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Bounded Linear Transformations on Hypernormed Vector Spaces

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Abstract The aim of this paper is to generalize normed vector spaces to hypernormed vector spaces. In this regards, we define hypernorm on hypervector space and investigate it's properties. Moreover, we introduce the notion of hypermetric space and use it to find a necessary and sufficient condition for two hypernorms on a hypervector space to be equivalent. Finally, we use hypernormed vector spaces to define bounded linear transformations and present some results related to them.

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1. Introduction

After introducing the notion of hyperstructure in 1934 by F. Marty in [1], several researchers were involved in this research area. Some worked on the theoretical part of this area, where they generalized many algebraic structures' concepts such as hyperfields, hypervector spaces (see [2–6]). Other researchers worked on applications of this topic in different fields of sciences (see [7]).

In [5], Roy et al. defined hypernorms on hypervector spaces over the real hyperfield. In our paper, we extend their definition of hypernorms and consider hypernorms on hypervector spaces over any valued hyperfield. The remainder part of this paper is constructed as follows: After an Introduction, in Section 2 we present some definitions related to hyperstructures and results related to both: hyperabsolute values of hyperfields and hypervector spaces over hyperfields that are proved by the authors in [2, 3]. In Section 3, we define hypernorms on hypervector spaces, investigate their properties and find a necessary and sufficient condition for two hypernorms to be equivalent. Finally, in Section 4, we discuss bounded linear transformation over hypervector spaces and find relationships between bounded and continuous linear transformations.

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Throughout this paper, $\mathbb{R}$ is the set of real numbers, K is a Krasner hyperfield, $\overrightarrow{0}$ is the zero of the hypervector space V, $\underline{0}$ is the additive identity of K, $|\cdot|$ is the standard absolute value of real numbers and $/\cdot/$ is hyperabsolute value of K.

2. Preliminaries

In this section, we present some definitions related to hyperstructures and results related to both: hyperabsolute values of hyperfields and hypervector spaces over hyperfields that are used throughout this paper.

Let H be a non-empty set. Then, a mapping $\circ : H \times H \to \mathcal{P}^*(H)$ is called a *binary hyperoperation* on H, where $\mathcal{P}^*(H)$ is the family of all non-empty subsets of H. The couple $(H, \circ)$ is called a *hypergroupoid*. In this definition, if A and B are two non-empty subsets of H and $x \in H$, then we define $A \circ B = \bigcup_{a\in A, b\in B} a \circ b$, $x \circ A = \{x\} \circ A$ and $A \circ x = A \circ \{x\}$. A hypergroupoid $(H, \circ)$ is called: a *semihypergroup* if for every $x, y, z \in H$, we have $x \circ (y \circ z) = (x \circ y) \circ z$; a *quasihypergroup* if for every $x \in H$, $x \circ H = H = H \circ x$ (this condition is called the reproduction axiom); a *hypergroup* if it is a semihypergroup and a quasihypergroup. A *Krasner hyperring* is an algebraic structure $(R, +, \cdot)$ which satisfies the following axiom: (1) $(R, +)$ is a commutative hypergroup; (2) there exists $0 \in R$ such that $0 + x = \{x\}$ for all $x \in R$; (3) for every $x \in R$ there exists unique $x' \in R$ such that $0 \in x + x'$; (x' is denoted by $-x$); (4) $z \in x + y$ implies that $y \in -x + z$ and $x \in z - y$; (5) $(R, \cdot)$ is a semigroup having zero as a bilaterally absorbing element, i.e., $x \cdot 0 = 0 \cdot x = 0$; (6) the multiplication "$\cdot$" is distributive with respect to the hyperoperation "$+$". Note that every ring is a Krasner hyperring. A subhyperring A of a Krasner hyperring $(R, +, \cdot)$ is a *hyperideal* of R if $r \cdot a \in A$ ($a \cdot r \in A$) for all $a \in A, r \in R$. A commutative Krasner hyperring $(R, +, \cdot)$ with identity element "1" is a *hyperfield* if $(R \setminus \{0\}, \cdot)$ is a group. Different examples of finite and infinite hyperfields were constructed.

Example 2.1. Let $S = \{0, 1, 2\}$ and define $(S, +)$ and $(S, \cdot)$ by the following tables:

$+$	0	1	2
0	0	1	2
1	1	1	S
2	2	S	2

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Then $(S, +, \cdot)$ is a hyperfield.

We present the following examples of infinite hyperfields from [8, 9].

Example 2.2. (**Triangle hyperfield**) Let $\mathbb{V}$ be the set of non-negative real numbers with the following hyperoperations:

$$a \oplus b = \{c \in \mathbb{V} : |a - b| \leq c \leq a + b\},$$

and

$$a \odot b = ab.$$

Then $(\mathbb{V}, \oplus, \odot)$ is a hyperfield. Here, the additive identity $\underline{0} = 0$ and $-a = a$ for all $a \in \mathbb{V}$.

Example 2.3. (**Tropical hyperfield**) Let $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ with the following hyperoperations:

$$a \oplus b = \begin{cases} \max\{a, b\}, & \text{if } a \neq b; \\ \{c \in \mathbb{T} : c \leq a\}, & \text{if } a = b. \end{cases},$$

and

$$a \odot b = a + b.$$

Then $(\mathbb{T}, \oplus, \odot)$ is a hyperfield. Here, the additive identity $\underline{0} = -\infty$ and $-a = a$ for all $a \in \mathbb{V}$. Moreover, the multiplicative identity is 0.

In [2], the authors defined hyperabsolute values of hyperfields as a generalization of the notion of absolute values of fields. They presented some results related to it and provided some examples.

Definition 2.4. [2] Let K be a hyperfield and $\mathbb{R}_{\geq 0}$ be the set of non-negative real numbers. A *hyperabsolute value* of K is a function

$$/\cdot/ : K \longrightarrow \mathbb{R}_{\geq 0}$$

satisfying the following conditions for all $x, y \in K$:

(1) $/x/ = 0$ if and only if $x = 0$;
(2) $/xy/ = /x//y/$;
(3) $sup/z/_{z \in x+y} \leq /x/ + /y/$. (Triangle inequality)

Proposition 2.5. [2] *Let K be a finite hyperfield and $/\cdot/$ be a hypersbolute value of K. Then $/\cdot/$ is the trivial hyperabsolute value of K. i.e., for all $x \in K$,*

$$/x/ = \begin{cases} 0, & \textit{if } x = \underline{0}; \\ 1, & \textit{otherwise.} \end{cases}$$

Example 2.6. [2] Let $(\mathbb{V}, \oplus, \odot)$ be the Triangle hyperfield and define $/\cdot/$ of $\mathbb{V}$ as follows: For all $x \in \mathbb{V}$, $/x/ = x$. Then $/\cdot/$ is a hyperabsolute value of $\mathbb{V}$.

In [6], Tallini defined hypervector spaces over fields and many authors used her definition to define new concepts such as basis, linear transformation, hypernorm and many other. In [3], the authors presented a different definition of hypervector spaces. They defined hypervector spaces over hyperfields, studied their properties and presented some examples.

Definition 2.7. [3] Let K be a Krasner hyperfield. A canonical hypergroup $(V, +)$ together with a map $\cdot : K \times V \to V$, is called a *hypervector space* over K if for all $a, b \in F$ and $x, y \in V$, the following conditions hold: (1) $a \cdot (x + y) = a \cdot x + a \cdot y$; (2) $(a + b) \cdot x = a \cdot x + b \cdot x$; (3) $a \cdot (b \cdot x) = (ab) \cdot x$; (4) $a \cdot (-x) = (-a) \cdot x = -(a \cdot x)$; (5) $x = 1 \cdot x$.

Example 2.8. [3] Let $(K, +, \cdot)$ be a hyperfield, E be a non-empty set and K^E be the set of all functions from E to K. Then K^E is a hypervector space over K. Where $\star : K \times K^E \longrightarrow K^E$ is defined as follows: For all $f, g \in K^E, k, x \in K$,

$$(f + g)(x) = f(x) + g(x) \text{ and } (k \star f)(x) = k \cdot f(x).$$

Proposition 2.9. [3] *Let K be a hyperfield and $(V, +)$ be a hypervector space over K. A non-empty subset $W \subseteq V$ is subhyperspace of V if and only if $a \cdot x + b \cdot y \subseteq W$ for all $x, y \in W$ and $a, b \in K$.*

Definition 2.10. [3] A subset $S = \{v_1, v_2, \cdots, v_n\}$ of a hypervector space V over a hyperfield K is said to *basis* for V if it is linearly independent and it spans V. We say that V is *finite dimensional* if it has a finite basis. Otherwise, it is called *infinite dimensional*.

3. Hypernormed Vector Spaces and Equivalent Hypernorms

Inspired by the definition of norm of a vector space, we define hypernorm of hypervector space, prove its properties and present some examples.
Throughout this section, V is a hypervector space over a valued hyperfield K with hyperabsolute value $/\cdot/$.

3.1. Hypernormed Vector Spaces

A hypernormed vector space is a hypervector space where each vector is associated with a "length".

Definition 3.1. Let K be a hyperfield and V a hypervector space over K. A *hypernorm* on V is a function

$$||\cdot|| : V \longrightarrow \mathbb{R}$$

satisfying the following conditions for all $x, y \in V$, $a \in K$:

(1) If $||x|| = 0$ then $x = \overrightarrow{0}$;
(2) $||ax|| = /a/||x||$;
(3) $sup||z||_{z\in x+y} \leq ||x|| + ||y||$. (Triangle inequality)

Moreover, $(V, ||\cdot||)$ is called *hypernormed vector space*.

Proposition 3.2. *Let K be a finite hyperfield and V be a hypervector space over K. If there exists $r > 0$ such that for all $x \in V$:*

$$||x|| = \begin{cases} r, & \text{if } x \neq \overrightarrow{0}; \\ 0, & \text{if } x = \overrightarrow{0}. \end{cases}$$

Then $||\cdot||$ is a hypernorm on V.

Proof. It is clear that if $||x|| = 0$ then $x = \overrightarrow{0}$. Proposition 2.5 asserts that $/k/ = 1$ for all $k \in K \setminus \{\underline{0}\}$. We get now that $||kx|| = r = /k/||x||$. Let $z \in x + y$. If $x = y = \overrightarrow{0}$ then $z = \overrightarrow{0}$ and hence $||z|| = 0 \leq ||x|| + ||y||$. If $x \neq \overrightarrow{0}$ or $y \neq \overrightarrow{0}$ then either $||z|| = 0$ or $||z|| = r$. Thus, $||z|| \leq ||x|| + ||y||$. ∎

Example 3.3. Let K be a valued hyperfield. Define $||\cdot||$ on K^E as follows: for all $f \in K^E$,

$$||f|| = \sup\{/f(x)/ : x \in E\}.$$

We show that $||\cdot||$ is a hypernorm on K^E. Let $f, g \in K^E$ and $k \in K$.

- If $||f|| = 0$ then $/f(x)/ = 0$ for all $x \in E$. It follows, from Definition 2.4, that $f(x) = 0$ for all $x \in E$ and hence, f is the zero of K^E.
- $||kf|| = \sup\{/af(x)/ : x \in E\} = \sup\{/a//f(x)/ : x \in E\} = /a/\sup\{/f(x)/ : x \in E\} = /a/||f||$.
- $||f+g|| = \sup\{/f(x)+g(x)/ : x \in E\} \leq \sup\{/f(x)/ : x \in E\} + \sup\{/g(x)/ : x \in E\} = ||f|| + ||g||$.

Example 3.4. Let V be a finite dimensional hypervector space over a valued hyperfield K and $B = \{e_1, \ldots, e_n\}$ be a basis for V. Define $||\cdot||$ on V as follows: for all $v \in V$,

$$||v|| = ||\sum_{i=1}^{n} c_i e_i|| = \max\{/c_i/ : 1 \leq i \leq n\}.$$

Then it easy to see that $||\cdot||$ is a hypernorm on V.

Definition 3.5. Let K be a hyperfield and V a hypervector space over K. A *semi-hypernorm* of V is a function

$$||\cdot|| : V \longrightarrow \mathbb{R}$$

satisfying the following conditions for all $x, y \in V$, $a \in K$:

(1) If $||\overrightarrow{0}|| = 0$;
(2) $||ax|| = /a/||x||$;
(3) $sup||z||_{z\in x+y} \leq ||x|| + ||y||$. (Triangle inequality)

Example 3.6. Let K be a valued hyperfield, $a \in E$ and $|E| \geq 2$. Define $||\cdot||$ on K^E as follows: for all $f \in K^E$,

$$||f|| = /f(a)/.$$

Then it is clear that $||\cdot||$ is a semi-hypernorm on K^E that is not a hypernorm.

Proposition 3.7. *Let $||\cdot||$ be a (semi) hypernorm on V. Then*

(1) $||\overrightarrow{0}|| = 0$;
(2) $||-x|| = ||x||$;
(3) $\inf ||kx - ky|| = /k/ \inf ||x - y||$;
(4) $||x|| \geq 0$.

Proof.

- Proof of 1. Since $0v = \overrightarrow{0}$ ([3]) for all $v \in V$, it follows that $||\overrightarrow{0}|| = ||0v|| = /0/||v|| = 0$.
- Proof of 2. In [2], the authors proved that $/-1/ = 1$ We get that $||-x|| = ||(-1)x|| = /-1/||x|| = ||x||$;
- Proof of 3. Let $z \in x - y$, $t \in kx - ky$ such that $||z|| = \inf ||x - y||$ and $||t|| = \inf ||kx - ky||$. Since $z \in x - y$, it follows that $kz \in k(x - y) = kx - ky$. Thus, $||t|| = \inf ||kx - ky|| \leq ||kz|| = /k/||z||$. Since $t \in kx - ky$, it follows that $k^{-1}t \in k^{-1}(kx - ky) = x - y$. Thus, $||z|| = \inf ||x - y|| \leq ||k^{-1}t|| = \frac{1}{/k/}||t||$.
- Proof of 4. $\overrightarrow{0} \in -x + x$ for all $x \in V$. The Triangle inequality implies that $0 = ||\overrightarrow{0}|| \leq ||-x|| + ||x|| = 2||x||$. ∎

Proposition 3.8. *(Generalized Triangle inequality.) Let V be a hypervector space, n be a positive integer greater than 1, $||\cdot||$ be a (semi) hypernorm on V and $x_i \in V$ for all $i = 1, 2, \ldots, n$. Then*

$$sup||z||_{z\in x_1+x_2+\ldots+x_n} \leq ||x_1|| + ||x_2|| + \ldots + ||x_n||.$$

Proof. We prove by induction on the value of n. For $n = 2$, the proof follows from Definitions 3.1 and 3.5, Condition 3. Suppose that $sup||z||_{z\in x_1+x_2+\ldots+x_{n-1}} \leq ||x_1|| + ||x_2|| + \ldots + ||x_{n-1}||$ and let $t \in x_1 + x_2 + \ldots + x_{n-1} + x_n$. Then there exists $z \in x_1 + x_2 + \ldots + x_{n-1}$ such that $t \in z + x_n$. Condition 3 in both Definitions 3.1 and 3.5, asserts that $||z|| \leq ||t|| + ||x_n||$. Thus, $||z|| \leq ||x_1|| + ||x_2|| + \ldots + ||x_{n-1}|| + ||x_n||$. ∎

Proposition 3.9. *Let V be a hypervector space, $||\cdot||$ be a (semi) hypernorm on V and $x, y \in V$. Then*

$$inf||z||_{z\in x-y} \geq |||x|| - ||y|||.$$

Proof. Suppose that $z \in x - y$. By using the definition of a hypervector space, we obtain that $x \in z + y$ and $y \in x - z$. The Triangle inequality implies that $||x|| \leq ||z|| + ||y||$ and that $||y|| \leq ||x|| + ||-z|| = ||x|| + ||z||$. We get that $||x|| - ||y|| \leq ||z||$ and that $||y|| - ||x|| \leq ||z||$. Thus, $||z|| \geq |||x|| - ||y|||$. ■

Proposition 3.10. *Let $||\cdot||$ be a semi-hypernorm on V and define N as follows:*

$$N = \{v \in V : ||v|| = 0\}.$$

Then N is a subhyperspace of V.

Proof. Let $u, v \in N$, $a, b \in K$. Then $0 \leq ||au+bv|| \leq ||au||+||bv|| = /a/||u||+/b/||v|| = 0$. We get that $||au + bv|| = 0$ and hence, $au + bv \subseteq N$. ■

Proposition 3.11. *Let $||\cdot||$ be a semi-hypernorm on V and $N = \{v \in V : ||v|| = 0\}$. Define $||\cdot||'$ on V/N as follows:*

$$||v + N||' = ||v||.$$

Then $||\cdot||'$ is a hypernorm on the quotient hypevector space V/N.

Proof. We show that the conditions of Definition 3.1 are satisfied for $||\cdot||'$. Let $u, v \in V$ and $a \in K$.

- Condition 1. Let $||v + N||' = 0$. We get that $||v|| = 0$, and hence, $v \in N$. Thus, $v + N = N$.
- Condition 2. $||a(v + N)||' = ||av + N||' = ||av|| = /a/|v|| = /a/||v + N||'$.
- Condition 3. $\{\sup ||z + N||' : z + N \in u + N + v + N\} = \{\sup ||z|| : z + n' \subseteq u + v + n$ for some $n, n' \in N\}$. The latter implies that there exists $t \in z + n'$ such that $t \in u + v + n$. We get now that $z \in t - n' \subseteq u + v + n - n'$. Since $||z|| \leq ||u|| + ||v|| + ||n|| + ||n'||$ and $||n'|| = ||n|| = 0$, it follows that $||z + N||' \leq ||u + N||' + ||v + N||'$.

Therefore, $||\cdot||'$ is a hypernorm on V/N. ■

3.2. Equivalent Hypernorms

Definition 3.12. Let V be a hypervector space and $d : V \times V \longrightarrow R_{\geq 0}$. Then (V, d) is called *hypermetric space* if for all $x, y, z \in V$, the following conditions are satisfied.

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$.

Example 3.13. Let V be any hypervector space and define $d : V \times V \longrightarrow R_{\geq 0}$ as follows:

$$d(x, y) = \begin{cases} 0, & \text{if } x = y; \\ 1, & \text{otherwise.} \end{cases}$$

Then (V, d) is a hypermetric space.

Proposition 3.14. *Let $(V, ||\cdot||)$ be a hypernormed vector space. Define d on V as follows:*

$$d(x, y) = \inf ||x - y||.$$

Then (V, d) is a hypermetric space.

Proof. Let $x, y, z \in V$. We show that conditions of Definition 3.12 are satisfied.

- Having $\overrightarrow{0} \in x - x$ for all $x \in V$ implies that $d(x, x) = \inf ||x - x|| = ||\overrightarrow{0}|| = 0$. Let $d(x, y) = \inf ||x - y|| = 0$. Then there exists $z \in x - y$ such that $||z|| = 0$. The latter implies that $z = \overrightarrow{0}$ and having $\overrightarrow{0} \in x - y$ implies that $x = y$.
- Since $\inf ||kx-ky|| = /k/ \inf ||x-y||$ for all $k \in K$ (Proposition 3.7) and $/-1/ = 1$, it follows that $d(x, y) = \inf ||x - y|| = \inf ||y - x|| = d(y, x)$.
- To prove that $d(x, z) = \inf ||x - z|| \leq d(x, y) + d(y, z) = \inf ||x - y|| + \inf ||y - z||$, we may use a proof that is similar to that of Theorem 3.4 in [5]. ∎

In what follows, the hypermetric space that we are using is that induced by the hypernorm on the hypervector space V, i.e., $d(x, y) = \inf ||x - y||$ for all $x, y \in V$.

Definition 3.15. Let (V, d) be a hypermetric space. We define the *open balls*, $B_r(x)$ in V as follows: For $x \in V, r > 0$,

$$B_r(x) = \{y \in V : d(x, y) < r\}.$$

Open subsets in V are defined to be union of open balls in V.

Example 3.16. Let K be a finite hyperfield and $(V, ||\cdot||)$ be the hypernormed vector space over K that is defined in Proposition 3.2. Let $x \in V$ and $s > 0$. Then

$$B_s(x) = \begin{cases} \{x\}, & \text{if } s \leq r; \\ V, & \text{if } s > r. \end{cases}$$

Thus, $||\cdot||$ induces the power set topology on V. (Every singleton set in V is open).

Definition 3.17. Let V be a hypernormed vector space and $||\cdot||_1, ||\cdot||_2$ be hypernorms on V. Then $||\cdot||_1$ and $||\cdot||_2$ are *equivalent hypernorms* on V if they induce same metric topology in V.

Lemma 3.18. *Let $||\cdot||$ and $||\cdot||'$ be hypernorms on V. If there are positive real numbers A, B such that $A||x|| \leq ||x||' \leq B||x||$ for all $x \in V$. Then $||\cdot||$ and $||\cdot||'$ are equivalent hypernorms on V.*

Proof. It suffices to show that open balls in $(V, ||\cdot||)$ are open subsets in $(V, ||\cdot||')$ and vice-versa. Let $x \in V$ and $r, s > 0$, $B_r(x) = \{y \in V : \inf ||x - y|| < r\}$ is an open ball in $(V, ||\cdot||)$ and $B'_s(x) = \{y \in V : \inf ||x - y||' < s\}$ is an open ball in $(V, ||\cdot||')$. We need to show that $B_r(x)$ is open in $(V, ||\cdot||')$ and that $B'_s(x)$ is open in $(V, ||\cdot||)$. Since $||x|| < \frac{r}{B}$ implies that $||x||' < r$, it follows that $\inf ||x - y|| < \frac{r}{B}$ implies that $\inf ||x - y||' < r$. Thus, any open ball around x in $(V, ||\cdot||')$ contains an open ball around x in $(V, ||\cdot||)$. We deduce that any open ball in $(V, ||\cdot||')$ is an open subset in $(V, ||\cdot||)$. In a similar manner, we prove that any open ball around x in $(V, ||\cdot||)$ contains an open ball around x in $(V, ||\cdot||')$. ∎

Lemma 3.19. *Let K be a hyperfield with non-trivial hyperabsolute value, V be a hypervector space over K and $||\cdot||, ||\cdot||'$ be equivalent hypernorms on V. Then there are positive real numbers A, B such that $A||x|| \leq ||x||' \leq B||x||$ for all $x \in V$.*

Proof. Let $||\cdot||, ||\cdot||'$ be equivalent hypernorms on V. Then $B_1(\overrightarrow{0})$ is open in $(V, ||\cdot||')$ and $B'_1(\overrightarrow{0})$ is open in $(V, ||\cdot||)$. We get that there exist $r, s > 0$ such that

$$\{v \in V : ||v||' < r\} \subseteq \{v \in V : ||v|| < 1\} = B_1(\overrightarrow{0}),$$

and

$$\{v \in V : ||v|| < s\} \subseteq \{v \in V : ||v||' < 1\} = B'_1(\overrightarrow{0}).$$

Having $/\cdot/$ a non-trivial hyperabsolute value on K implies that there exist $\gamma \in K$ with $/\gamma/ \neq 1$. If $/\gamma/ < 1$ then $/\gamma^{-1}/ = \frac{1}{/\gamma/} > 1$. Thus, there exists $\gamma \in K$ such that $/\gamma/ > 1$. One can easily see that $/\gamma/^n \longrightarrow \infty$ as $n \longrightarrow \infty$ and $/\gamma/^n \longrightarrow 0$ as $n \longrightarrow -\infty$ (as $/\gamma/ > 1$). We deduce that every positive real number is either equal to $/\gamma/^n$ for some $n \in \mathbb{N}$ or between $/\gamma/^n$ and $/\gamma/^{n+1}$ for some $n \in \mathbb{N}$. Let $v \in V$. Then there exist $n \in \mathbb{N}$ such that

$$/\gamma/^n \leq \frac{1}{s}||v|| \leq /\gamma/^{n+1}.$$

We get that $||\frac{1}{\gamma^{n+1}}v|| = \frac{1}{/\gamma^{n+1}/}||v|| < s$. Since $\{v \in V : ||v|| < s\} \subseteq \{v \in V : ||v||' < 1\}$, it follows that $||\frac{1}{\gamma^{n+1}}v||' < 1$. The latter implies that $||v||' < /\gamma^{n+1}/ = /\gamma//\gamma/^n \leq \frac{/\gamma/}{s}||v||$. Thus, $B = \frac{/\gamma/}{s}$. In a similar manner, we find $A = \frac{/\gamma/}{r}$. ∎

Theorem 3.20. *Let K be a hyperfield with non-trivial hyperabsolute value, V be a hypervector space over K. Then $||\cdot||, ||\cdot||'$ be equivalent hypernorms on V if and only if there are positive real numbers A, B such that $A||x|| \leq ||x||' \leq B||x||$ for all $x \in V$.*

Proof. The proof results from Lemmas 3.18 and 3.19. ∎

Corollary 3.21. *The relation $||\cdot||_1 \sim ||\cdot||_2 \Leftrightarrow ||\cdot||_1$ and $||\cdot||_2$ are equivalent hypernorms on V, is an equivalence relation.*

Proof. The proof is straightforward by using Theorem 3.20. ∎

4. Bounded Linear Transformations

In this section, we define bounded linear transformation over hypernormed vector spaces, use the definitions of Cauchy and convergent sequences and find the relationships between them and bounded (continuous) linear transformations.

Definition 4.1. [3] Let U, V be two hypervector spaces over a hyperfield K and $T : U \to V$. Then T is a *linear transformation* if for all $x, y \in U$ and $a \in K$: (1) $T(x + y) = T(x) + T(y)$; (2) $T(ax) = aT(x)$.

Proposition 4.2. [3] *Let U, V be two hypervector spaces over a hyperfield K and $T : U \to V$. Then T is a linear transformation if and only if $T(ax + by) = aT(x) + bT(y)$ for all $x, y \in U$ and $a, b \in K$:*

Proposition 4.3. [3] *Let $T : U \to V$ be a linear transformation. Then $ker(T) = \{x \in U : T(x) = \overrightarrow{0}\}$ is a subhyperspace of U.*

Definition 4.4. [5] Let B be a hypernormed vector space, $||\cdot||$ be a hypernorm on V and (x_n) be a sequence in V. Then (x_n) is said to converge to $x \in V$ $(x_n \longrightarrow x)$ if the following assertion holds:

For every $\epsilon > 0$, there exists a natural number N such that $\inf ||x_n - x|| < \epsilon$ for all $n \geq N$.

Definition 4.5. [5] Let B be a hypernormed vector space, $||\cdot||$ be a hypernorm on V and (x_n) be a sequence in V. Then (x_n) is said to be *Cauchy* if the following assertion holds:

For every $\epsilon > 0$, there exists a natural number N such that $\inf ||x_m - x_n|| < \epsilon$ for all $m, n \geq N$.

Example 4.6. Let V be a hypernormed vector space, $||\cdot||$ be a hypernorm on V and $x \in V$. The constant sequence (x) is convergent. This is because $d(x,x) = \inf ||x - x|| = 0 < \epsilon$.

Proposition 4.7. [5] *Let V be a hypernormed vector space, $||\cdot||$ be a hypernorm on V. Then every convergent sequence in V is Cauchy.*

Proposition 4.8. *Let V be a hypernormed vector space, $||\cdot||$ be a hypernorm on V. Then every convergent sequence in V has a unique limit.*

Proof. Let x and y be two limits for (x_n) and let $\epsilon > 0$. Then there exists a natural number N such that $\inf ||x_n - x|| < \epsilon/2$ and $\inf ||x_n - y|| < \epsilon/2$ for all $n \geq N$. Since $\inf ||x - y|| \leq \inf ||x_n - x|| + \inf ||x_n - y|| < \epsilon$, it follows that there exists $z \in x - y$ such that $||z|| = 0$. We get that $\overrightarrow{0} = z \in x - y$. Thus, $x = y$. ■

Definition 4.9. [5] Let V be a hypernormed vector space, $||\cdot||$ be a hypernorm on V. A sequence (x_n) in V is said to be bounded if there exists $M > 0$ such that $||x_n|| \leq M$ for all $n \in \mathbb{N}$.

Proposition 4.10. [5] *Let V be a hypernormed vector space, $||\cdot||$ be a hypernorm on V. Then every convergent sequence in V is bounded.*

Proposition 4.11. [5] *Let V be a hypernormed vector space, $||\cdot||$ be a hypernorm on V. Then every Cauchy sequence in V is bounded.*

We present an example of a bounded sequence in a hypernormed space that is not convergent.

Example 4.12. Let $E = \{a, b\}$ and $S = \{0, 1, 2\}$ be the hyperfield defined in Example 2.1. Let $f, g : E \longrightarrow K$ be defined by: $f(a) = 0, f(b) = 1, g(a) = 1, g(b) = 0$. Define the hypernorm $||\cdot||$ defined in Example 3.4 on $K^E = \{0, f, g, 2f, 2g, f + g, f + 2g, 2f + g, 2f + 2g\}$. Let $\{f, g, f, g, \ldots\}$ be a sequence in K^E. It is clear $\{f, g, f, g, \ldots\}$ is bounded as $||f|| = ||g|| = 1$. To prove that it is not convergent, it suffices to show that it is not Cauchy. The latter is clear as $\inf ||f - g|| = 1$.

Proposition 4.13. *Let V, W be hypervector spaces over K and $L(V, W)$ be the set of all linear transformations from V to W. Then $L(V, W)$ is a hypervector space over K.*

Proof. One can easily see that the properties of hypervector space presented in Definition 2.7 are satisfied for $L(V, W)$. ■

Definition 4.14. Let $(V, ||\cdot||_1), (W, ||\cdot||_2)$ be hypernormed vector spaces over K and $T \in L(V, W)$. Then T is a *bounded linear transformation* if there exists $M > 0$ such that $||T(x)||_2 \leq M||x||_1$ for all $x \in V$.

Proposition 4.15. *Let $(V, ||\cdot||_1), (W, ||\cdot||_2)$ be hypernormed vector spaces over K and $BL(V, W)$ be the set of all bounded linear transformations from V to W. Then $BL(V, W)$ is a subhyperspace of $L(V, W)$.*

Proof. Let $S, T \in BL(V, W)$, $x \in V$ and $a, b \in K$. Then there exist $M > 0$ such that $||T(x)||_2, ||S(x)||_2 \leq M$. We need to show that $aS + bT \in BL(V, W)$. $||(aT + bS)(x)||_2 = ||aT(x) + bS(x)||_2$. Since $(W, ||\cdot||_2)$ is a hypernormed vector space, it follows that $||(aT + bS)(x)||_2 \leq /a/||T(x)||_2 + /b/||S(x)||_2 \leq (/a/ + /b/)M$. Thus, $aS + bT$ is bounded. ∎

Definition 4.16. Let $(V, ||\cdot||_1), (W, ||\cdot||_2)$ be hypernormed vector spaces over K and $T \in BL(V, W)$. The *norm* of T is defined as follows:

$$||T|| = \sup\{\frac{||T(x)||_2}{||x||_1} : x \in V \setminus \{\overrightarrow{0}\}\}.$$

Proposition 4.17. *Let $(V, ||\cdot||_1), (W, ||\cdot||_2)$ be hypernormed vector spaces over K and $T : V \longrightarrow W$ be a bounded linear transformation. Then $||T|| = \sup\{||T(x)||_2 : ||x||_1 = 1$ and $x \in V\}$. Moreover, $||T|| = \min\{M > 0 : ||T(x)||_2 \leq M||x||_1$ for all $x \in V\}$.*

Proof. The proof is straightforward. ∎

Proposition 4.18. *Let T, S be a bounded linear transformations such that $S \circ T$ is defined. Then $||S \circ T|| \leq ||S||||T||$.*

Proof. Let $T : (U, ||\cdot||_1) \longrightarrow (V, ||\cdot||_2), T : (V||\cdot||_2) \longrightarrow (W, ||\cdot||_3)$ and $x \in U$. We have that $||S \circ T(x)||_3 = ||S(T(x))||_3 \leq ||S||||T(x)||_2 \leq ||S||||T||||x||_1$. Thus, $||S \circ T|| \leq ||S||||T||$. ∎

Definition 4.19. Let $(V, ||\cdot||_1), (W, ||\cdot||_2)$ be hypernormed vector spaces over K, $x_0 \in V$ and $T : V \longrightarrow W$. Then T is said to be *continuous* at x_0 if for all $\epsilon > 0$ there exist $\delta > 0$ such that the following holds:

$$||x - x_0|| < \delta \Longrightarrow ||T(x) - T(x_0)|| < \epsilon.$$

Theorem 4.20. *Let T be a linear transformation. Then T is bounded if and only if T is continuous at $\overrightarrow{0}$.*

Proof. If T is the zero transformation, we are done. We suppose that T is not the zero transformation so that $||T|| \neq 0$. Let T be a bounded linear transformation, $v \in V$ and $\epsilon > 0$. Take $\delta = \frac{\epsilon}{||T||} > 0$. If $||v||_1 < \delta$ then $||T(v)||_2 < \epsilon$.

Let T be continuous at $\overrightarrow{0}$. Then for every $\epsilon > 0$ there exists $\delta > 0$ such that $||v||_1 < \delta$ implies $||T(v)||_2 < \epsilon$. Let $u = \frac{\delta v}{2||v||_1}$. Then $||u||_1 = \frac{\delta}{2} < \delta$ and

$$||T(v)||_2 = ||T(\frac{\delta v}{2||v||})||_2 \frac{2||v||_1}{\delta} < \frac{2\epsilon}{\delta}||v||_1.$$

We obtain that $||T|| < \frac{2\epsilon}{\delta}$ and hence, T is bounded. ∎

Theorem 4.21. *Let T be a linear transformation. Then T is continuous if and only if T is continuous at $\overrightarrow{0}$.*

Proof. If T is continuous then T is continuous at $\overrightarrow{0}$.

Let T be continuous at $\overrightarrow{0}$, $\delta > 0$ and $y \in V$ such that $\inf ||x - y||_1 < \delta$. We have that $||T(x) - T(y)||_2 = ||T(x - y)||_2 = \{||T(z)||_2 : z \in x - y\}$. Since $\inf ||x - y||_1 < \delta$, it follows that there exist $z \in x - y$ such that $||z||_1 = \inf ||x - y||_1 < \delta$. Having T continuous at $\overrightarrow{0}$ implies that $||T(z)||_2 < \epsilon$. Thus, $\inf ||T(x) - T(y)||_2 \leq ||T(z)||_2 < \epsilon$. ∎

Theorem 4.22. *Let T be a linear transformation. Then the following are equivalent:*

(1) *T is continuous;*
(2) *T is continuous at $\overrightarrow{0}$;*
(3) *T is bounded.*

Proof. The proof results from Theorems 4.20 and 4.21. ∎

Proposition 4.23. *Let $T : V \longrightarrow W$ be a continuous linear transformation and (x_n) be a Cauchy sequence in V. Then $(T(x_n))$ is a Cauchy sequence in W.*

Proof. If T is the zero transformation, we are done. We assume that T is not the zero transformation so that $||T|| \neq 0$. Since T is a continuous linear transformation, it follows that T is bounded. Let (x_n) be a Cauchy sequence in V and $\epsilon > 0$ such that $\inf ||x_n - x_m||_1 < \frac{\epsilon}{||T||}$. We get that $\inf ||T(x_n) - T(x_m)||_2 = \inf ||T(x_n - x_m)||_2 \leq ||T|| \inf ||x_n - x_m||_1 < \epsilon$. ∎

Theorem 4.24. *Let $T : V \longrightarrow W$ be a linear transformation and (x_n) be a sequence in V. Then T is continuous if and only of the following holds:*

$$x_n \longrightarrow x \Longrightarrow T(x_n) \longrightarrow T(x).$$

Proof. Let T be a continuous linear transformation, $x \in V$, $\epsilon > 0$ and $x_n \longrightarrow x$ in V. For every $\delta > 0$, there exists $N \in \mathbb{N}$ such that $\inf ||x_n - x||_1 < \delta$ for all $n \geq N$. Since T is continuous at x, it follows that $\inf ||T(x_n) - T(x)||_2 < \epsilon$ for all $n \geq N$. Thus, $T(x_n) \longrightarrow T(x)$.

Let $y \in V$ and $x_n \longrightarrow x \Longrightarrow T(x_n) \longrightarrow T(x)$. Suppose, to get contradiction, that T is not continuous at y. Then there exists $\epsilon > 0$ such that for every $\delta > 0$, the following holds

$$\inf ||x - y||_1 < \delta \Longrightarrow \inf ||T(x) - T(y)|| > \epsilon.$$

By setting $\delta = \frac{1}{n}$, we get that

$$\inf ||x_n - y||_1 < \frac{1}{n} \Longrightarrow \inf ||T(x_n) - T(y)|| > \epsilon.$$

The latter contradicts our hypothesis. ∎

Definition 4.25. Let $T : V \longrightarrow W$ be a linear transformation and M be a subhyperspace of V. M is said to be *closed subhyperspace* if $(x_n) \subseteq M$ and $x_n \longrightarrow x$ then $x \in M$.

Proposition 4.26. *Let $T : V \longrightarrow W$ be a continuous linear transformation. Then $Ker(T)$ is a closed subhyperspace of V.*

Proof. Proposition 4.3 asserts that $Ker(T)$ is a subhyperspace of V. Let $(x_n) \subseteq Ker(T)$ such that $x_n \longrightarrow x$. Since T is continuous, it follows by Theorem 4.24 that $\overrightarrow{0} = T(x_n) \longrightarrow T(x)$. And having $\overrightarrow{0} \longrightarrow \overrightarrow{0}$, we get by using the uniqueness of limits (Proposition 4.8) that $T(x) = \overrightarrow{0}$. Thus, $x \in Ker(T)$. ∎

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