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Introduction to Intuitionistic Fuzzy b-Metric Spaces and Fixed Point Results

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Abstract The main purpose of the present paper is to introduce and study the notion of intuitionistic fuzzy b-metric spaces (shortly, IFbMS). In this way, we generalize both the notion of intuitionistic fuzzy metric spaces and fuzzy b-metric spaces. Further, the formulation and proof of intuitionistic fuzzy b-metric versions of some conventional theorems regarding fixed points via intuitionistic fuzzy sets are presented. In order to show the strength of these results, some motivating examples are established as well.

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1. Introduction

Fixed point results provide tremendous circumstances in the study of mathematical analysis under which the solutions of linear and non-linear operator equations can be approximated. The theory itself is a beautiful mixture of analysis, topology, and geometry. As a result, the theory of fixed points has been revealed as a very powerful and important tool in the study of nonlinear phenomena. In particular, fixed point techniques have been applied in such diverse fields as biology, chemistry, economics, engineering, game theory, and physics.

In 1922, the Polish mathematician Stefan Banach formulated and proved a theorem which ensured the existence and uniqueness of a fixed point in a complete metric space X of the self map f on X with contractive condition $d(fx, fy) \leq \alpha\, d(x, y)$, where $\alpha \in (0, 1)$. This result is known as Banach's fixed point theorem. In 1962 Edelstein used the compact metric space with contractive condition $d(fx, fy) < d(x, y)$ to show the existence of unique fixed point. Since contractive condition deduces the uniform continuity of an operator f, so it was a natural question to raise the concern about existence of fixed point in the absence of continuity of f. In 1968 Kannan answered this question by the introduction of

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Kannan contractive condition. Dhompongsa and Kumam [1], gave an elementary proof of the Brouwer fixed point theorem in 2019.

Fuzzy sets were introduced by Zadeh [2] in 1965 to represent/manipulate data and information possessing nonstatistical uncertainties. It was specifically designed to mathematically represent uncertainty and vagueness and to provide formalized tools for dealing with the imprecision intrinsic to many problems. In 1975 Kramosil and Michalek [3] have introduced and studied the notion of fuzzy metric space with the help of continuous t-norm, which is modified by George and Veeramani [4] in 1994 in order to generate a Hausdorff topology induced by fuzzy metric. Ljubiša D. R. Kočinac [5] defined subspaces of fuzzy metric spaces and introduced some boundedness properties in connection with fuzzy metric to investigate selection principles in these spaces (see also [6]). Abdullahi et al. [7] and Gupta et al. [8], found L-fuzzy fixed points and fixed point results in V-fuzzy metric space respectively. Gregori et al. [9], worked on fixed point theorems in extended fuzzy metrics.

The concept of b-metric was introduced by Bakhtin [10]. The class of b-metric spaces is larger than that of metric spaces. Shoaib et al [11] formulated and proved fixed point theorems for fuzzy mappings in a b-metric space. Kumam [12], Phiangsungnoen et al. [13, 14] worked on fixed point theorems for fuzzy mapping in b-metric spaces. Chaipornjareansri [15] and Konwar and Debnath [16], established and proved fixed point and coincidence point theorems for expansive mappings in partial b-metric spaces. Mukheimer [17], formulated and proved some common fixed point theorems in complex valued b-metric spaces. Recently in 2016 Nădăban [18] introduced the concept of fuzzy b-metric space and agreed that the study of operators in fuzzy b-metric spaces will obtain a lot of applications both in Mathematics as well as in Engineering and Computer Science. Many wonderful and valuable fixed point results in b-metric spaces and fuzzy metric spaces have been established and proved (see [12–14, 19–23]). Moreover, some valuable fixed point results in extended fuzzy metrics [9], V-fuzzy metric spaces [8] and parametric and fuzzy b-metric spaces [24] have been formulated and proved. Singh et al. [25], worked on n-tupled coincidence and fixed point results in partially ordered G-metric spaces. Hussain et al. [24], established fixed point results for various contractions in parametric and fuzzy b-metric spaces. In 2004, Park [26], using the idea of intuitionistic fuzzy sets, defined the notion of intuitionistic fuzzy metric spaces with the help of continuous t-norm and continuous t-conorm as a generalization of fuzzy metric space due to George and Veeramani [4, 27]. In 2006, C. Alaca, D. Turkoglu and C. Yildiz [28] extended fixed points theorems in intuitionistic fuzzy metric spaces. Konwar and Debnath [29], fomulated coincidence point results for contractions in intuitionistic fuzzy n-normed linear spaces. In this paper we have established some conventional fixed point theorems in the setting of complete intuitionistic fuzzy b- metric spaces. The structure of the paper is as follows:

After the preliminaries, in section 3, the notion of intuitionistic fuzzy b-metric spaces has been defined and this concept is explained with the help of a comprehensible example. The conceptual definitions of convergent sequence, Cauchy sequence and topology induced by an intuitionistic fuzzy b-metric space are presented as well. In section 4 we formulate and prove our main results concerning fixed point theorems of contractive mappings in IFbMS and establish some non-trivial examples to justify the validity of our results.

2. Preliminaries

For the reader's convenience, some definitions and results are recalled.

The concept of b-metric space was introduced by I. A. Bakhtin [10] and extensively used by S. Czerwic [30].

Definition 2.1 ([30]). Let X be an arbitrary non empty set and $s \geq 1$ be a given real number. A function $d : X \times X \longrightarrow [0, \infty)$ is a *b-metric on* X if, for all $x, y, z \in X$ the following conditions are satisfied:
(b_1) $d(x, y) = 0 \Longleftrightarrow x = y$;
(b_2) $d(x, y) = d(y, x)$;
(b_3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.
The triple (X, d, s) will be called *b-metric space.*

Example 1 [31]: The space $l_p(0 < p < 1)$,

$$l_p = \{(x_n) \subset R : \sum_{n=1}^{\infty} |x_n|^p < \infty\},$$

together with a function $d : l_p \times l_p \to R$

$$d(x, y) = \big(\sum_{n=1}^{\infty} |x_n - y_n|^p\big)^{\frac{1}{p}},$$

where $x = (x_n),\ y = (y_n) \in l_p$ is a b-metric space. By an elementry calculation we obtain that

$$d(x, z) \leq 2^{\frac{1}{p}}\big[d(x, y) + d(y, z)\big].$$

Here $s = 2^{\frac{1}{p}} > 1$.

Example 2 [31]: The space $L_p(0 < p < 1)$, of all real functions $x(t), t \in [0, 1]$ such that $\int_0^1 |x(t)|^p dt < \infty$, is b-metric space if we take

$$d(x, y) = \big[\int_0^1 |x(t) - y(t)|^p dt\big]^{\frac{1}{p}},$$

for each $x, y \in L_p$.

Remark: Note that a (usual) metric space is evidently a b-metric space.

However Czerwik [30, 32] has shown that a b-metric on X need not be a metric on X.

Definition 2.2 ([2]). Let X be an arbitrary non-empty set. A *fuzzy set* in X is a function with domain X and values in $[0, 1]$. If A is a fuzzy set and $x \in X$, then the function-value $A(x)$ is called the *grade of membership* of x in A. $F(X)$ stands for the collection of all fuzzy sets in X unless and until stated otherwise.

Definition 2.3 ([33]). Let X be a non-empty set. An *intuitionistic fuzzy set* is defined as:

$$A = \{x \in X : \langle\mu_A(x), \nu_A(x)\rangle\},$$

where $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ denote the degree of membership and degree of non-membership of each element x to the set A respectively such that

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1,\ \forall x \in X.$$

Definition 2.4 ([28]). A binary operation $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is called *continuous triangular norm (t-norm)* if it satisfies the following conditions:
1. $*$ is associative and commutative;
2. $*$ is continuous;
3. $a * 1 = a, \forall a \in [0,1]$;
4. if $a \leq c$ and $b \leq d$ with $a, b, c, d \in [0,1]$, then $a * b \leq c * d$.

Example: Three basic t-norms are defined as follows:
(1) The minimum t-norm, $a *_1 b = \min(a, b)$,
(2) The product t-norm, $a *_2 b = a.b$,
(3) The Lukasiewicz t-norm, $a *_3 b = \max(a + b - 1, 0)$.

Definition 2.5 ([28]). A binary operation $\diamondsuit : [0,1] \times [0,1] \longrightarrow [0,1]$ is called *continuous triangular conorm (t-conorm)* if it satisfies the following conditions:
1. $\diamondsuit$ is associative and commutative;
2. $\diamondsuit$ is continuous;
3. $a \diamondsuit 0 = a, \forall a \in [0,1]$;
4. $a \diamondsuit b \leq c \diamondsuit d$, whenever $a \leq c$ and $b \leq d$ $\forall a, b, c, d \in [0,1]$.

Example: Three basic t-conorms are given below:
(1) $a \diamondsuit_1 b = min(a + b, 1)$;
(2) $a \diamondsuit_2 b = a + b - ab$;
(3) $a \diamondsuit_3 b = max(a, b)$.

Definition 2.6 ([3]). (Kramosil and Michalek) The triple $(X, M, *)$ is said to be *fuzzy metric space* if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0, \infty)$ such that $\forall x, y, z \in X$ we have:
(M1) $M(x, y, 0) = 0$;
(M2) $M(x, y, t) = 1, \forall t > 0$ iff $x = y$;
(M3) $M(x, y, t) = M(y, x, t)$ $\forall$ $t \geq 0$;
(M4) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s), \forall$ $t, s \geq 0$;
(M5) $M(x, y, .) : [0, \infty) \longrightarrow [0, 1]$ is left continuous and $\lim_{t\to\infty} M(x, y, t) = 1$.

Lemma ([34]). *Let $(X, M, *)$ be a fuzzy metric space. Then $M(x, y, .) : [0, \infty) \to [0, 1]$ is non-decreasing, $\forall x, y \in X$.*

Example [4]: Let (X, d) be a metric space and $a*b = ab$ $(or$ $a*b = min(a, b)), \forall a, b \in [0, 1]$ and let M_d be fuzzy set on $X^2 \times [0, \infty)$, defined as follows:

$$M_d(x, y, t) = \begin{cases} \frac{t}{t+d(x,y)}, & \text{if} \quad t > 0 \\ 0, & \text{if} \quad t = 0. \end{cases}$$

This metric is called *standard fuzzy metric induced by a metric d.*

Definition 2.7 ([5]). (Subspace) If $(X, M, *)$ is a fuzzy metric space and $Y \subset X$, then $(Y, M_Y, *)$, where $M_Y = M \lceil Y^2 \times (0, \infty)$, is also a fuzzy metric space and it is called the *fuzzy metric subspace* (or shortly fm-subspace) of $(X, M, *)$.

Definition 2.8 ([18]). Let X be a nonempty set. Let $s \geq 1$ be a given real number and $*$ be a continuous t-norm. A fuzzy set M on $X \times X \times [0, \infty)$ is called *fuzzy b-metric* if, for all $x, y, z \in X$ the following conditions hold:
(bM1) $M(x, y, 0) = 0$;

(bM2) $M(x,y,t) = 1, \forall t \geq 0$ if and only if $x = y$;
(bM3) $M(x,y,t) = M(y,x,t), \forall t \geq 0$;
(bM4) $M(x,z,s(t+u)) \geq M(x,y,t) * M(y,z,u), \forall t,u \geq 0$;
(bM5) $M(x,y,.) : [0,\infty) \to [0,1]$ is left continuous and $\lim_{t\to\infty} M(x,y,t) = 1$.
The quardruple $(X, M, *, s)$ is said to be *fuzzy b-metric space.*

Remark: The class of fuzzy b-metric spaces is larger than the class of fuzzy metric spaces, since a fuzzy b-metric space is fuzzy metric space when $s = 1$.

Example [18]: Let (X,d,s) be a b-metric space and $a * b = min(a,b)$, $\forall a,b \in [0,1]$ and let M_d be a fuzzy set on $X^2 \times [0,\infty)$, defined as follows:

$$M_d(x,y,t) = \begin{cases} \frac{t}{t+d(x,y)}, & \text{if} \quad t > 0 \\ 0, & \text{if} \quad t = 0. \end{cases}$$

Then $(X, M_d, *, s)$ is standard fuzzy b-metric space.

Theorem ([18])**.** *Let $(X,M,*,s)$ be a fuzzy b-metric space. For $x \in X$, $r \in (0,1)$, $t > 0$, an open ball is defined as:*

$$B(x,r,t) = \{y \in X : M(x,y,t) > 1 - r\}.$$

Then

$$\tau_M = \{U \subset X : x \in U \quad iff \quad \exists\, t > 0,\ r \in (0,1) : B(x,r,t) \subseteq U\}$$

is a topology on X, where $P(X)$ is the power set of X.

Definition 2.9 ([18])**.** Let $s \geq 1$ be a given real number. A function $f : R \to R$ will be called *s-nondecreasing* if $t < u$ implies that $f(t) \leq f(su)$.

Proposition ([18])**.** *Let $(X,M,*,s)$ be a fuzzy b-metric space. Then $M(x,y,.) : [0,\infty) \to [0,1]$ is s-nondecreasing, $\forall x,y \in X$.*

3. Intuitionistic Fuzzy b-Metric Space

Definition 3.1. A 6-tuple $(X, M, N, *, \diamondsuit, s)$ is said to be an *intuitionistic fuzzy b-metric space (IFbMS)*, if X is an arbitrary set, $s \geq 1$ is a given real number, $*$ is a continuous t-norm, $\diamondsuit$ is a continuous t-conorm, M and N are fuzzy sets on $X^2 \times [0,\infty)$ satisfying the following conditions: For all $x,y,z \in X$,
(a) $M(x,y,t) + N(x,y,t) \leq 1$;
(b) $M(x,y,0) = 0$;
(c) $M(x,y,t) = 1, \forall\, t > 0$ iff $x = y$;
(d) $M(x,y,t) = M(y,x,t), \forall\, t > 0$;
(e) $M(x,z,s(t+u)) \geq M(x,y,t) * M(y,z,u), \forall\, t,u > 0$;
(f) $M(x,y,.) : [0,\infty) \longrightarrow [0,1]$ is left continuous and $\lim_{t\to\infty} M(x,y,t) = 1$;
(g) $N(x,y,0) = 1$;
(h) $N(x,y,t) = 0, \forall\, t > 0$ iff $x = y$;
(i) $N(x,y,t) = N(y,x,t), \forall\, t > 0$;
(j) $N(x,z,s(t+u)) \leq N(x,y,t) \diamondsuit N(y,z,u), \forall\, t,\ u > 0$;
(k) $N(x,y,.) : [0,\infty) \to [0,1]$ is right continuous and $\lim_{t\to\infty} N(x,y,t) = 0$.
Here, $M(x,y,t)$ and $N(x,y,t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t respectively.

Example 3.2. Let (X, d, s) be a b-metric space and $a * b = min(a, b)$, $a \diamondsuit b = max(a, b)$ $\forall a, b \in [0, 1]$ and let M_d , N_d be fuzzy sets on $X^2 \times [0, \infty)$, defined as follows:

$$M_d(x, y, t) = \begin{cases} \frac{t}{t+d(x,y)}, & \text{if} \quad t > 0 \\ 0, & \text{if} \quad t = 0, \end{cases}$$

and

$$N_d(x, y, t) = \begin{cases} \frac{d(x,y)}{t+d(x,y)}, & \text{if} \quad t > 0 \\ 1, & \text{if} \quad t = 0. \end{cases}$$

We check only axioms (e) and (j) of definition (3.1), because verifying the other conditions is standard. Let $x, y, z \in X$ and $t, s > 0$. Without restraining the generality we assume that

$$M_d(x, y, t) \leq M_d(y, z, u) \quad \text{and} \quad N_d(x, y, t) \geq N_d(y, z, u)$$

Thus

$$\frac{t}{t + d(x, y)} \leq \frac{u}{u + d(y, z)} \quad \text{and} \quad \frac{d(x, y)}{t + d(x, y)} \geq \frac{d(y, z)}{u + d(y, z)},$$

i.e., $td(y, z) \leq ud(x, y)$. On the other hand,

$$\begin{aligned} M_d(x, z, s(t + u)) &= \frac{s(t + u)}{s(t + u) + d(x, z)} \\ &\geq \frac{s(t + u)}{s(t + u) + s[d(x, y) + d(y, z)]} \\ &= \frac{t + u}{t + u + d(x, y) + d(y, z)}. \end{aligned}$$

Also,

$$\begin{aligned} N_d(x, z, s(t + u)) &= \frac{d(x, z)}{s(t + u) + d(x, z)} \\ &\leq \frac{s[d(x, y) + d(y, z)]}{s(t + u) + s[d(x, y) + d(y, z)]} \\ &= \frac{d(x, y) + d(y, z)}{t + u + d(x, y) + d(y, z)}. \end{aligned}$$

We will prove that

$$\frac{t + u}{t + u + d(x, y) + d(y, z)} \geq \frac{t}{t + d(x, y)}$$

and

$$\frac{d(x, y) + d(y, z)}{t + u + d(x, y) + d(y, z)} \leq \frac{d(x, y)}{t + d(x, y)}.$$

Hence we will obtain that

$$M_d(x, z, s(t + u)) \geq M_d(x, y, t) = M_d(x, y, t) * M_d(y, z, u)$$

and

$$N_d(x, z, s(t + u)) \leq N_d(x, y, t) = N_d(x, y, t) \diamondsuit N_d(y, z, u).$$

what had to be verified. We remark that

$$\frac{t+u}{t+u+d(x,y)+d(y,z)} \geq \frac{t}{t+d(x,y)}$$
$$\Leftrightarrow t^2+ut+td(x,y)+ud(x,y) \geq t^2+ut+td(x,y)+td(y,z)$$
$$\Leftrightarrow ud(x,y) \geq td(y,z),$$

which is true. Also,

$$\frac{d(x,y)+d(y,z)}{t+u+d(x,y)+d(y,z)} \leq \frac{d(x,y)}{t+d(x,y)}$$
$$\Leftrightarrow td(x,y)+td(y,z)+d(x,y)d(y,z)+(d(x,y))^2$$
$$\leq td(x,y)+ud(x,y)+d(x,y)d(y,z)+(d(x,y))^2$$
$$\Leftrightarrow td(y,z) \leq ud(x,y),$$

which is true. Hence $(X, M_d, N_d, *, \diamondsuit, s)$ is (standard) intuitionistic fuzzy b-metric space.

Definition 3.3. Let $s \geq 1$ be a given real number. A function $f : R \to R$ will be called *s-nondecreasing* if $t < u$ implies that $f(t) \leq f(su)$ and f is called *s-nonincreasing* if $t < u$ implies that $f(t) \geq f(su)$.

Proposition 3.4. *In an intuitionistic fuzzy b-metric space* $(X, M, N, *, \diamondsuit, s)$*,* $M(x, y, .) : [0, \infty) \to [0, 1]$ *is s-nondecreasing and* $N(x, y, .) : [0, \infty) \to [0, 1]$ *is s-nonincreasing for all* $x, y \in X$*.*

Proof. For $0 < t < u$, we have

$$\begin{aligned} M(x,y,su) &= M(x,y,s(u-t+t) \\ &\geq M(x,x,u-t) * M(x,y,t) \\ &= 1 * M(x,y,t) \\ &= M(x,y,t). \end{aligned}$$

Also,

$$\begin{aligned} N(x,y,su) &= N(x,y,s(u-t+t) \\ &\leq N(x,x,u-t) \diamondsuit N(x,y,t) \\ &= 0 \diamondsuit N(x,y,t) \\ &= N(x,y,t). \end{aligned}$$

∎

Definition 3.5. Let $(X, M, N, *, \diamondsuit, s)$ be an intuitionistic fuzzy b-metric space.

(a) A sequence $\{x_n\}$ in X is said to be *convergent* if there exists $x \in X$ such that $\lim_{n\to\infty} M(x_n, x, t) = 1$ and $\lim_{n\to\infty} N(x_n, x, t) = 0, \forall\ t > 0$. In this case x is called the *limit* of the sequence $\{x_n\}$ and we write $\lim_{n\to\infty} x_n = x$, or $x_n \to x$.

(b) A sequence $\{x_n\}$ in $(X, M, *, \diamondsuit, s)$ is said to be a *Cauchy sequence* if for every $\epsilon \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that

$$M(x_n, x_m, t) > 1 - \epsilon \ \ \textit{and} \ \ N(x_n, x_m, t) < \epsilon, \ \forall m, n \geq n_0 \ \textit{and} \ t > 0.$$

(c) The space X is said to be *complete* if and only if every Cauchy sequence is convergent and it is called *compact* if every sequence has a convergent subsequence.

Definition 3.6. Let $(X, M, N, *, \diamondsuit, s)$ be an intuitionistic fuzzy b-metric space. We define an *open ball* $B(x, r, t)$ with center $x \in X$ and radius r, $0 < r < 1$, $t > 0$ as

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r, N(x, y, t) < r\}.$$

Definition 3.7. Let $(X, M, N, *, \diamondsuit, s)$ be an intuitionistic fuzzy b-metric space and A be a subset of X. A is said to be *open* if, for each $x \in A$, there is an open ball B(x, r, t) contained in A.

Theorem ([26])**.** *Every open ball is an open set.*

Result: Let $(X, M, N, *, \diamondsuit, s)$ be an intuitionistic fuzzy b-metric space. Define $\tau_{M,N}$ as:

$$\tau_{M,N} = \{A \subset X : x \in A \; iff \; \exists \; t > 0 \; and \; r \in (0, 1) \; : \; B(x, r, t) \subset A\},$$

then $\tau_{M,N}$ is a *topology* on X, where $P(X)$ is the power set of X.

4. Fixed Point Theorems

Theorem 4.1. (Intuitionistic fuzzy b-metric Banach contraction theorem)
Let $(X, M, N, *, \diamondsuit, s)$ be a complete intuitionistic fuzzy b-metric space. Let $T : X \to X$ be a mapping satisfying

$$M(Tx, Ty, kt) \geq M(x, y, t), \tag{4.1}$$

$$N(Tx, Ty, kt) \leq N(x, y, t). \tag{4.2}$$

for all $x, y \in X$ where $0 < k < 1$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be an arbitrary element and let $\{x_n\}$ be a sequence in X such that, $x_n = T^n x_0$ $(n \in \mathbb{N})$. Then

$$\begin{aligned} M(x_n, x_{n+1}, kt) &= M(T^n x_0, T^{n+1} x_0, kt) \\ &\geq M(T^{n-1} x_0, T^n x_0, t) \\ &= M(x_{n-1}, x_n, t) \\ &\geq M(T^{n-2} x_0, T^{n-1} x_0, t/k) \\ &= M(x_{n-2}, x_{n-1}, t/k) \\ &... \geq M(x_0, x_1, t/k^{n-1}). \end{aligned}$$

Clearly, $1 \geq M(x_n, x_{n+1}, kt) \geq M(x_0, x_1, t/k^{n-1}) \to 1$, when $n \to \infty$. Thus

$$Lim_{n\to\infty} M(x_n, x_{n+1}, kt) = 1$$

and

$$\begin{aligned} N(x_n, x_{n+1}, kt) &= N(T^n x_0, T^{n+1} x_0, kt) \\ &\leq N(T^{n-1} x_0, T^n x_0, t) \\ &= N(x_{n-1}, x_n, t) \\ &\leq N(T^{n-2} x_0, T^{n-1} x_0, t/k) \\ &= N(x_{n-2}, x_{n-1}, t/k) \\ &... \leq N(x_0, x_1, t/k^{n-1}), \end{aligned}$$

for all n and $t > 0$. Clearly, $0 \le N(x_n, x_{n+1}, kt) \le N(x_0, x_1, t/k^{n-1}) \to 0$, when $n \to \infty$. Thus

$$Lim_{n\to\infty} N(x_n, x_{n+1}, kt) = 0.$$

Let $\tau_n(t) = M(x_n, x_{n+1}, t)$, $\mu_n(t) = N(x_n, x_{n+1}, t)$ for all $n \in \mathbb{N} \cup \{0\}, t > 0$.
Next we show that the sequence $\{x_n\}$ is a Cauchy sequence. If it is not, then there exists $0 < \epsilon < 1$ and two sequences $p(n)$ and $q(n)$ such that for every $n \in \mathbb{N} \cup \{0\}$, $t > 0$, $p(n) > q(n) \ge n$,

$$M(x_{p(n)}, x_{q(n)}, t) \le 1 - \epsilon \text{ and } N(x_{p(n)}, x_{q(n)}, t) \ge \epsilon$$

and

$$M(x_{p(n)-1}, x_{q(n)-1}, t) > 1 - \epsilon, \; M(x_{p(n)-1}, x_{q(n)}, t) > 1 - \epsilon$$

and

$$N(x_{p(n)-1}, x_{q(n)-1}, t) < \epsilon, \; N(x_{p(n)-1}, x_{q(n)}, t) < \epsilon.$$

Now,

$$\begin{aligned} 1 - \epsilon &\ge M(x_{p(n)}, x_{q(n)}, t) \\ &\ge M(x_{p(n)-1}, x_{p(n)}, t/2s) * M(x_{p(n)-1}, x_{q(n)}, t/2s) \\ &> \tau_{p(n)-1}(t/2s) * (1 - \epsilon), \end{aligned}$$

$$\begin{aligned} \epsilon &\le N(x_{p(n)}, x_{q(n)}, t) \\ &\le N(x_{p(n)-1}, x_{p(n)}, t/2s) \Diamond N(x_{p(n)-1}, x_{q(n)}, t/2s) \\ &< \mu_{p(n)-1}(t/2s) \Diamond \epsilon. \end{aligned}$$

Since $\tau_{p(n)-1}(t/2s) \to 1$ as $n \to \infty$ and $\mu_{p(n)-1}(t/2s) \to 0$ as $n \to \infty$ for every t, therefore for $n \to \infty$, we have

$$\begin{aligned} 1 - \epsilon \ge M(x_{p(n)}, x_{q(n)}, t) > 1 - \epsilon, \\ \epsilon \le N(x_{p(n)}, x_{q(n)}, t) < \epsilon. \end{aligned}$$

Clearly, this leads to a contradiction. Hence x_n is a Cauchy sequence in X. Since X is complete so there exist a point y in X such that

$$\lim_{n\to\infty} x_n = y.$$

Now,

$$\begin{aligned} M(y, Ty, t) &\ge M(y, x_{n+1}, t/2s) * M(x_{n+1}, Ty, t/2s) \\ &= M(y, x_{n+1}, t/2s) * M(Tx_n, Ty, t/2s) \\ &\ge M(y, x_{n+1}, t/2s) * M(x_n, y, t/2sk). \end{aligned}$$

The case when $n \to \infty$, we have

$$M(y, Ty, t) \ge 1 * 1 = 1$$

and

$$\begin{aligned} N(y,Ty,t) &\leq N(y,x_{n+1},t/2s)\diamondsuit N(x_{n+1},Ty,t/2s) \\ &= N(y,x_{n+1},t/2s)\diamondsuit N(Tx_n,Ty,t/2s) \\ &\leq N(y,x_{n+1},t/2s)\diamondsuit N(x_n,y,t/2sk). \end{aligned}$$

On $n \to \infty$, we have

$$N(y,Ty,t) \leq 0\diamondsuit 0 = 0.$$

By (c) and (h)of definition (3.1), we have,

$$y = Ty.$$

For uniqueness of fixed point, let y, z be two fixed points of the mapping T, then, $y = Ty$ and $z = Tz$ and

$$\begin{aligned} 1 \geq M(y,z,t) &= M(Ty,Tz,t) \\ &\geq M(y,z,t/k) \\ &= M(Ty,Tz,t/k) \\ &\geq M(y,z,t/k^2) \\ &\geq ... \\ &\geq M(y,z,t/k^n) \to 1, as\ n \to \infty. \end{aligned}$$

Also,

$$\begin{aligned} 0 \leq N(y,z,t) &= N(Ty,Tz,t) \\ &\leq N(y,z,t/k) \\ &= N(Ty,Tz,t/k) \\ &\leq N(y,z,t/k^2) \\ &\leq ... \\ &\leq N(y,z,t/k^n) \to 0, as\ n \to \infty. \end{aligned}$$

By (c) and(h) of definition (3.1),

$$y = z.$$

■

Corollary. *Let (X,d) be a complete metric space and $T : X \to X$ be a map which satisfies the following condition for all $x, y \in X$ and $0 < k < 1$:*

$$d(Tx,Ty) \leq kd(x,y) \tag{4.3}$$

Then T has unique fixed point in X.

Proof. We consider the corresponding intuitionistic fuzzy b-metric space $(X, M, N, *, \diamondsuit, s)$ where

$$M(x,y,t) = \frac{t}{t + d(x,y)} \quad \text{and} \quad N(x,y,t) = \frac{d(x,y)}{t + d(x,y)}.$$

(4.3) $\Rightarrow$ (4.2)

$$d(Tx,Ty) \leq kd(x,y)$$

$$\frac{d(Tx,Ty)}{k} \leq d(x,y). \tag{4.4}$$

Note that for $a,b,c,d \geq 0$, if $\frac{a}{b} \leq \frac{c}{d}$ then $\frac{a}{b+a} \leq \frac{c}{d+c}$. It follows that,

$$\frac{d(Tx,Ty)}{kt + d(Tx,Ty)} \leq \frac{d(x,y)}{t + d(x,y)}.$$

Hence $N(Tx,Ty,kt) \leq N(x,y,t)$.
(4.3) $\Rightarrow$ (4.1) From ineq. (4.4)

$$\frac{k}{d(Tx,Ty)} \geq \frac{1}{d(x,y)} \quad \Rightarrow \quad \frac{kt}{kt + d(Tx,Ty)} \geq \frac{t}{t + d(x,y)}$$

Hence $M(Tx,Ty,kt) \geq M(x,y,t)$. ■

In support of above theorem we furnish the following example.

Example 1: Let $X = [0,1]$ and $M,N : X^2 \times [0,\infty) \to [0,1]$ be fuzzy sets on $X^2 \times [0,\infty)$. For all $x,y \in X$ and $t \in [0,\infty)$, define

$$M(x,y,t) = \begin{cases} \frac{t}{t+|x-y|}, & \text{if} \quad t > 0 \\ 0, & \text{if} \quad t = 0 \end{cases}$$

and

$$N(x,y,t) = \begin{cases} \frac{|x-y|}{t+|x-y|}, & \text{if} \quad t > 0 \\ 1, & \text{if} \quad t = 0 \end{cases}$$

Clearly, $(X,M,N,*,\diamondsuit,s)$ is a complete intuitionistic fuzzy b-metric space, where $a * b = min(a,b)$, $a\diamondsuit b = max(a,b)$ $\forall a,b \in [0,1]$. Let $T : X \to X$ be such that

$$Tx = \frac{x}{8}.$$

Then for $k = \frac{1}{4}$,

$$\begin{aligned} M(Tx,Ty,\frac{t}{4}) = \frac{\frac{t}{4}}{\frac{t}{4} + |Tx - Ty|} &= \frac{\frac{t}{4}}{\frac{t}{4} + \frac{|x-y|}{8}} \\ &= \frac{t}{t + 4(\frac{|x-y|}{8})} \\ &\geq \frac{t}{t + |x-y|} \\ &= M(x,y,t) \end{aligned}$$

and

$$\begin{aligned} N(Tx,Ty,\frac{t}{4}) = \frac{|Tx - Ty|}{\frac{t}{4} + |Tx - Ty|} &= \frac{\frac{|x-y|}{8}}{\frac{t}{4} + \frac{|x-y|}{8}} \\ &= \frac{\frac{1}{2}(|x-y|)}{t + \frac{1}{2}(|x-y|)} \\ &\leq \frac{|x-y|}{t + |x-y|} \\ &= N(x,y,t). \end{aligned}$$

Hence T satisfies the contractive condition of Theorem 4.1 to obtain a fixed point.

Theorem 4.2. *Let $(X, M, N, *, \diamondsuit, s)$ be a complete intuitionistic fuzzy b-metric space with $*$ t-norm and $\diamondsuit$ conorm defined as $a * b = min\{a, b\}$ and $a\diamondsuit b = max\{a, b\}$ respectively. Also suppose that $M(x, y, .)$ is strictly increasing and $N(x, y, .)$ is strictly decreasing respectively. Let $A : X \to X$ be a self map which satisfies the following conditions for all $x, y \in X$*

$$M(Ax, Ay, kt) \geq M(x, Ax, t) * M(y, Ay, t) \tag{4.5}$$

$$N(Ax, Ay, kt) \leq N(x, Ax, t)\diamondsuit N(y, Ay, t), \tag{4.6}$$

where $t > 0$, $0 < k < 1$. Then A has a unique fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point. Consider a sequence $x_n = Ax_{n-1}$ of points in X. Then

$$\begin{aligned} M(x_n, x_{n+1}, kt) &= M(Ax_{n-1}, Ax_n, kt) \\ &\geq M(x_{n-1}, Ax_{n-1}, t) * M(x_n, Ax_n, t) \\ &= M(x_{n-1}, x_n, t) * M(x_n, x_{n+1}, t), \end{aligned}$$

Since $M(x, y, .)$ is strictly increasing function, $kt < t$ and if

$$min\{M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t)\} = M(x_n, x_{n+1}, t),$$

then we reach to a contradiction

$$M(x_n, x_{n+1}, kt) \geq M(x_n, x_{n+1}, t).$$

Therefore,

$$\begin{aligned} M(x_n, x_{n+1}, kt) &\geq M(x_{n-1}, x_n, t) \\ &= M(Ax_{n-2}, Ax_{n-1}, t) \\ &\geq M(x_{n-1}, Ax_{n-1}, t/k) * M(x_{n-2}, Ax_{n-2}, t/k) \\ &= M(x_{n-1}, x_n, t/k) * M(x_{n-2}, x_{n-1}, t/k) \\ &= M(x_{n-2}, x_{n-1}, t/k) \\ &... \\ &\geq M(x_0, x_1, t/k^{n-1}). \end{aligned}$$

Clearly, $1 \geq M(x_n, x_{n+1}, kt) \geq M(x_0, x_1, t/k^{n-1}) \to 1$, when $n \to \infty$. Thus

$$Lim_{n\to\infty} M(x_n, x_{n+1}, kt) = 1.$$

Now,

$$\begin{aligned} N(x_n, x_{n+1}, kt) &= N(Ax_{n-1}, Ax_n, kt) \\ &\leq N(x_{n-1}, Ax_{n-1}, t)\diamondsuit N(x_n, Ax_n, t) \\ &= N(x_{n-1}, x_n, t)\diamondsuit N(x_n, x_{n+1}, t). \end{aligned}$$

Since $N(x,y,.)$ is strictly decreasing function, $kt<t$, by the same arguement $N(x_n,x_{n+1},kt)\le N(x_n,x_{n+1},t)$ is not possible. Therefore,

$$\begin{aligned}N(x_n,x_{n+1},kt) &\le N(x_{n-1},x_n,t)\\ &= N(Ax_{n-2},Ax_{n-1},t)\\ &\le N(x_{n-1},Ax_{n-1},t/k)\diamondsuit N(x_{n-2},Ax_{n-2},t/k)\\ &= N(x_{n-1},x_n,t/k)\diamondsuit N(x_{n-2},x_{n-1},t/k)\\ &= N(x_{n-2},x_{n-1},t/k)\\ &\dots\\ &\le N(x_0,x_1,t/k^{n-1}).\end{aligned}$$

Clearly, $0\le N(x_n,x_{n+1},kt)\le N(x_0,x_1,t/k^{n-1})\to 0$, when $n\to\infty$. Hence

$$Lim_{n\to\infty}N(x_n,x_{n+1},kt)=0.$$

Let $\tau_n(t)=M(x_n,x_{n+1},t)$ and $\mu_n(t)=N(x_n,x_{n+1},t)$ for all $n\in\mathbb{N}\cup\{0\}$, $t>0$. Clearly, $\lim_{t\to\infty}\tau_n(t)=1$, and $\lim_{t\to\infty}\mu_n(t)=0$. Next, we show that the sequence $\{x_n\}$ is a Cauchy sequence. If it is not, then there exists $0<\epsilon<1$ and two sequences $\{p(n)\}$ and $\{q(n)\}$ such that for every $n\in\mathbb{N}\cup\{0\}$, $t>0$, $p(n)>q(n)\ge n$, $M(x_{p(n)},x_{q(n)},t)\le 1-\epsilon$ and $N(x_{p(n)},x_{q(n)},t)\ge\epsilon$ and

$$M(x_{p(n)-1},x_{q(n)-1},t)>1-\epsilon, M(x_{p(n)-1},x_{q(n)},t)>1-\epsilon$$

and

$$N(x_{p(n)-1},x_{q(n)-1},t)<\epsilon, N(x_{p(n)-1},x_{q(n)},t)<\epsilon.$$

Now,

$$\begin{aligned}1-\epsilon &\ge M(x_{p(n)},x_{q(n)},t)\\ &\ge M(x_{p(n)-1},x_{p(n)},t/2s)*M(x_{p(n)-1},x_{q(n)},t/2s)\\ &> \tau_{p(n)-1}(t/2s)*(1-\epsilon)\end{aligned}$$

and

$$\begin{aligned}\epsilon &\le N(x_{p(n)},x_{q(n)},t)\\ &\le N(x_{p(n)-1},x_{p(n)},t/2s)\diamondsuit N(x_{p(n)-1},x_{q(n)},t/2s)\\ &< \mu_{p(n)-1}(t/2s)\diamondsuit\epsilon.\end{aligned}$$

Since $\tau_{p(n)-1}(t/2s)\to 1$ as $n\to\infty$ and $\mu_{p(n)-1}(t/2s)\to 0$ as $n\to\infty$ for every t. It follows that

$$1-\epsilon\ge M(x_{p(n)},x_{q(n)},t)>1-\epsilon$$

and

$$\epsilon\le N(x_{p(n)},x_{q(n)},t)<\epsilon.$$

Clearly, this leads to a contradiction. Hence x_n is a cauchy sequence in X. Since X is complete so there exists $y\in X$ such that

$$\lim_{n\to\infty}x_n=y.$$

Assume that $y\ne Ay$, then there exists $t>0$ such that $M(y,Ay,t)\ne 1$ or $N(y,Ay,t)\ne 0$. For this $t>0$,

$$M(Ax_n,Ay,kt)\ge M(x_n,Ax_n,t)*M(y,Ay,t),$$

by contractive condition (4.5). That is

$$M(x_{n+1}, Ay, kt) \geq M(x_n, x_{n+1}, t) * M(y, Ay, t).$$

In limiting case as $n \to \infty$,

$$M(y, Ay, kt) \geq M(y, Ay, t).$$

As $M(y, Ay, t) \neq 1$, the above inequality yields a contradiction to the fact that $M(x, y, .)$ is strictly increasing. Moreover,

$$N(Ax_n, Ay, kt) \leq N(x_n, Ax_n, t) \diamondsuit N(y, Ay, t),$$

by contractive condition (4.6). That is,

$$N(x_{n+1}, Ay, kt) \leq N(x_n, x_{n+1}, t) \diamondsuit N(y, Ay, t).$$

In limiting case as $n \to \infty$,

$$N(y, Ay, kt) \leq N(y, Ay, t).$$

As $N(y, Ay, t) \neq 0$, the above inequality yields a contradiction to the fact that $N(x, y, .)$ is strictly decreasing. Hence,

$$y = Ay.$$

For uniqueness, let y *and* z be two fixed points of A. So, $y = Ay$ and $z = Az$. Then

$$M(y, Ay, t) = 1, M(z, Az, t) = 1$$

and

$$N(y, Ay, t) = 0, N(z, Az, t) = 0; \forall\ t > 0.$$

Now,

$$\begin{aligned} 1 \geq M(y, z, t) = M(Ay, Az, t) &\geq M(y, Ay, t/k) * M(z, Az, t/k) \\ &= 1 * 1 = 1, \end{aligned}$$

$$\begin{aligned} 0 \leq N(y, z, t) = N(Ay, Az, t) &\leq N(y, Ay, t/k) \diamondsuit N(z, Az, t/k) \\ &= 0 \diamondsuit 0 = 0. \end{aligned}$$

From (c) and (h) of definition 3.1, we have

$$z = y.$$

■

Corollary. *Let* (X, d) *be a complete metric space and* $T : X \to X$ *be a map which satisfies the following condition for all* $x, y \in X$ *and* $0 < k < 1$*:*

$$d(Tx, Ty) \leq \frac{k}{2}[d(x, Tx) + d(y, Ty)] \tag{4.7}$$

Then T *has unique fixed point in* X*.*

Proof. We consider the corresponding intuitionistic fuzzy b-metric space $(X, M, N, *, \diamondsuit, s)$ where,

$$M(x, y, t) = \begin{cases} \frac{t}{t+d(x,y)}, & \text{if} \quad t > 0 \\ 0, & \text{if} \quad t = 0 \end{cases}$$

and

$$N(x,y,t) = \begin{cases} \frac{d(x,y)}{t+d(x,y)}, & \text{if} \quad t > 0 \\ 1, & \text{if} \quad t = 0. \end{cases}$$

Replacing A with T in inequalities (4.5) and (4.6).
Now, (4.7) $\Rightarrow$ (4.5). If otherwise, then from (4.5), for some $t > 0$,

$$M(Tx,Ty,kt) < min\{M(x,Tx,t), M(y,Ty,t)\},$$

i.e.,

$$\frac{t}{t + \frac{1}{k}d(Tx,Ty)} < min\{\frac{t}{t + d(x,Tx)}, \frac{t}{t + d(y,Ty)}\}$$

This implies that

$$t + \frac{1}{k}d(Tx,Ty) > t + d(x,Tx)$$

and

$$t + \frac{1}{k}d(Tx,Ty) > t + d(y,Ty)$$

$$\Rightarrow \quad \frac{2}{k}d(Tx,Ty) > [d(x,Tx) + d(y,Ty)]$$

or

$$d(Tx,Ty) > \frac{k}{2}[d(x,Tx) + d(y,Ty)]$$

which is contradiction to (4.7).
Now (4.7) $\Rightarrow$ (4.6)

$$\begin{aligned} d(Tx,Ty) \leq \frac{k}{2}\{d(x,Tx) + d(y,Ty)\} &\leq \frac{2k}{2}max\{d(x,Tx), d(y,Ty)\} \\ &\leq kmax\{d(x,Tx), d(y,Ty)\} \end{aligned}$$

$$\frac{d(Tx,Ty)}{k} \leq max\{d(x,Tx), d(y,Ty)\}.$$

Without loss of generality, we assume that $max\{d(x,Tx), d(y,Ty)\} = d(x,Tx)$. This implies that

$$\frac{d(Tx,Ty)}{k} \leq d(x,Tx)$$

so

$$\frac{d(Tx,Ty)}{kt} \leq \frac{d(x,Tx)}{t}$$

and

$$\frac{d(Tx,Ty)}{kt + d(Tx,Ty)} \leq \frac{d(x,Tx)}{t + d(x,Tx)}$$

Then,

$$\frac{d(Tx,Ty)}{kt + d(Tx,Ty)} \leq max\{\frac{d(x,Tx)}{t + d(x,Tx)}, \frac{d(y,Ty)}{t + d(y,Ty)}\}.$$

Hence

$$N(Tx,Ty,kt) \leq max\{N(x,Tx,t), N(y,Ty,t)\}.$$

■

In support of theorem 4.2 we establish an example.

Example 2: Let $X=[0,1]$ and $M,N : X^2\times[0,\infty)\to[0,1]$ be fuzzy sets on $X^2\times[0,\infty)$. For all $x,y\in X$ and $t\in[0,\infty)$, define

$$M(x,y,t)=\begin{cases}\frac{t}{t+|x-y|}, & \text{if} \quad t>0\\ 0, & \text{if} \quad t=0\end{cases}$$

and

$$N(x,y,t)=\begin{cases}\frac{|x-y|}{t+|x-y|}, & \text{if} \quad t>0\\ 1, & \text{if} \quad t=0.\end{cases}$$

Clearly, $(X,M,N,*,\diamondsuit,s)$ is a complete intuitionistic fuzzy b-metric space, where $a*b=min(a,b)$, $a\diamondsuit b=max(a,b)$ $\forall a,b\in[0,1]$. Let $T:X\to X$ be such that

$$Tx=\tfrac{x}{30}.$$

Then for $k=\frac{2}{3}$,

$$M(Tx,Ty,\frac{2t}{3})=\frac{\frac{2t}{3}}{\frac{2t}{3}+|Tx-Ty|}=\frac{\frac{2t}{3}}{\frac{2t}{3}+(|x-y|)/30}$$
$$=\frac{t}{t+\frac{3}{2}(\frac{|x-y|}{30})}.$$

Now, as $x,y\in[0,1]$

$$|\frac{x-y}{30}|\leq|\frac{x+y}{30}|\leq\frac{1}{3}|\frac{3x}{30}+\frac{3y}{30}|$$
$$\leq\frac{1}{3}|\frac{29x}{30}+\frac{29y}{30}|$$
$$\leq\frac{1}{3}(|\frac{29x}{30}|+|\frac{29y}{30}|)$$

$$\Rightarrow 2(\frac{3}{2})|\frac{x-y}{30}|\leq|\frac{29x}{30}|+|\frac{29y}{30}|. \tag{4.8}$$

Note that if $a,\ b,\ c\geq0, 2a\leq b+c$, then $a\leq Max\{b,c\}$. Otherwise $2a>b+c$. It follows that

$$\frac{3}{2}|\frac{x-y}{30}|\leq|\frac{29x}{30}| \quad \text{or} \quad \frac{3}{2}|\frac{x-y}{30}|\leq|\frac{29y}{30}|.$$

Without loss of generality assume that $x\geq y$, then

$$min\{\frac{t}{t+\frac{29x}{30}},\frac{t}{t+\frac{29y}{30}}\}=\frac{t}{t+\frac{29x}{30}}$$

and

$$max\{\frac{29x}{30},\frac{29y}{30}\}=\frac{29x}{30}$$
$$\Rightarrow\frac{3}{2}|\frac{x-y}{30}|\leq\frac{29x}{30}$$
$$\Rightarrow\frac{1}{\frac{3}{2}|\frac{x-y}{30}|}\geq\frac{1}{\frac{29x}{30}}$$

$$\Rightarrow \frac{t}{t+\frac{3}{2}|\frac{x-y}{30}|} \geq \frac{t}{t+\frac{29x}{30}}$$

$$\Rightarrow \frac{\frac{2t}{3}}{\frac{2t}{3}+|Tx-Ty|} \geq min\{\frac{t}{t+|x-Tx|}, \frac{t}{t+|y-Ty|}\}$$

$$\Rightarrow M(Tx,Ty,kt) \geq M(x,Tx,t) * M(y,Ty,t).$$

Moreover,

$$N(Tx,Ty,kt) = \frac{|\frac{x-y}{30}|}{kt+|\frac{x-y}{30}|}$$

From inequality (4.8),

$$|\frac{x-y}{30}| \leq \frac{2}{3}max\{\frac{29x}{30}, \frac{29y}{30}\}$$

As it is assumed that $x \geq y$, therefore,

$$|\frac{x-y}{30}| \leq \frac{2}{3}(\frac{29x}{30})$$

$$\frac{3}{2}(|\frac{x-y}{30}|) \leq \frac{29x}{30}$$

$$1+\frac{t}{\frac{3}{2}(|\frac{x-y}{30}|)} \geq 1+\frac{t}{\frac{29x}{30}}$$

$$\frac{\frac{3}{2}(|\frac{x-y}{30}|)+t}{\frac{3}{2}(|\frac{x-y}{30}|)} \geq \frac{\frac{29x}{30}+t}{\frac{29x}{30}}$$

$$\frac{\frac{2t}{3}+|Tx-Ty|}{|Tx-Ty|} \geq \frac{t+|x-Tx|}{|x-Tx|}$$

$$\frac{|Tx-Ty|}{\frac{2t}{3}+|Tx-Ty|} \leq \frac{|x-Tx|}{t+|x-Tx|}$$

$$N(Tx,Ty,kt) \leq N(x,Tx,t)\diamondsuit N(y,Ty,t).$$

Hence T satisfies the contractive conditions of Theorem (4.2) to obtain a fixed point.

Theorem 4.3. *Let $(X, M, N, *, \diamondsuit, s)$ be a complete intuitionistic fuzzy b-metric space with $*$ t-norm and $\diamondsuit$ conorm defined as $a*b = min\{a,b\}$, $a\diamondsuit b = max\{a,b\}$, $M(x,y,.)$ and $N(x,y,.)$ are strictly increasing and strictly decreasing functions respectively. Let $A: X \to X$ be a self mapping on X. If for all $x,y \in X$, $0 < k < 1/2s$, A satisfies the following conditions:*

$$M(Ax,Ay,kt) \geq M(x,Ay,t) * M(y,Ax,t) \tag{4.9}$$

$$N(Ax,Ay,kt) \leq N(x,Ay,t)\diamondsuit N(y,Ax,t), \tag{4.10}$$

where $t > 0$. Then A has a unique fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point, such that $x_n = Ax_{n-1}$ is a sequence in X.

$$\begin{aligned} M(x_n, x_{n+1}, kt) &= M(Ax_{n-1}, Ax_n, kt) \\ &\geq M(x_{n-1}, Ax_n, t) * M(x_n, Ax_{n-1}, t) \\ &= M(x_{n-1}, x_{n+1}, t) * M(x_n, x_n, t). \end{aligned}$$

Since $M(x_n, x_n, t) = 1$. So,

$$M(x_n, x_{n+1}, kt) \geq M(x_{n-1}, x_{n+1}, t).$$

By using (e) of definition (3.1), we have

$$M(x_n, x_{n+1}, kt) \geq M(x_{n-1}, x_n, t/2s) * M(x_n, x_{n+1}, t/2s).$$

Since $M(x, y, .)$ is strictly increasing function and $kt < t/2s$. If

$$min\{M(x_{n-1}, x_n, t/2s), M(x_n, x_{n+1}, t/2s)\} = M(x_n, x_{n+1}, t/2s)$$

then we reach to a contradiction

$$M(x_n, x_{n+1}, kt) \geq M(x_n, x_{n+1}, t/2s).$$

Therefore,

$$M(x_n, x_{n+1}, kt) \geq M(x_{n-1}, x_n, t/2s),$$

continuing this process, we have

$$M(x_n, x_{n+1}, kt) \geq M(x_0, x_1, t/(2s)^n k^{n-1}).$$

Clearly, $1 \geq M(x_n, x_{n+1}, kt) \geq M(x_0, x_1, t/(2s)^n k^{n-1}) \to 1$, when $n \to \infty$. Thus,

$$Lim_{n\to\infty} M(x_n, x_{n+1}, kt) = 1.$$

Moreover,

$$\begin{aligned} N(x_n, x_{n+1}, kt) &= N(Ax_{n-1}, Ax_n, kt) \\ &\leq N(x_{n-1}, Ax_n, t) \diamondsuit N(x_n, Ax_{n-1}, t) \\ &= N(x_{n-1}, x_{n+1}, t) \diamondsuit N(x_n, x_n, t). \end{aligned}$$

Since $N(x_n, x_n, t) = 0$ So,

$$N(x_n, x_{n+1}, kt) \leq N(x_{n-1}, x_{n+1}, t).$$

By using (j) of definition (3.1), we have

$$N(x_n, x_{n+1}, kt) \leq N(x_{n-1}, x_n, t/2s) \diamondsuit N(x_n, x_{n+1}, t/2s).$$

Since $N(x, y, .)$ is strictly decreasing function, $kt < t/2s$, by the same arguement

$$N(x_n, x_{n+1}, kt) \leq N(x_n, x_{n+1}, t/2s)$$

is not possible, Therefore,

$$N(x_n, x_{n+1}, kt) \leq N(x_{n-1}, x_n, t/2s),$$

continuing this process, we have

$$N(x_n, x_{n+1}, kt) \leq N(x_0, x_1, t/(2s)^n k^{n-1}).$$

Clearly, $0 \leq N(x_n, x_{n+1}, kt) \leq N(x_0, x_1, t/(2s)^n k^{n-1}) \to 0$, when $n \to \infty$. Thus,

$$Lim_{n\to\infty} N(x_n, x_{n+1}, kt) = 0.$$

Let $\tau_n(t) = M(x_n, x_{n+1}, t)$ and $\mu_n(t) = N(x_n, x_{n+1}, t)$ for all $n \in \mathbb{N} \cup \{0\}$ and $t > 0$. Clearly, $\lim_{t\to\infty} \tau_n(t) = 1$, and $\lim_{t\to\infty} \mu_n(t) = 0$. Next, we show that the sequence $\{x_n\}$ is a Cauchy sequence. If it is not. Then there exists $0 < \epsilon < 1$ and two sequences $p(n)$ and $q(n)$ such that for every $n \in \mathbb{N} \cup \{0\}$, $t > 0$, $p(n) > q(n) \geq n$,

$$M(x_{p(n)}, x_{q(n)}, t) \leq 1 - \epsilon \text{ and } N(x_{p(n)}, x_{q(n)}, t) \geq \epsilon$$

and

$$M(x_{p(n)-1}, x_{q(n)-1}, t) > 1 - \epsilon,\ M(x_{p(n)-1}, x_{q(n)}, t) > 1 - \epsilon$$

and

$$N(x_{p(n)-1}, x_{q(n)-1}, t) < \epsilon,\ N(x_{p(n)-1}, x_{q(n)}, t) < \epsilon.$$

Now,

$$\begin{aligned} 1 - \epsilon &\geq M(x_{p(n)}, x_{q(n)}, t) \\ &\geq M(x_{p(n)-1}, x_{p(n)}, t/2s) * M(x_{p(n)-1}, x_{q(n)}, t/2s) \\ &> \tau_{p(n)-1}(t/2s) * (1 - \epsilon) \end{aligned}$$

and

$$\begin{aligned} \epsilon &\leq N(x_{p(n)}, x_{q(n)}, t) \\ &\leq N(x_{p(n)-1}, x_{p(n)}, t/2s) \diamondsuit N(x_{p(n)-1}, x_{q(n)}, t/2s) \\ &< \mu_{p(n)-1}(t/2s) \diamondsuit \epsilon. \end{aligned}$$

Since $\tau_{p(n)-1}(t/2s) \to 1$ as $n \to \infty$ and $\mu_{p(n)-1}(t/2s) \to 0$ as $n \to \infty$ for every t, it follows that,

$$\begin{gathered} 1 - \epsilon \geq M(x_{p(n)}, x_{q(n)}, t) > 1 - \epsilon, \\ \epsilon \leq N(x_{p(n)}, x_{q(n)}, t) < \epsilon. \end{gathered}$$

Clearly, this leads to a contradiction. Hence x_n is a cauchy sequence in X. Since X is complete so there exist $y \in X$ such that

$$\lim_{n\to\infty} x_n = y.$$

Assume that $y \neq Ay$, then there exists $t > 0$ such that $M(y, Ay, t) \neq 1$ or $N(y, Ay, t) \neq 0$. For this $t > 0$,

$$M(Ax_n, Ay, kt) \geq M(x_n, Ay, t) * M(y, Ax_n, t),$$

by ineq. (4.9). That is

$$M(x_{n+1}, Ay, kt) \geq M(x_n, Ay, t) * M(y, Ax_n, t).$$

In limiting case as $n \to \infty$,

$$M(y, Ay, kt) \geq M(y, Ay, t) * M(y, Ay, t) = M(y, Ay, t).$$

As $M(y, Ay, t) \neq 1$, the above inequality yields a contradiction to the fact that $M(x, y, .)$ is strictly increasing. Moreover,

$$N(Ax_n, Ay, kt) \leq N(x_n, Ay, t) \diamondsuit N(y, Ax_n, t),$$

by ineq. (4.10). That is

$$N(x_{n+1}, Ay, kt) \leq N(x_n, Ay, t) \diamondsuit N(y, Ax_n, t).$$

In limiting case as $n \to \infty$,

$$N(y, Ay, kt) \leq N(y, Ay, t) \diamondsuit N(y, Ay, t) = N(y, Ay, t).$$

As $N(y, Ay, t) \neq 0$, the above inequality yields a contradiction to the fact that $N(x, y, .)$ is strictly deccreasing. Hence

$$y = Ay.$$

For uniqueness, let y, z be two fixed points of A. So, $y = Ay$ and $z = Az$. Then

$$M(y, Ay, t) = 1,\ M(z, Az, t) = 1$$

and

$$N(y, Ay, t) = 0,\ N(z, Az, t) = 0,\ \forall\ t > 0.$$

Now,

$$\begin{aligned}
1 \geq M(y, z, t) = M(Ay, Az, t) &\geq M(y, Az, t/k) * M(z, Ay, t/k) \\
&= M(y, z, t/k) * M(z, y, t/k) \\
&= M(y, z, t/k) \\
&= M(Ay, Az, t/k) \\
&\geq M(y, Az, t/k^2) * M(z, Ay, t/k^2) \\
&\geq ... \\
&\geq M(y, z, t/k^n) \to 1, as\ n \to \infty.
\end{aligned}$$

and

$$\begin{aligned}
0 \leq N(y, z, t) = N(Ay, Az, t) &\leq N(y, Az, t/k) \Diamond N(z, Ay, t/k) \\
&= N(y, z, t/k) \Diamond N(z, y, t/k) \\
&= N(y, z, t/k) \\
&= N(Ay, Az, t/k) \\
&\leq N(y, Az, t/k^2) \Diamond N(z, Ay, t/k^2) \\
&\leq ... \\
&\leq N(y, z, t/k^n) \to 0, as\ n \to \infty.
\end{aligned}$$

Now from (c) and (h) of definition (3.1), we have

$$z = y.$$

■

Corollary. *Let (X, d) be a complete metric space and $A : X \to X$ be a map which satisfies the following condition for all $x, y \in X$ and $0 < k < 1$:*

$$d(Ax, Ay) \leq \frac{k}{2}[d(x, Ay) + d(y, Ax)] \tag{4.11}$$

Then A has unique fixed point in X.

In the following, Zamfirescu [35] type result in a fuzzy b-metric space has been established. A special case (Banach contraction theorem) of this immenent result has been desired by Nădăban [18]. We believe that it is right to say in [18] that this kind of work may be of interest for researchers working in the fields belonging to computer science and information technology, communications, computational intelligence methods and advanced decision support systems.

Theorem 4.4. *Let $(X, M, *, s)$ be a complete fuzzy b-metric space. Let $T : X \to X$ be a mapping on X. If for $x, y \in X$ and $t > 0$, any one of the following is satisfied:*

(i) $M(Tx, Ty, kt) \geq M(x, y, t)$ *for $0 < k < 1$ and $*$ is any continuous t-norm;*

(ii) $M(Tx, Ty, kt) \geq M(x, Tx, t) * M(y, Ty, t)$ *for $0 < k < 1$, $t > 0$, $a * b = min\{a, b\}, \forall a, b \in [0, 1]$ and $M(x, y, .)$ is strictly increasing function;*

(iii) $M(Tx, Ty, kt) \geq M(x, Ty, t) * M(y, Tx, t)$ *for $0 < k < 1/2s$, $t > 0$, $a * b = min\{a, b\}, \forall a, b \in [0, 1]$ and $M(x, y, .)$ is strictly increasing function;*

Then T has a unique fixed point in X.

Proof. (i),(ii) and (iii) are respectively special cases of Theorems 4.1, 4.2 and 4.3. ∎

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References

[1] S. Dhompongsa, P. Kumam, An elementary proof of the Brouwer fixed point theorem, Thai J. Math. 17 (2) (2019) 539–542.

[2] L. Zadeh, Fuzzy sets, Inf. Control 8 (1965) 338–353.

[3] O. Kramosil, J. Michalek, Fuzzy metric and statistical metric spaces, Kybernetica 11 (1975) 326–334.

[4] A. George, P. Veeramani, On some results in fuzzy metric spaces, Fuzzy Sets and Systems 64 (1994) 395–399.

[5] L.D.R. Kočinac, Selection properties in fuzzy metric spaces, Filomat 26 (2) (2012) 305–312.

[6] L.D.R. Kočinac, Some covering properties in topological and uniform spaces, Proc. Steklov Inst. Math. 252 (2006) 122–137.

[7] M.S. Abdullahi, A. Azam, P. Kumam, Edelstein type L-fuzzy fixed point theorems, Thai J. Math. (2019) 21–33.

[8] V. Gupta, M. Verma, M.S. Khan, Some modified fixed point results in V-fuzzy metric spaces, Advances in Fuzzy Systems 2019 (2019) Article ID 6923937.

[9] V. Gregori, J. Miñana, D. Miravet, Extended fuzzy metrics and fixed point theorems, Mathematics 7 (303) (2019) 1–14.

[10] I.A. Bakhtin, The contraction mapping principle in quasi-metric spaces, Funct. Anal. Unianowsk Gos. Ped. Inst. 30 (1989) 26–37.

[11] A. Shoaib, P. Kumam, A. Shahzad, S. Phiangsungnoen, Q. Mahmood, Fixed point results for fuzzy mapping in a b-metric space, Fixed Point Theory Appl. 2018 (2) (2018).

[12] W. Kumam, P. Sukprasert, P. Kumam, A. Shoaib, A. Shahzad, Q. Mahmood, Some fuzzy fixed point results for fuzzy mappings in complete b-metric spaces, Cogent Mathematics & Statistics 5 (2018) Article ID 1458933.

[13] S. Phiangsungnoen, P. Kumam, Fuzzy fixed point theorems for multivalued fuzzy contractions in b-metric spaces, J. Nonlinear Sci. Appl. 8 (2015) 55–63.

[14] S. Phiangsungnoen, P. Kumam, On stability of fixed point inclusion for multivalued type contraction mappings in dislocated b-metric spaces with application, Math. Meth. Appl. Sci. (2018) 1–14.

[15] S. Chaipornjareansri, Fixed point and coincidence point theorems for expansive mappings in partial b-metric spaces, Thai J. Math. (2018) 169–185.

[16] N. Konwar, P. Debnath, Fixed point and coincidence point theorems for expansive mappings in partial b-metric spaces, Thai J. Math. (2017) 169–185.

[17] A.A. Mukheimer, Some common fixed point theorems in complex valued b-metric spaces, Sci. World J. 2014 (2014) Article ID 587825.

[18] S. Nădăban, Fuzzy b-Metric spaces, Int. J. Comput. Commun. Control 11 (2) (2016) 273–281.

[19] D. Hunwisai, P. Kumam, Fuzzy fixed point theorems for multivalued fuzzy F-contraction mappings in b-metric spaces, Communications in Mathematics and Applications 7(3) (2016) 179–187.

[20] A. Roldan, E. Karapinar, S. Manro, Some new fixed point theorems in fuzzy metric spaces, J. Intell. Fuzzy Syst. 27 (5) (2014) 2257–2264.

[21] W. Sintunavarat, P. Kumam, Common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces, J. Appl. Math. 2011 (2011) Article ID 637958.

[22] W. Sintunavarat, P. Kumam, Common fixed points for R-weakly commuting in fuzzy metric spaces, Annali dell'Universita'di Ferrara 58 (2) (2012) 389–406.

[23] P. Sukprasert, P. Kumam, D. Thongtha, K. Sombut, Fixed point results on generalized $(\psi, \phi)_s$-contractive mappings in rectangular b-metric spaces, Communications in Mathematics and Applications 7 (3) (2016) 207–216.

[24] N. Hussain, P. Salimi and V. Parvaneh, Fixed point results for various contractions in parametric and fuzzy b-metric spaces, J. Nonlinear Sci. Appl. 8 (2015) 719–739.

[25] D. Singh, V. Chauhan, V. Joshi, S.S. Tomar, On n-tupled coincidence and fixed point results in partially ordered G-metric spaces, Thai J. Math. 17 (2) (2019) 321–342.

[26] J.H. Park, Intuitionistic fuzzy metric spaces, Chaos Solitons Fractals 22 (2004) 1039–1046.

[27] A. George, P. Veeramani, On some results of analysis for fuzzy metric spaces, Fuzzy Sets and Systems 90 (1997) 365–368.

[28] C. Alaca, D. Turkoglu, C. Yildiz, Fixed points in intuitionistic fuzzy metric spaces, Chaos Solitons Fractals 29 (2006) 1073–1078.

[29] N. Konwar, P. Debnath, Some results on coincidence points for contractions in intuitionistic fuzzy n-normed linear space, Thai J. Math. 17 (1) (2019) 43–62.

[30] S. Czerwik, Contraction mappings in b-metric space, Acta Math. Inf. Univ. Ostraviensis 1 (1993) 5-11.

[31] M. Boriceanu, Fixed Point theory for multivalued generalized contraction on a set with two b-metrics, studia Univ Babes, Bolya: Math. LIV. 3 (2009) 1–14.

[32] S. Czerwik, Non-linear set-valued contraction mapping in b-metric spaces, Atti Sem Math. Fic. Univ. Modena, 46 (2) (1998) 263–276.

[33] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1986) 87–96.

[34] M. Grabiec, Fixed points in fuzzy metric spaces, Fuzzy Sets and System 27 (1988) 385–389.

[35] T. Zamfirescu, Fixed point theorems in metric spaces. Arch. Math. 23 (1972) 292–298.