_θ(x,y,t) $*$ N_θ(y,z,s), for all t,s $\geq$ 0;
(5) N_θ(x,y,$\cdot$) : [0,∞) $\longrightarrow$ [0,1] is left continuous and $\lim_{t\longrightarrow\infty}$ N(x,y,t) = 1.

In this paper, we will use B for denoting Banach space and ϑ for the zero of Banach space.

Definition 2.8. [9] Let H be a subset of B. Then, H is a cone if;

(1) H is closed, non empty, and H $\neq$ ϑ ;
(2) If a,b $\in$ [0,∞) and u,v $\in$ H, then au + bv $\in$ H;
(3) If both u and -u are in H, then u = ϑ.

For a given cone H $\subset$ B, a partial ordering $\prec$ on B via H is defined by u $\prec$ v if and only if v - u $\in$ H. Here, u $\prec$ v stands for u > v and u $\neq$ v, whereas u $\ll$ v stands for v - u $\in$ int(H). Throughout this paper, we will assume that each cone has nonempty interior.

Definition 2.9. [25] Let X be a non empty set and H be a cone on B. Consider $*$ be a continuous t-norm and N be a fuzzy set on X $\times$ X $\times$ int(H) satisfying the following conditions, for all x,y,z $\in$ X and t,s $\in$ int(H);

(1) N(x,y,t) > 0 and N (x,y,t) = 1 $\Leftrightarrow$ x = y;
(2) N(x,y,t) = N(y,x,t);
(3) N(x,z,t+s) $\geq$ N(x,y,t) $*$ N(y,z,s);
(4) N(x,y,$\cdot$) : int(H) $\longrightarrow$ [0,1] is continuous.

Thus, the triplet (X,N,$*$) is said to be Fuzzy Cone Metric Space.

Definition 2.10. [25] Consider a fuzzy cone metric space (X,N,$*$) and let x $\in$ X and $\{x_n\}$ be a sequence in X. Then,

(1) $\{x_n\}$ is said to converge to x if for t $\gg$ ϑ and α $\in$ (0,1), there exists a natural number n_1 such that N(x_n , x, t) > 1 - α for all n > n_1. It is denoted as $\lim_{n\longrightarrow\infty}$ x_n = x;
(2) $\{x_n\}$ is said to be cauchy sequence if for α $\in$ (0,1) and t $\gg$ ϑ, there exists natural number n_1 such that N(x_n , x_m, t) > 1 - α for all n, m > n_1;
(3) (X,N,$*$) is said to be complete fuzzy cone metric space if every Cauchy sequence is convergent in X;

(4) $\{x_n\}$ is said to be fuzzy cone contractive if there exists $\alpha \in$ (0,1) such that $\dfrac{1}{N(x_{n+1}, x_{n+2}, t)} - 1 \leq \alpha \left(\dfrac{1}{N(x_n, x_{n+1}, t)} - 1 \right)$ for all t $\gg \vartheta$, n $\in$ N.

Definition 2.11. [33] Let X be a non empty set and $*$ be a continuous t-norm, N is a fuzzy set on X $\times$ X $\times$ int(H) where H is the cone of B (Real Banach Space). A quadruple (X,N,$*$,b) is said to be fuzzy cone b-metric space if the following conditions are satisfied, for all x,y,z $\in$ X and t,s $\in$ int(H) and b $\geq$ 1;
FCNB1: N(x,y,t) $>$ 0 and N(x,y,0) = 0;
FCNB2: N(x,y,t) = 1 $\Leftrightarrow$ x = y;
FCNB3: N(x,y,t) = N(y,x,t);
FCNB4: N(x,z,b(t+s)) $\geq$ N(x,y,t) $*$ N(y,z,s);
FCNB5: N(x,y,$\cdot$) : int(H) $\longrightarrow$ [0,1] is continuous and $\lim_{t \longrightarrow \infty}$ N(x,y,t) = 1.

Definition 2.12. [33] Consider a fuzzy cone b-metric space (X,N,$*$,b) with b $\geq$ 1 and let x $\in$ X and $\{x_n\}$ be a sequence in X. Then,

(1) $\{x_n\}$ is said to b-converge to x if for t $\gg \vartheta$ and $\alpha \in$ (0,1), there exists a natural number n_1 such that N(x_n , x, t) $>$ 1 - α for all n $>$ n_1. It is denoted as $\lim_{n \longrightarrow \infty}$ x_n = x;
(2) $\{x_n\}$ is said to be cauchy sequence if for $\alpha \in$ (0,1) and t $\gg \vartheta$, there exists natural number n_1 such that N(x_n , x_m, t) $>$ 1 - α for all n, m $>$ n_1;
(3) (X,N,$*$,b) is said to be complete fuzzy cone b-metric space if every Cauchy sequence is b-convergent in X;
(4) A self mapping R: X $\longrightarrow$ X is said to be fuzzy cone b-contractive if there exists $\alpha \in$ (0,1) such that $\dfrac{1}{N(Rx, Ry, t)}$ - 1 $\leq \alpha \left(\dfrac{1}{N(x, y, t)} - 1 \right)$ for all t $\gg \vartheta$, n $\in$ N, where α is known as contraction constant of R;
(5) $\{x_n\}$ is said to be fuzzy cone b-contractive if there exists $\alpha \in$ (0,1) such that $\dfrac{1}{N(x_{n+1}, x_{n+2}, t)}$ - 1 $\leq \alpha \left(\dfrac{1}{N(x_n, x_{n+1}, t)} - 1 \right)$ for all t $\gg \vartheta$, n $\in$ N.

3. Main Result

We now introduce extended fuzzy cone b-metric space which is one of the generalizations of b-metric space. In this definition, last condition of fuzzy cone b-metric (Definition 2.8) has been reformed in new manner.

Definition 3.1. Let X be a non empty set with $*$ as continuous t-norm, θ : X $\times$ X $\longrightarrow$ [1,∞), N_θ be a fuzzy set on X $\times$ X $\times$ int(H) where H is a cone of B, the real banach space. A quadruple (X, N_θ, $*$, θ) is said to be extended fuzzy cone b-metric space if the following conditions are satisfied, for all x,y,z $\in$ X and t,s $\in$ int(H) and
(EFCbM1) N_θ(x,y,t) $>$ 0 and N_θ(x,y,0) = 0;
(EFCbM2) N_θ(x,y,t) = 1 $\Leftrightarrow$ x = y;
(EFCbM3) N_θ(x,y,t) = N_θ(y,x,t);
(EFCbM4) N_θ(x,z,θ(x,z)(t+s)) $\geq$ N_θ(x,y,t) $*$ N_θ(y,z,s);
(EFCbM5) N_θ(x,y,$\cdot$) : int(H) $\longrightarrow$ [0,1] is continuous and $\lim_{t \longrightarrow \infty}$ N_θ(x,y,t) = 1.

Example 3.2. Let E = R^2 and H = $\{(r_1, r_2); r_1, r_2 \geq 0\}$ is a normal cone with normal constant $k = 1$.

Let X $= C([a,b], R)$ be the space of all real valued continuous functions defined on [a,b]. It is an extended fuzzy cone b-metric space by considering $N_\theta : X \times X \times int(H) \longrightarrow [0,1]$ defined as;

$$N_\theta(x,y,t) = e^{-\frac{\sup_{\tau\in[a,b]} |x(\tau)-y(\tau)|^2}{\|t\|}} \quad \text{for all } x,y \in X \text{ and } t \geq 0.$$

where $\theta(x,y) = |x(\tau)| + |y(\tau)| + 2$.

(EFCbM1)

$$N_\theta(x,y,0) = e^{-\frac{\sup_{\tau\in[a,b]} |x(\tau)-y(\tau)|^2}{\|0\|}} = e^{-\infty} = \frac{1}{\infty} = 0$$

and $N_\theta(x,y,t) > 0, t > 0$.

(EFCbM2) $N_\theta(x,y,t) = 1$ for all $t > 0 \Leftrightarrow x = y$
i.e. for all $\tau \in [a,b], x(\tau) = y(\tau)$

$$N_\theta(x,y,t) = e^{-\frac{\sup_{\tau\in[a,b]} |x(\tau)-y(\tau)|^2}{\|t\|}} = \frac{1}{e^0} = 1.$$

(EFCbM3)

$$\begin{aligned} N_\theta(x,y,t) &= e^{-\frac{\sup_{\tau\in[a,b]} |x(\tau)-y(\tau)|^2}{\|t\|}} \\ &= e^{-\frac{\sup_{\tau\in[a,b]} |y(\tau)-x(\tau)|^2}{\|t\|}} \\ &= e^{-\frac{\sup_{\tau\in[a,b]} |y(\tau)-x(\tau)|^2}{\|t\|}} \\ &= N_\theta(y,x,t). \end{aligned}$$

(EFCbM4)

$$t + s \leq \theta(x,z)(t+s) \text{ for } \theta(x,z) \geq 1$$

Since, $t \leq t+s$ and $s \leq t+s$

$$\Rightarrow t \leq \theta(x,z)(t+s), s \leq \theta(x,z)(t+s)$$

and hence, $\|t\| \leq \|\theta(x,z)(t+s)\|$ and $\|s\| \leq \|\theta(x,z)(t+s)\|$

$$\Rightarrow \frac{\|\theta(x,z)(t+s)\|}{\|t\|} \geq 1 \text{ and } \frac{\|\theta(x,z)(t+s)\|}{\|s\|} \geq 1$$

Now, $|x-z|^2 = |(x-y)-(z-y)|^2 \leq |x-y|^2 + |z-y|^2$

$$= |x-y|^2 + |y-z|^2$$

$$\Rightarrow |x-z|^2 \leq |x-y|^2 + |y-z|^2$$

$$\Rightarrow |x-z|^2 \leq |x-y|^2 \frac{\|\theta(x,z)(t+s)\|}{\|t\|} + |y-z|^2 \frac{\|\theta(x,z)(t+s)\|}{\|s\|}$$

$$\Rightarrow -|x-z|^2 \geq -|x-y|^2 \frac{\|\theta(x,z)(t+s)\|}{\|t\|} - |y-z|^2 \frac{\|\theta(x,z)(t+s)\|}{\|s\|}$$
$$\Rightarrow -\frac{|x-z|^2}{\|\theta(x,z)(t+s)\|} \geq -\frac{|x-y|^2}{\|t\|} - \frac{|y-z|^2}{\|s\|}$$

Thus, $-\frac{|x(\tau)-z(\tau)|^2}{\|\theta(x,z)(t+s)\|} \geq -\frac{|x(\tau)-y(\tau)|^2}{\|t\|} - \frac{|y(\tau)-z(\tau)|^2}{\|s\|}$

$$\Rightarrow -\frac{\sup_{\tau\in[a,b]}|x(\tau)-z(\tau)|^2}{\|\theta(x,z)(t+s)\|} \geq -\frac{\sup_{\tau\in[a,b]}|x(\tau)-y(\tau)|^2}{\|t\|} - \frac{\sup_{\tau\in[a,b]}|y(\tau)-z(\tau)|^2}{\|s\|}$$
$$\Rightarrow e^{-\frac{\sup_{\tau\in[a,b]}|x(\tau)-z(\tau)|^2}{\|\theta(x,z)(t+s)\|}} \geq e^{-\frac{\sup_{\tau\in[a,b]}|x(\tau)-y(\tau)|^2}{\|t\|}} e^{-\frac{\sup_{\tau\in[a,b]}|y(\tau)-z(\tau)|^2}{\|s\|}}$$

Hence, N_θ(x,z,θ(x,z)(t+s)) $\geq$ N_θ(x,y,t) $*$ N_θ(y,z,s).

(EFCbM5)
Define $f : int(H) \longrightarrow (0,\infty)$ as f(t) = $\|t\| = \sqrt{{r_1}^2 + {r_2}^2}$ and g: $(0,\infty) \longrightarrow [0,1]$ as g(s) $= e^{-\frac{\sup_{\tau\in[a,b]}|x(\tau)-y(\tau)|^2}{s}}$.

Then, N_θ(x,y,$\cdot$) : int(H) $\longrightarrow$ [0,1] is composition of the both functions f and g. Since, both f and g are continuous. Therefore, N_θ(x,y,$\cdot$) is also continuous and

$$\begin{aligned}\lim_{t\longrightarrow\infty} N_\theta(x,y,t) &= \lim_{t\longrightarrow\infty} e^{-\frac{\sup_{\tau\in[a,b]}|x(\tau)-y(\tau)|^2}{\|t\|}}\\ &= e^{-\frac{\sup_{\tau\in[a,b]}|x(\tau)-y(\tau)|^2}{\infty}}\\ &= e^{-0}\\ &= \frac{1}{e^0} = 1.\end{aligned}$$

Hence, it satisfies all the conditions of Definition 3.1 and is an extended fuzzy cone b-metric space.

Definition 3.3. Consider an extended fuzzy cone b-metric space (X,N_θ,$*$,θ) where θ:X $\times$ X $\longrightarrow$ [1,∞) and let x $\in$ X and $\{x_n\}$ be a sequence in X. Then,

(1) $\{x_n\}$ is said to converge to x if for t $\gg \vartheta$ and $\alpha \in$ (0,1), there exists a natural number n_1 such that $N_\theta(x_n$, x, t) $>$ 1 - α for all n $> n_1$. It is denoted as $\lim_{n\longrightarrow\infty} x_n$ = x;
(2) $\{x_n\}$ is said to be cauchy sequence if for $\alpha \in$ (0,1) and t $\gg \vartheta$, there exists natural number n_1 such that $N_\theta(x_n$, x_m, t) $>$ 1 - α for all n, m $> n_1$;
(3) (X,N_θ,$*$,θ) is said to be complete extended fuzzy cone b-metric space if every Cauchy sequence is convergent in X;
(4) $\{x_n\}$ is said to be extended fuzzy cone b-contractive if there exists $\alpha \in$ (0,1) such that $\frac{1}{N_\theta(x_{n+1}, x_{n+2}, t)}$ - 1 $\leq \alpha \left(\frac{1}{N_\theta(x_n, x_{n+1}, t)} - 1\right)$ for all t $\gg \vartheta$, where n $\in$ N.

Definition 3.4. Let (X,N_θ,$*$,θ) be an extended fuzzy cone b-metric space and R:X $\longrightarrow$ X be a self mapping. Then, R is said to be an extended fuzzy cone b-contractive if there

exists $\alpha \in(0,1)$ such that
$\frac{1}{N_\theta(Rx,Ry,t)}$ - $1 \leq \alpha \left(\frac{1}{N_\theta(x,y,t)} - 1\right)$ for all t $\gg \vartheta$ (ϑ denotes the zero of B), n $\in$ N, where α is known as contraction constant of R.

Definition 3.5. Let (X,N_θ,$*$,θ) be an extended fuzzy cone b-metric space and P, Q : X $\longrightarrow$ X are self mappings. Then, P and Q are said to be extended fuzzy cone b-contractive if there exists $\alpha \in(0,1)$ such that
$\frac{1}{N_\theta(Px,Qy,t)}$ - $1 \leq \alpha \left(\frac{1}{N_\theta(x,y,t)} - 1\right)$ for all t $\gg \vartheta$ (ϑ denotes the zero of B), n $\in$ N, where α is known as contraction constant of both P and Q.

Theorem 3.6. *Extended Fuzzy Cone b-Banach Contraction Theorem: Let (X,N_θ,$*$,θ) be complete extended fuzzy cone b-metric space in which extended fuzzy cone b-contractive sequences are Cauchy and S,T:X $\longrightarrow$ X be extended fuzzy cone b-contractive mappings where S(X) $\subseteq$ T(X). Then, S and T have unique common fixed point.*

Proof. Let $x_0 \in$ X. Consider a sequence $\{x_n\}$ defined as x_{2n+1} = Sx_{2n},x_{2n+2} = Tx_{2n+1} for n = 0,1,2,...

At first, we will show that the subsequence $\{x_{2n}\}$ of sequence $\{x_n\}$ is Cauchy.

$$\begin{aligned}\frac{1}{N_\theta(x_{2n+1},x_{2n+2},t)} - 1 &= \frac{1}{N_\theta(Sx_{2n},Tx_{2n+1},t)} - 1\\ &\leq \alpha\left(\frac{1}{N_\theta(x_{2n},x_{2n+1},t)} - 1\right)\\ &= \alpha\left(\frac{1}{N_\theta(Sx_{2n-1},Tx_{2n},t)} - 1\right)\\ &\leq \alpha\cdot\alpha\left(\frac{1}{N_\theta(x_{2n-1},x_{2n},t)} - 1\right)\end{aligned}$$

$\Rightarrow \frac{1}{N_\theta(x_{2n+1},x_{2n+2},t)}$ - $1 \leq \alpha^2 \left(\frac{1}{N_\theta(x_{2n-1},x_{2n},t)} - 1\right)$

Continuing in the same way, we get $\Rightarrow \frac{1}{N_\theta(x_{2n+1},x_{2n+2},t)}$ - $1 \leq \alpha^{2n+1} \left(\frac{1}{N_\theta(x_0,x_1,t)} - 1\right)$

Thus, $\{x_{2n}\}$ is a Cauchy sequence in X. Therefore, $\{x_{2n}\}$ converges to some y in X. Then, by Theorem 2.10 of [25], we have

$\frac{1}{N_\theta(Sx_{2n},Tx_{2n+1},t)}$ - $1 \leq \alpha \left(\frac{1}{N_\theta(x_{2n+1},x_{2n+2},t)} - 1\right) \leq \alpha \left(\frac{1}{N_\theta(y,y,t)} - 1\right)$

$\frac{1}{N_\theta(Sx_{2n},Tx_{2n+1},t)} = 1$

Sx_{2n} = Tx_{2n+1}

$Sy = Ty$ as n$\longrightarrow \infty$.

Thus, y is a coincidence point of S and T. Now, we will prove y to be a fixed point of S and T.

$\frac{1}{N_\theta(Sy,Tx_{2n},t)}$ - $1 \leq \alpha \left(\frac{1}{N_\theta(y,x_{2n},t)} - 1\right)$

as $x_{2n} \longrightarrow$ y and Tx_{2n} = x_{2n+1}

$\frac{1}{N_\theta(Sy,x_{2n+1},t)}$ - $1 \leq \alpha \left(\frac{1}{N_\theta(y,y,t)} - 1\right)$

$\dfrac{1}{N_\theta(Sy, x_{2n+1}, t)}$ - 1 $\leq$ 0
$\Rightarrow$ Sy = y.

Uniqueness of fixed point: Let u is also a fixed point of S and T. Then, Su = Tu = u.
$\left(\dfrac{1}{N_\theta(y,u,t)} - 1\right) = \left(\dfrac{1}{N_\theta(Sy,Tu,t)} - 1\right) \leq \alpha \left(\dfrac{1}{N_\theta(y,u,t)} - 1\right)$
(1 - α) $\left(\dfrac{1}{N_\theta(y,u,t)} - 1\right) \leq 0$
$\Rightarrow N_\theta$(y, u, t) = 1
$\Rightarrow$ y = u.
Thus, y is a unique fixed point of S and T. ∎

Corollary 3.7. *Let (X,N_θ,$*$,θ) be complete extended fuzzy cone b-metric space in which extended fuzzy cone b-contractive sequence is Cauchy and T:X $\longrightarrow$ X be an extended fuzzy cone b-contractive mapping. Then, S and T have unique common fixed point.*

Proof. If we put S = T in Theorem 3.6, then the obtained result will give us extended fuzzy cone b-Banach contraction theorem. ∎

Definition 3.8. Let (X,N_θ,$*$,θ) be an extended fuzzy cone b-metric space. A self mapping T: X$\longrightarrow$ X is called V-contraction if it satisfies the following conditions:

$$\begin{aligned}\frac{1}{N_\theta(Tx,Ty,t)} - 1 \quad \leq \quad & a\left(\frac{1}{N_\theta(x,Tx,t) * N_\theta(y,Tx,t)} - 1\right) + b\left(\frac{1}{N_\theta(y,Ty,t)} - 1\right) \\ & + k\left(\frac{1}{\min(N_\theta(x,Ty,t), N_\theta(y,Tx,t))} - 1\right)\end{aligned}$$

for all x, y $\in$ X where k $\geq$ 0, a,b $\in$ [0,1) and a + b < 1 with a < 1 - k.

Theorem 3.9. *Let (X,N_θ,$*$,θ) be an extended fuzzy cone b-metric space. A self mapping T: X$\longrightarrow$ X is a V-contraction given by*

$$\begin{aligned}\frac{1}{N_\theta(Tx,Ty,t)} - 1 \quad \leq \quad & a\left(\frac{1}{N_\theta(x,Tx,t) * N_\theta(y,Tx,t)} - 1\right) + b\left(\frac{1}{N_\theta(y,Ty,t)} - 1\right) \\ & + k\left(\frac{1}{\min(N_\theta(x,Ty,t), N_\theta(y,Tx,t))} - 1\right). \qquad (3.1)\end{aligned}$$

Then, T has a unique common fixed point in X.

Proof. Let $x_0 \in$ X. Define a sequence $\{x_n\}$ by x_n = Tx_{n-1} for n $\geq$ 0. Then, by given contraction (3.1) for t $\geq$ 0 and n $\geq$ 1,

$$\begin{aligned}\frac{1}{N_\theta(Tx,Ty,t)} - 1 \quad \leq \quad & a\left(\frac{1}{N_\theta(x,Tx,t) * N_\theta(y,Tx,t)} - 1\right) + b\left(\frac{1}{N_\theta(y,Ty,t)} - 1\right) \\ & + k\left(\frac{1}{\min(N_\theta(x,Ty,t), N_\theta(y,Tx,t))} - 1\right).\end{aligned}$$

We have,

$$\frac{1}{N_\theta(x_n, x_{n+1}, t)} - 1 \quad = \quad \frac{1}{N_\theta(Tx_{n-1}, Tx_n, t)} - 1$$

$$\leq a\left(\frac{1}{N_\theta(x_{n-1},Tx_{n-1},t)*N_\theta(x_n,Tx_{n-1},t)}-1\right) +b\left(\frac{1}{N_\theta(x_n,Tx_n,t)}-1\right) +k\left(\frac{1}{\min(N_\theta(x_{n-1},Tx_n,t),N_\theta(x_n,Tx_{n-1},t))}-1\right)$$

$$\Rightarrow \frac{1}{N_\theta(x_n,x_{n+1},t)}-1 \leq a\left(\frac{1}{N_\theta(x_{n-1},x_n,t)*N_\theta(x_n,x_n,t)}-1\right) +b\left(\frac{1}{N_\theta(x_n,x_{n+1},t)}-1\right) +k\left(\frac{1}{\min(N_\theta(x_{n-1},x_{n+1},t),N_\theta(x_n,x_n,t))}-1\right).$$

$\Rightarrow \frac{1}{N_\theta(x_n,x_{n+1},t)}$ - 1 $\leq a\left(\frac{1}{N_\theta(x_{n-1},x_n,t)}-1\right)+b\left(\frac{1}{N_\theta(x_n,x_{n+1},t)}-1\right)+k(0)$

$\Rightarrow \left(\frac{1}{N_\theta(x_n,x_{n+1},t)}-1\right)$ (1 - b) $\leq a\left(\frac{1}{N_\theta(x_{n-1},x_n,t)}-1\right)$

$\Rightarrow \left(\frac{1}{N_\theta(x_n,x_{n+1},t)}-1\right)\leq \frac{a}{(1-b)}\left(\frac{1}{N_\theta(x_{n-1},x_n,t)}-1\right)$

Take h = $\frac{a}{(1-b)}$ as a + b < 1 implies a < 1 - b.

This implies

$\left(\frac{1}{N_\theta(x_n,x_{n+1},t)}-1\right)\leq h\left(\frac{1}{N_\theta(x_{n-1},x_n,t)}-1\right)\leq h^n\left(\frac{1}{N_\theta(x_0,x_1,t)}-1\right).$

Since, $\{x_n\}$ is an extended fuzzy cone b-contractive sequence and we get, $\lim_{n\longrightarrow\infty} N_\theta(x_n,x_{n+1},t)=1$ for t > 0.

Now, for m > n $\geq n_0$,

$\frac{1}{N_\theta(x_n,x_m,t)}$ - 1

$$\leq \left(\frac{1}{N_\theta(x_n,x_{n+1},t)}-1\right)+\left(\frac{1}{N_\theta(x_{n+1},x_{n+2},t)}-1\right)+\dots +\left(\frac{1}{N_\theta(x_{m-1},x_m,t)}-1\right)$$

$$\leq h^n\left(\frac{1}{N_\theta(x_0,x_1,t)}-1\right)+h^{n+1}\left(\frac{1}{N_\theta(x_0,x_1,t)}-1\right)+\dots +h^{m-1}\left(\frac{1}{N_\theta(x_0,x_1,t)}-1\right)$$

$$= (h^n+h^{n+1}+\dots+h^{m-1})\left(\frac{1}{N_\theta(x_0,x_1,t)}-1\right)\longrightarrow 0 \text{ as } n\longrightarrow\infty$$

$\lim_{n\longrightarrow\infty} N_\theta(x_n,u,t)=1$ and $\lim_{n\longrightarrow\infty} x_n=$ u for t $\geq$ 0.

$$\frac{1}{N_\theta(x_{n+1},Tu,t)}-1 = \frac{1}{N_\theta(Tx_n,Tu,t)}-1$$

$$\begin{aligned}
&\leq a\left(\frac{1}{N_\theta(x_n,Tx_n,t)*N_\theta(u,Tx_n,t)}-1\right)\\
&\quad +b\left(\frac{1}{N_\theta(u,Tu,t)}-1\right)\\
&\quad +k\left(\frac{1}{\min(N_\theta(x_n,Tu,t),N_\theta(u,Tx_n,t))}-1\right)\\
&= a\left(\frac{1}{N_\theta(x_n,x_{n+1},t)*N_\theta(u,x_{n+1},t)}-1\right)\\
&\quad +b\left(\frac{1}{N_\theta(u,Tu,t)}-1\right)\\
&\quad +k\left(\frac{1}{\min(N_\theta(x_n,Tu,t),N_\theta(u,x_{n+1},t))}-1\right).
\end{aligned}$$

Applying n $\longrightarrow \infty$,

$$\begin{aligned}
\frac{1}{N_\theta(u,Tu,t)}-1 &= a\left(\frac{1}{N_\theta(u,u,t)*N_\theta(u,u,t)}-1\right)\\
&\quad +b\left(\frac{1}{N_\theta(u,Tu,t)}-1\right)\\
&\quad +k\left(\frac{1}{\min(N_\theta(u,Tu,t),N_\theta(u,u,t))}-1\right)\\
&= 0+b\left(\frac{1}{N_\theta(u,Tu,t)}-1\right)+k(0)\\
\Rightarrow \frac{1}{N_\theta(u,Tu,t)}-1 &= b\left(\frac{1}{N_\theta(u,Tu,t)}-1\right)\ \text{for } t\geq 0.
\end{aligned}$$

As b < 1 since, a + b < 1. Thus, u is a fixed point of T.

Uniqueness of fixed point: Let v is another fixed point of T.

$$\begin{aligned}
\frac{1}{N_\theta(u,v,t)}-1 &= \frac{1}{N_\theta(Tu,Tv,t)}-1\\
&\leq a\left(\frac{1}{N_\theta(u,Tu,t)*N_\theta(v,Tu,t)}-1\right)\\
&\quad +b\left(\frac{1}{N_\theta(v,Tv,t)}-1\right)\\
&\quad +k\left(\frac{1}{\min(N_\theta(u,Tv,t),N_\theta(v,Tu,t))}-1\right)\\
&= a\left(\frac{1}{N_\theta(u,u,t)*N_\theta(v,u,t)}-1\right)\\
&\quad +b\left(\frac{1}{N_\theta(v,v,t)}-1\right)\\
&\quad +k\left(\frac{1}{\min(N_\theta(u,v,t),N_\theta(v,u,t))}-1\right)
\end{aligned}$$

$$\Rightarrow \frac{1}{N_\theta(u,v,t)} - 1 \leq a\left(\frac{1}{N_\theta(v,u,t)} - 1\right) + 0 + k\left(\frac{1}{N_\theta(u,v,t)} - 1\right)$$
$$\Rightarrow \frac{1}{N_\theta(u,v,t)} - 1 \leq (a+k)\left(\frac{1}{N_\theta(v,u,t)} - 1\right).$$

As a < 1 - k ⇒ a + k < 1.

$$\Rightarrow N_\theta(u,v,t) = 1 \Rightarrow u = v.$$

Hence, u is a unique fixed point of T. ■

4. Application

Here, we are going to apply our proved results on Fredholm Integral Equations (FIE's). Let U = C([0,δ], R) be the space of all real valued continuous functions on [0,δ] where 0 < $\delta \in$ R. Thus, the Fredholm Integral equations are;

$$\phi(\tau) = \int_0^\delta k_1(\tau, s, \phi(s))ds \tag{4.1}$$

$$\psi(\tau) = \int_0^\delta k_2(\tau, s, \psi(s))ds \tag{4.2}$$

for all ϕ, $\psi \in$ U, where $\tau \in [0,\delta]$ and k_1, k_2:$[0,\delta] \times [0,\delta] \times R \longrightarrow R$. Thus, the induced metric d_θ: U × U $\longrightarrow R$ be defined as;

$$d_\theta(\phi,\psi) = \sup_{\tau\in[0,\delta]} |\phi(\tau) - \psi(\tau)|^2 = \|\phi - \psi\| \tag{4.3}$$

where θ: U × U $\longrightarrow$ [1,∞) defined as $\theta(\phi,\psi) = 1 + \phi + \psi$.

The binary operation $*$ is defined as c $*$ d = cd, for all c, d $\in [0,\delta]$. An extended fuzzy cone b-metric $N_\theta : U \times U \times (0,\infty) \longrightarrow [0,1]$ is given by

$$N_\theta(\phi,\psi,t) = \frac{t}{t + d_\theta(\phi,\psi)} \tag{4.4}$$

for t > 0, for all $\phi, \psi \in$ U. As it can be easily verified that N_θ satisfies all the properties of extended fuzzy cone b-metric space and hence, a complete extended fuzzy cone b-metric space.

Theorem 4.1. *Given two Fredholm Integral Equations are*

$$\phi(\tau) = \int_0^\delta k_1(\tau, s, \phi(s))ds \tag{4.5}$$

$$\psi(\tau) = \int_0^\delta k_2(\tau, s, \psi(s))ds \tag{4.6}$$

where $\tau \in [0,1]$ and $\phi, \psi \in U$. Assume that $K_1, K_2 : [0,1] \times [0,1] \times R \longrightarrow R$ are such that $A_\phi, B_\psi \in B$ for every $\phi, \psi \in B$ where

$$A_\phi(\tau) = \int_0^\delta k_1(\tau, s, \phi(s))ds \tag{4.7}$$

$$A_\psi(\tau) = \int_0^\delta k_2(\tau, s, \psi(s))ds. \tag{4.8}$$

If there exists $\beta \in (0,1)$ such that for all $\phi, \psi \in U$,

$$\|A_\phi - A_\psi\| \leq \beta M(T, \phi, \psi) \tag{4.9}$$

where

$$\begin{aligned} M(T, \phi, \psi) = \max \Big\{ & \|y - Ty\|, \|x - Ty\|, \|y - Tx\|, \\ & \frac{t\|x - Tx\| + t\|y - Tx\| + \|x - Tx\| \cdot \|y - Tx\|}{t^2} \Big\}. \end{aligned} \tag{4.10}$$

Then, the two integral equations defined in (4.5)-(4.6), have a unique common solution in U.

Proof. Define T: E $\longrightarrow$ E by

$$T(\phi) = A_\phi, T(\psi) = A_\psi \tag{4.11}$$

The FIE in (4.5)-(4.6) have a common solution if and only if T has a fixed point in U. Now, we have to show that Theorem 3.9 is applied to the integral operator T.

Then, for all $\phi, \psi \in U$, we have the following four cases:

Case 1: If $M(T, \phi, \psi) = \|y - Ty\|$ in (4.10), then from (4.4) and (4.9), we get

$$\begin{aligned} \frac{1}{N_\theta(T\phi, T\psi, t)} - 1 &= \frac{1}{\frac{t}{t + d_\theta(T\phi, T\psi)}} - 1 \\ &= \frac{t + d_\theta(T\phi, T\psi)}{t} - 1 \\ &= \frac{d_\theta(T\phi, T\psi)}{t} \\ &= \frac{d_\theta(A_\phi, A_\psi)}{t} \\ &\leq \frac{\beta M(T, \phi, \psi)}{t} \\ &= \frac{\beta \|y - Ty\|}{t} \end{aligned}$$

$$\Rightarrow \frac{1}{N_\theta(T\phi, T\psi, t)} - 1 \leq \frac{\beta \|y - Ty\|}{t} \tag{4.12}$$

Now,

$$\begin{aligned}\frac{1}{N_\theta(y,Ty,t)}-1&=\frac{1}{\frac{t}{t+d_\theta(y,Ty)}}\\&=\frac{t+d_\theta(y,Ty)}{t}-1\\&=\frac{d_\theta(y,Ty)}{t}\\&=\frac{\|y-Ty\|}{t}\end{aligned}$$

$$\Rightarrow \frac{1}{N_\theta(y,Ty,t)}-1=\frac{\|y-Ty\|}{t}. \tag{4.13}$$

Thus, from (4.12) and (4.13), we get

$$\frac{1}{N_\theta(T\phi,T\psi,t)}-1\leq\beta\left(\frac{1}{N_\theta(y,Ty,t)}-1\right) \tag{4.14}$$

for $t \gg \vartheta$ and for all $\phi, \psi \in U$ such that T$\phi \neq$ Tψ. It is clear that the inequality (4.14) holds if T$\phi =$ Tψ. Thus, the integral operator T satisfy all the conditions of Theorem 3.9 with $\beta = b$ and $a = k = 0$ in V-contraction and have a fixed point i.e. (4.5)-(4.6) have a common solution in U.

Case 2: If $M(T,\phi,\psi)=\dfrac{t\|x-Tx\|+t\|y-Tx\|+\|x-Tx\|\cdot\|y-Tx\|}{t^2}$ in (4.10), then from (4.4) and (4.9), we get

$$\begin{aligned}\frac{1}{N_\theta(T\phi,T\psi,t)}-1&=\frac{1}{\frac{t}{t+d_\theta(T\phi,T\psi)}}-1\\&=\frac{d_\theta(T\phi,T\psi)}{t}\\&=\frac{d_\theta(A_\phi,A_\psi)}{t}\\&\leq\frac{\beta M(T,\phi,\psi)}{t}\\&=\beta\left(\frac{t\|x-Tx\|+t\|y-Tx\|+\|x-Tx\|\cdot\|y-Tx\|}{t^2}\right)\end{aligned}$$

$$\Rightarrow \frac{1}{N_\theta(T\phi,T\psi,t)}-1\leq\beta\left(\frac{t\|x-Tx\|+t\|y-Tx\|+\|x-Tx\|\cdot\|y-Tx\|}{t^2}\right) \tag{4.15}$$

Now,

$$\frac{1}{N_\theta(x,Tx,t)*N_\theta(y,Tx,t)}-1=\frac{1}{\left(\frac{t}{t+d_\theta(x,Tx)}\right)\left(\frac{t}{t+d_\theta(y,Tx)}\right)}-1$$

$$= \frac{(t + d_\theta(x, Tx))(t + d_\theta(y, Tx))}{t^2} - 1$$
$$= \frac{td_\theta(x, Tx) + td_\theta(y, Tx) + d_\theta(x, Tx)d_\theta(y, Tx)}{t^2}$$

$$\Rightarrow \frac{1}{N_\theta(x, Tx, t) * N_\theta(y, Tx, t)} - 1 = \frac{t\|x - Tx\| + t\|y - Tx\| + \|x - Tx\| \cdot \|y - Tx\|}{t^2}. \tag{4.16}$$

Thus, from (4.15) and (4.16), we get

$$\frac{1}{N_\theta(T\phi, T\psi, t)} - 1 \leq \beta \left(\frac{1}{N_\theta(x, Tx, t) * N_\theta(y, Tx, t)} - 1 \right) \tag{4.17}$$

for $t \gg \vartheta$ and for all $\phi, \psi \in U$ such that $\mathrm{T}\phi \neq \mathrm{T}\psi$. It is clear that the inequality (4.17) holds if $\mathrm{T}\phi = \mathrm{T}\psi$. Thus, the integral operator T satisfy all the conditions of Theorem 3.9 with $\beta = a$ and $b = k = 0$ in V-contraction and have a fixed point i.e. (4.5)-(4.6) have a common solution in U.

Case 3: If $M(T, \phi, \psi) = \|x - Ty\|$ in (4.10), then from (4.4) and (4.9), we get

$$\frac{1}{N_\theta(T\phi, T\psi, t)} - 1 = \frac{1}{\frac{t}{t + d_\theta(T\phi, T\psi)}} - 1$$
$$= \frac{d_\theta(T\phi, T\psi)}{t}$$
$$= \frac{d_\theta(A_\phi, A_\psi)}{t}$$
$$\leq \frac{\beta M(T, \phi, \psi)}{t}$$
$$= \frac{\beta\|x - Ty\|}{t}$$

$$\Rightarrow \frac{1}{N_\theta(T\phi, T\psi, t)} - 1 \leq \frac{\beta\|x - Ty\|}{t}. \tag{4.18}$$

Now, if $\min(N_\theta(x, Ty, t), N_\theta(y, Tx, t)) = N_\theta(x, Ty, t)$, then

$$\frac{1}{\min(N_\theta(x, Ty, t), N_\theta(y, Tx, t))} - 1 = \frac{1}{N_\theta(x, Ty, t)} - 1$$
$$= \frac{d_\theta(x, Ty)}{t}$$
$$= \frac{\|x - Ty\|}{t}$$

$$\Rightarrow \frac{1}{\min(N_\theta(x, Ty, t), N_\theta(y, Tx, t))} - 1 = \frac{\|x - Ty\|}{t}. \tag{4.19}$$

Thus, from (4.18) and (4.19), we get

$$\Rightarrow \frac{1}{N_\theta(T\phi, T\psi, t)} - 1 \leq \beta \left(\frac{1}{\min(N_\theta(x, Ty, t), N_\theta(y, Tx, t))} - 1 \right) \tag{4.20}$$

for $t \gg \vartheta$ and for all $\phi, \psi \in U$ such that $\mathrm{T}\phi \neq \mathrm{T}\psi$. It is clear that the inequality (4.20) holds if $\mathrm{T}\phi = \mathrm{T}\psi$. Thus, the integral operator T satisfy all the conditions of Theorem 3.9 with $\beta = k$ and $a = b = 0$ in V-contraction and have a fixed point i.e. (4.5)-(4.6) have a common solution in U.

Case 4: If $M(T, \phi, \psi) = \|y - Tx\|$ in (4.10), then from (4.4) and (4.9), we get

$$\begin{aligned} \frac{1}{N_\theta(T\phi, T\psi, t)} - 1 &= \frac{1}{\frac{t}{t + d_\theta(T\phi, T\psi)}} - 1 \\ &= \frac{d_\theta(T\phi, T\psi)}{t} \\ &= \frac{d_\theta(A_\phi, A_\psi)}{t} \\ &\leq \frac{\beta M(T, \phi, \psi)}{t} \\ &= \frac{\beta \|y - Tx\|}{t} \end{aligned}$$

$$\Rightarrow \frac{1}{N_\theta(T\phi, T\psi, t)} - 1 \leq \frac{\beta \|y - Tx\|}{t}. \tag{4.21}$$

Now, if $\min(N_\theta(x, Ty, t), N_\theta(y, Tx, t)) = N_\theta(y, Tx, t)$, then

$$\begin{aligned} \frac{1}{\min(N_\theta(x, Ty, t), N_\theta(y, Tx, t))} - 1 &= \frac{1}{N_\theta(y, Tx, t)} - 1 \\ &= \frac{d_\theta(y, Tx)}{t} \\ &= \frac{\|y - Tx\|}{t} \end{aligned}$$

$$\Rightarrow \frac{1}{\min(N_\theta(x, Ty, t), N_\theta(y, Tx, t))} - 1 = \frac{\|y - Tx\|}{t}. \tag{4.22}$$

Thus, from (4.21) and (4.22), we get

$$\Rightarrow \frac{1}{N_\theta(T\phi, T\psi, t)} - 1 \leq \beta \left(\frac{1}{\min(N_\theta(x, Ty, t), N_\theta(y, Tx, t))} - 1 \right) \tag{4.23}$$

for $t \gg \vartheta$ and for all $\phi, \psi \in U$ such that $\mathrm{T}\phi \neq \mathrm{T}\psi$. It is clear that the inequality (4.23) holds if $\mathrm{T}\phi = \mathrm{T}\psi$. Thus, the integral operator T satisfy all the conditions of Theorem 3.9 with $\beta = k$ and $a = b = 0$ in V-contraction and have a fixed point i.e. (4.5)-(4.6) have a common solution in U.

Hence, combining the results of all cases, the given Fredholm Integral equations have a solution in U.

Uniqueness of solution: Let us consider that u is common solution of given integral equations under T in U i.e. u is fixed under T, implies T(u) = u. Let v be another fixed point under T i.e. T(v) = v.

For uniqueness,

$$\begin{aligned}
\frac{1}{N_\theta(u,v,t)} - 1 &= \frac{1}{N_\theta(Tu,Tv,t)} - 1 \\
&\leq a\left(\frac{1}{N_\theta(u,Tu,t) * N_\theta(v,Tu,t)} - 1\right) \\
&\quad + b\left(\frac{1}{N_\theta(v,Tv,t)} - 1\right) \\
&\quad + k\left(\frac{1}{\min(N_\theta(u,Tv,t), N_\theta(v,Tu,t))} - 1\right) \\
&= a\left(\frac{1}{N_\theta(u,u,t) * N_\theta(v,u,t)} - 1\right) \\
&\quad + b\left(\frac{1}{N_\theta(v,v,t)} - 1\right) \\
&\quad + k\left(\frac{1}{\min(N_\theta(u,v,t), N_\theta(v,u,t))} - 1\right) \\
&= a\left(\frac{1}{N_\theta(v,u,t)} - 1\right) + b(0) + k\left(\frac{1}{N_\theta(u,v,t)} - 1\right) \\
&= (a+k)\left(\frac{1}{N_\theta(u,v,t)} - 1\right) \\
\Rightarrow \frac{1}{N_\theta(u,v,t)} - 1 &\leq (a+k)\left(\frac{1}{N_\theta(u,v,t)} - 1\right)
\end{aligned}$$

As a < 1 - k ⇒ a + k < 1.

$$\Rightarrow N_\theta(u,v,t) = 1 \Rightarrow u = v.$$

Hence, u is a unique fixed point of T which implies the uniqueness of solution of given integral equations. ∎

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