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Approximate Optimality Conditions and Approximate Duality Conditions for Robust Multiobjective Optimization Problems

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Abstract In this paper, following robust optimization framework, we consider approximate robust optimal solutions for a nonsmooth multi objective optimization problem with uncertainty of data. Some necessary and sufficient conditions are investigated proving different result for approximate optimal solutions depending on Fritz-John type. The concept of generalized convexity is defined, and the relationship with sufficient optimality theorem. Moreover, by using robust optimization approach(worst-case approach), we establish optimality theorems and duality theorem for two robust approximate optimal solutions of an uncertain multi-objective optimization.

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1. Introduction

In the real-world, optimization problems are often uncertain. The reason of uncertainty include, they are not know exactly when problem is solved. Actually, there have been two prominent approaches to dealing with uncertain optimization problems, that is, robust optimization and stochastic programming. In this paper, robust optimization is becoming more and more popular in solving scalar optimization problems with uncertain data. Robust optimization (worst case) has emerged as a remarkable deterministic framework for studying optimization problems with data uncertainty, in which the uncertain parameters belong to a known set.

Robust optimization is one of the most important approach for studying optimization problems with data uncertainty, see [1–11] and other references therein. Recently, the

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authers considered robust problem under without uncertain. We refer the researchers to [11–16] and the references therein for the robust approaches for uncertain multiobjective functions is considered in [8]. In addition, robust counterpart of an interest problem play an important role in uncertain optimization problem. Robust optimization can be reviewed as a kind of sentitive against pertubations in the decision space. In particular, robust optimization concentrates in the case no probability distribution information on the uncrtain paramiters is given. It is know that the operation of the solutions is judged by multiple objective that are conflict. Thus, it is interesting to deeply study the theory and applications of robust multiobjective optimization.

For investigate, the main concentrate is depended on finding the global optimum or global efficient solution, representing the best possible objective values. In the classes of certain problem have associated with an exact solution, but an exact solution does not exist for all problems. It is well known that the approximate solution (ϵ-solutions) provides type of solutions important optimization problem under uncertain and without uncertain, such as ϵ-(weakly)-optimality condition and ϵ-(weakly)-duality conditions. We introduce various heuristics to obtain good approximate solutions for difficult problems, this is, multiobjective problem and see [2, 12, 13, 15, 17–19]. Among many desirable properties of an approximate solution to multiobjective optimization problems, there are various different approaches. Recently, in the field, the approximate solution of a multiobjective problem studied by many authors. Several notations of approximality have been studied (see, [2, 14, 18, 20–22]).

Consider the following constrained multiobjective optimization problem of the form (P):

$$\min_{\mathbb{R}^m_+}\{f(x) \mid g_i(x) \leq 0,\ i = 1, \dots, p,\ x \in \Omega\}, \tag{P}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, i.e, $f := (f_k) = \{1, \dots, m\}$, $g : \mathbb{R}^n \to \mathbb{R}$, $g := (g_i), i \in I = \{1, \dots, p\}$ are vector functions with locally Lipchitz components defined on $\mathbb{R}^n$ and Ω is a closed subset of $\mathbb{R}^n$.

The multiobjective optimization problem (P) in the face of data uncertainty in the constrains can be taken by the problem:

$$\min_{\mathbb{R}^m_+}\{f(x) \mid g_i(x, w_i) \leq 0,\ i = 1, \dots, p,\ x \in \Omega\}, \tag{UP}$$

where w_i are uncertain parameters and $w_i \in \Omega_i$, $i = 1, \dots, p$ for some convex compact sets $\Omega_i \subseteq \mathbb{R}^n$, and $f := f_k$, $k \in K = \{1, \dots, m\}$, $g_i : \mathbb{R}^n \times \Omega_i \to \mathbb{R}$, $i \in I = \{1, \dots, p\}$ are vector function with locally Lipschitz component defined on $\mathbb{R}^n$ and Ω is cloed subset of $\mathbb{R}^n$.

We will concentraet problem (UP), one usually associates with it is called robust counterpart:

$$\min_{\mathbb{R}^m_+}\{f(x) \mid x \in A\}, \tag{RP}$$

the set A is given by

$$A := \{x \in \mathbb{R}^n : g_i(x, w_i) \leq 0,\ \forall w_i \in \Omega_i,\ i = 1, \dots, p\} \cap \Omega, \tag{1.1}$$

where

$$C := \{x \in \mathbb{R}^n : g_i(x, w_i) \leq 0,\ \forall w_i \in \Omega_i,\ i = 1, \dots, p\}.$$

Nowadays, the multiobjective optimization problems of approximate solutions for convex case is widly investigated in the literature: see [2, 12, 13, 15, 17–19] and references therein. Here we are interested in the nonconvex case. Recently, Chuong and Kim [17] studied approximate Pareto solution for multiobjective optimization problems. Very, recently, Chuong [8] investigated robust optimality theorems and duality theorem for multiobjective problems in terms of generalized convexity and constarint in the face uncertainty.

The aim of this paper is to interest in the study of a concept of approximate solutions and in the study of the multiobjective optimization problems of solutions when both the multiobjective functions and the uncertaint set are purturbed. Based on previous analysis, in particular, motivated by the work reported in [8, 17], in this article, we study approximate optimality condition and approximate duality theorem for robust multiobjective optimization problems. Thus we show that the neccessary optimality condition and sufficient optimality conditions of robust multiobjective optimization for two approximate. The sufficient conditions formulated in terms of the generalized convex function and the neccessary in Fritz-John type.

The organization of the paper is as follows: In the next section contain some basic definition and concept of two approximate solutions of problem (UP). In Section 3, we purpose necessary conditions for local ϵ-(weakly) Pareto solution and local quasi-ϵ-(weakly) Pareto solution of robust multiobjective problem. Then, we focus on sufficient conditions for local quasi-ϵ-(weakly) Pareto solution of the considered problem with a new concept of generalized convexity objective functions. Finally, in Section 4 is devoted to describing duality ralation in robust for local quasi-ϵ-(weakly) Pareto solution under (strictly) generalized convexity functions.

2. Preliminaries

In this paper, we use the standard notation, please see [23, 24]. The symbol $\mathbb{R}^n_+$, $\mathbb{B}_{\mathbb{R}^n}$, $\mathbb{B}(x_0, r)$ stands for the nonnegative orthant of $\mathbb{R}^n$, closed unit ball in $\mathbb{R}^n$ and the open ball with center at x_0 and radius $r > 0$, respectively. Unless otherwise specified, let $S \subseteq \mathbb{R}^n$ be a nonempty set, whose the convex hull of S are denoted by $\operatorname{co} S$, while the notation $x \to^S x_0$ means that $x \to x_0$ with $x \in S$. The definition of polar cone of $\Omega \subseteq \mathbb{R}^n$ is the set

$$\Omega^\circ := \{x^* \in \mathbb{R}^n \mid \langle x^*, x\rangle \leq 0, \ \forall x \in \Omega\}. \tag{2.1}$$

Let a point $x_0 \in S$ be given. The set S is said to be closed around x_0 if there is a neighborhood U of x_0 such that $S \cap U$ is closed. Moreover, the set S is said to be locally closed if it is closed around every $x_0 \in S$.

Given a set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, we denote by

$$\operatorname{Lim\,sup}_{x\to x_0} F(x) := \{x^* \in \mathbb{R}^n \mid \quad \exists \{x_n\} \to x_0 \text{ and } x_n^* \to x^* \text{ with } x_n^* \in F(x_n) \text{ for all } n \in \mathbb{N}\}.$$

the sequential Painlevé-Kuratowski upper/outer limit of F as $x \to x_0$.

Let $\Omega \subseteq \mathbb{R}^n$ be closed around $\bar{x} \in \Omega$. The Fréchet/regular normal cone to Ω at $\bar{x} \in \Omega$ is defined by

$$\hat{N}(\bar{x}; \Omega) := \Big\{x^* \in \mathbb{R}^n \mid \limsup_{x \to^{\Omega} \bar{x}} \frac{\langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \leq 0\Big\}, \tag{2.2}$$

If $\bar{x} \notin \Omega$, let $\hat{N}(\bar{x};\Omega) := \emptyset$.

The Mordukhovich/limiting normal cone $N(\bar{x};\Omega)$ to Ω at $\bar{x} \in \Omega$ is obtained from regular normal cones by taking the sequential Painlevé-Kuratowski upper limits as:

$$N(\bar{x};\Omega) := \text{Lim }\sup_{x \to^{\Omega} \bar{x}} \hat{N}(x;\Omega). \tag{2.3}$$

If $\bar{x} \notin \Omega$, we put $N(\bar{x};\Omega) := \emptyset$.

Note, the Mordukhovich normal cone enjoys the so-called robustness property (see [23], Page 11), that is,

$$N(\bar{x};\Omega) = \limsup_{x \to \bar{x}} N(x;\Omega). \tag{2.4}$$

For an extended real-valued function $\varphi : \mathbb{R}^n \to \bar{\mathbb{R}} := [-\infty, \infty]$, let

$$dom\,\varphi := \{x \in \mathbb{R}^n \mid \varphi(x) < \infty\},$$

and

$$epi\,\varphi := \{(x,\mu) \in \mathbb{R}^n \times \mathbb{R} \mid \mu \geq \varphi(x)\}.$$

The Mordukhovich/limiting subdifferentialof φ at $\bar{x} \in X$ with $\bar{x} \in \text{dom}(\varphi)$ is defined by

$$\partial\varphi(\bar{x}) := \left\{x^* \in \mathbb{R}^n \mid (x^*, -1) \in N((\bar{x}, \varphi(\bar{x})); epi\,\varphi)\right\}. \tag{2.5}$$

If $\bar{x} \in \text{dom}(\varphi)$, then one puts $\partial\varphi(\bar{x}) := \emptyset$.

Considering the indicator function $\delta(\cdot;\Omega)$ defined by $\delta(x;\Omega) := 0$ for $x \in \Omega$ and by $\delta(x;\Omega) := \infty$ otherwise, we have (see [23], Proposition 1.79):

$$N(\bar{x};\Omega) = \partial\delta(\bar{x};\Omega), \quad \forall \bar{x} \in \Omega. \tag{2.6}$$

The nonsmooth version of Fermat's rule (see [23], Proposition 1.114) is formulated as follows: If $\bar{x}$ is a local minimizer for φ, then

$$0 \in \partial\varphi(\bar{x}). \tag{2.7}$$

For locally Lipschitz function φ at $\bar{x}$ with modulus $K > 0$, i.e, there exists $r > 0$ such that

$$\|\varphi(x_1) - \varphi(x_2)\| \leq K\|x_1 - x_2\|, \ \forall x_1, x_2 \in B(\bar{x}, r),$$

we always have (see [23], Corollary 1.81)

$$\|x^*\| \leq K, \quad \forall x^* \in \partial\varphi(\bar{x}). \tag{2.8}$$

Note that ([17], Example 4, p. 198)

$$\partial\| \cdot - x_v\|(x_v) = \mathbb{B}_{\mathbb{R}^n}.$$

In what follows, we also use the limiting/Mordukhovich subdiffential sum rule.

Lemma 2.1. (See [23],Theorem 3.36) Let $\psi_i : \mathbb{R}^n \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}, i = 1, \ldots, n, n \geq 2$, be lower semicontinuous around $\bar{x} \in \mathbb{R}^n$, and let all but one of these functions be Lipschitz continuous around $\bar{x}$. Then one has

$$\partial(\varphi_1 + \varphi_2 + \cdots + \varphi_n)(\bar{x}) \subset \partial\varphi_1(\bar{x}) + \partial\varphi_2(\bar{x}) + \cdots + \partial\varphi_n(\bar{x}). \tag{2.9}$$

Lemma 2.2. ([8], Lemma 2.2). Let φ be Lipschitz continuous on an open set containing $[a,b] \subset \mathbb{R}^n$. Then one has

$$\langle x^*, b - a\rangle \geq \varphi(b) - \varphi(a) \ \text{ for some } x^* \in \partial\varphi(c), \ c \in [a, b)$$

where $[a,b] := \text{co}\,\{a,b\}$, and $[a,b) := \text{co}\,\{a,b\}\backslash\{b\}$.

For $\varphi : \mathbb{R}^n \to \bar{\mathbb{R}}$ locally Lipschitz at $\bar{x}$, the generalized directional derivative of φ at $\bar{x}$ in the direction $v \in \mathbb{R}^n$ is defined as follows:

$$\varphi^\circ(\bar{x}; v) := \limsup_{x \to \bar{x},\ \lambda \downarrow 0} \frac{\varphi(x + \lambda v) - \varphi(x)}{\lambda}. \tag{2.10}$$

In this case, the convexified/Clarke subdifferential of φ at $\bar{x}$ is the set

$$\partial^C \varphi(\bar{x}) := \{x \in \mathbb{R}^n \mid \langle x^*, v\rangle \le \varphi^\circ(\bar{x}; v)\ \forall v \in \mathbb{R}^n\}, \tag{2.11}$$

which is nonempty, and one has the relation (see [10, Proposition 2.1.2])

$$\varphi^\circ(\bar{x}; v) = \max\{\langle x^*, v\rangle \mid x^* \in \partial^C \varphi(\bar{x})\} \tag{2.12}$$

for each $v \in \mathbb{R}^n$.

Following [25], the relationship between the above subdifferentials of φ at $\bar{x} \in \mathbb{R}^n$ is as follows:

$$\partial \varphi(\bar{x}) \subset \partial^C \varphi(\bar{x}). \tag{2.13}$$

If φ is strictly differentiable at $\bar{x} \in \mathbb{R}^n$ with derivative $\triangledown \varphi(\bar{x})$, then one has

$$\partial \varphi(\bar{x}) = \partial^C \varphi(\bar{x}) = \{\triangledown \varphi(\bar{x})\}, \tag{2.14}$$

and further, in this case, it holds (see [10, Proposition 2.3.6]) that

$$\langle \triangledown \varphi(\bar{x}), v\rangle = \varphi^\circ(\bar{x}; v) = \lim_{\lambda \downarrow 0} \frac{\varphi(\bar{x} + \lambda v) - \varphi(\bar{x})}{\lambda} \tag{2.15}$$

for any $v \in \mathbb{R}^n$.

We recall the Ekeland variational principle (see [21]), which is needed for our investigation.

Lemma 2.3. (Ekeland Variational Principle) Let $(\mathbb{R}^n, \mathrm{d})$ be a complete metric space and $\varphi : \mathbb{R}^n \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ be a proper lowersemicontinuous function bounded from below. Let $\epsilon > 0$ and $x_0 \in \mathbb{R}^n$ be given such that $\varphi(x_0) \le \inf_{x \in X} \varphi(x) + \epsilon$. Then for any $\lambda > 0$ there is $\bar{x} \in \mathbb{R}^n$ satisfying

(i) $\varphi(\bar{x}) \le \varphi(x_0)$,
(ii) $d(\bar{x}, x_0) \le \lambda$,
(iii) $\varphi(\bar{x}) < \varphi(x) + \frac{\epsilon}{\lambda} d(x, \bar{x})$ for all $x \in \mathbb{R}^n \backslash \{\bar{x}\}$.

Finally in this section, we are extended concepts of two approximate Pareto solutions of considering problem (RP) (see [19, 20]). We relevent to consider author to some interest results in [12] for various characterizations of approximate strong/weak/proper Pareto solutions via scalarization methods.

Definition 2.4. Let $\epsilon := (\epsilon_1, \ldots, \epsilon_m) \in \mathbb{R}^m_+$.

(i) A point $\bar{x} \in C$ is said to be a local robust ϵ-Pareto solution for (UP), if it is a local ϵ-Pareto solution for (RP), we can write $\bar{x} \in \mathrm{loc}\ \epsilon\text{-S}(RP)$, i.e, $\bar{x} \in A$ and there exists neighborhood U of $\bar{x}$, there is no $x \in A \cap U$ such that

$$f_k(x) + \epsilon_k \le f_k(\bar{x}),\ \ \forall k \in K. \tag{2.16}$$

(ii) A point $\bar{x} \in C$ is said to be a local robust quasi-ϵ-Pareto solution for (UP) if it is a local quasi-ϵ-Pareto solution for (RP), we can write $\bar{x} \in \mathrm{loc\ quasi}\text{-}\epsilon\text{-S}(RP)$, i.e, $\bar{x} \in A$ and there exists neighborhood U of $\bar{x}$ and there is no $x \in A \cap U$ such that

$$f_k(x) + \epsilon_k \|x - \bar{x}\| \leq f_k(\bar{x}), \ \forall k \in K. \tag{2.17}$$

with at least one strict inequality.

It is worthy of the concept in equation (2.16) and (2.17) are strict, then one has the concept of definition local robust ϵ weakly Pareto solution and local robust quasi-ϵ-weakly Pareto solutions, respectively. We can write loc ϵ-$\mathrm{S}^w(RP)$ and loc quasi-ϵ-$\mathrm{S}^w(RP)$, respectively.

In the above definition, It is simple to see that every local ϵ-solution must be also a local robust solution. In contrast, the converse implication need not to be true.

3. Necessary Approximate Optimality Conditions

In this section, we discuss approaches for local robust approximate solutions and local robust apprximate quasi solution in robust multiobjective optimization problems. First, we will recall some concepts of function g_i and Fritz-John type necessary conditions for solving (UP).

The main concept is used to design Fritz-John neccessary conditions and suffcient conditions for local ϵ-(weakly) Pareto solutions of the considered problem (RP).

For each $i \in \{1, 2, \ldots, p\}$, the function g_i given in (1.1) is assumed to satisfy the following hypothese:

(H1) For a fixed $\bar{x} \in \mathbb{R}^n$, there exists $\delta_i^{x} > 0$ such that the function $w_i \in \Omega_i \mapsto g_i(x, w_i) \in \mathbb{R}$ is upper semicontinuous for each $x \in B(\bar{x}, \delta_i^{\bar{x}})$, and the functions $g_i(\cdot, w_i)$ $w_i \in \Omega_i$, are Lipschitz of given rank $K_i > 0$ on $B(\bar{x}, \delta_i^{\bar{x}})$, i.e,

$$\|g_i(x_1, w_i) - g_i(x_2, w_i)\| \leq K_i \|x_1 - x_2\| \ \ \forall x_1, x_2 \in B(\bar{x}, \delta_i^{\bar{x}}), \forall w_i \in \Omega_i. \tag{3.1}$$

(H2) The multifunction $(x, w_i) \in B(\bar{x}, \delta_i^{\bar{x}}) \times \Omega_i \rightrightarrows \partial_x g_i(x, w_i) \subseteq \mathbb{R}^n$ is closed at $(\bar{x}, \bar{w_i})$ for each $\bar{w_i} \in \Omega_i(\bar{x})$, where the symbol ∂_x stands for the limiting subdifferential operation with respect to x, and the notation $\Omega_i(\bar{x})$ signifies active indices in Ω_i at $\bar{x}$, i.e,

$$\Omega_i(\bar{x}) := \{w_i \in \Omega_i \mid g_i(\bar{x}, w_i) = G_i(\bar{x})\} \tag{3.2}$$

with $G_i(\bar{x}) := \sup_{w_i \in \Omega_i} g_i(\bar{x}, w_i)$.

Remark 3.1. Subgradients of supremum/max functions over compact set has studied by many reseahers.; [15, 24–26]. They use the hypothesis (1) guarantees that the function G_i, $i \in \{1, \ldots, p\}$, is defined and moreover, it implies by (3.1) that G_i is locally Lipschitz of rank K_i (see [18], Page 86).

The assumption (2) related to the closedness of the partial subdifferential operation with respect to the first variable is a comfoetabled property of subdifferentials for convex functions in the finite dimentional setting. It can be checked that under the hypothesis (1), this property holds for a more general class in [22] or [24].

The statements about problem (RP), for fixed $\bar{x} \in \mathbb{R}^n$ and $\epsilon \in \mathbb{R}^m_+ \backslash \{0\}$ we defined (see [13]) a real-valued function ψ on $\mathbb{R}^n$ as follows:

$$\psi(x) := \max_{k=1,\ldots,m,\ i=1,\ldots,p} \{f_k(x) - f_k(\bar{x}) + \epsilon_k, \ G_i(x)\} \tag{3.3}$$

Now we derive some Fritz-John necessary condition in a fuzzy form for local robust approximate (weakly) Pareto solutions of problem (UP).

Theorem 3.2. Suppose that $\bar{x} \in \text{loc}\,\epsilon\text{-}S^w(RP)$. For any $v > 0$, there exist $x_v \in \Omega$ and $\lambda_k \geq 0,\ k \in K = \{1, \dots, m\},\ \mu_i \geq 0,\ i \in I = \{1, \dots, p\}$ with $\sum_{k\in K} \lambda_k + \sum_{i\in I} \mu_i = 1,\ \|x_v - \bar{x}\| \leq v$ and

$$\begin{aligned}
0 \in & \sum_{k\in K} \lambda_k \partial f_k(x_v) + \sum_{i\in I} \mu_i \text{co}\Big[\bigcup_{w_i\in\Omega_i(x_v)} \partial_x g_i(x_v, w_i) \Big] \\
& + \frac{\max_{k\in K}\{\epsilon_k\}}{v} \mathbb{B}_{\mathbb{R}^n} + N(x_v; \Omega), \\
& \lambda_k[f_k(x_v) - f_k(\bar{x}) + \epsilon_k - \psi(x_v)] = 0,\ k \in K, \\
& \mu_i[\sup_{w_i\in\Omega_i} g_i(x_v, w_i) - \psi(x_v)] = 0,\ i \in I,
\end{aligned} \tag{3.4}$$

where the function ψ was defined in (3.3).

Proof. Let $\bar{x} \in \text{loc}\,\epsilon\text{-}S^w(RP)$. Then for all $i \in \{1, \dots, p\}$, we consider $\delta_i^{\bar{x}},\ K_i,\ \Omega_i(\bar{x})$, and $G_i(\bar{x})$, which satisfies (H1) and (H2). From Theorem 3.2 in [8], It can easily be shown that

$$\text{co}\Big[\bigcup_{w_i\in\Omega_i(x_v)} \partial_x g_i(x_v, w_i) \Big]$$

is closed.

In addition, due to this proof of Theorem 3.3 in [8], we obtain that

$$\partial G_i(\bar{x}) \subset \text{co}\Big[\bigcup_{w_i\in\Omega_i(x_v)} \partial_x g_i(x_v, w_i) \Big],\ i \in I. \tag{3.5}$$

is completed.

Since $\bar{x} \in \text{local}\,\epsilon\text{-}S^w(\text{RP})$. Then $\bar{x} \in A$, and there exists a neighborhood U of $\bar{x}$ and there is no $x \in A \cap U$ such that

$$f_k(x) + \epsilon_k < f_k(\bar{x}),\ \forall k \in K, \tag{3.6}$$

due to the face that the function ψ on $\mathbb{R}^n$ is defined by

$$\psi(x) := \max_{1\leq k\leq m,\ 1\leq i\leq p} \{f_k(x) - f_k(\bar{x}) + \epsilon_k,\ G_i(x)\},\ x \in \mathbb{R}^n.$$

We will show that

$$0 \leq \psi(x),\ \forall x \in U \cap \Omega. \tag{3.7}$$

In this case $x \in U \cap \Omega \cap C,$ where $A := C \cap \Omega$ then $\psi(x) \geq 0$. Assume to contrary $\psi(x) < 0$ leads to

$$f_k(x) - f_k(\bar{x}) + \epsilon < 0,\ \forall k \in K.$$

This contradicts to (3.6). Thus $\psi(x) \geq 0,\ \forall x \in U \cap \Omega$.

On the other hand, if $x \in U \cap \Omega \backslash C$, then there is $i_0 \in \{1, \dots, p\}$ such that $G_{i_0}(x) > 0$, thus $\psi(x) > 0$. From the inequality (3.7), we obtain that ψ is bounded from below on $U \cap \Omega$.

Moreover, since $\bar{x} \in A$, we have $\psi(\bar{x}) = \max_{k\in K}\{\epsilon_k\}$. Therefore, we get by (3.7) that

$$\psi(\bar{x}) \leq \inf_{x\in\Omega} \psi(x) + \max_{k\in K}\{\epsilon_k\}.$$

By applying Lemma 2.2, then for any $v > 0$, there exists $x_v \in U \cap \Omega$ such that $\|x_v - \bar{x}\| \le v$ and

$$\psi(x_v) \le \psi(x) + \frac{\max_{k\in K}\{\epsilon_k\}}{v}\|x - x_v\|, \ \forall x \in U \cap \Omega.$$

This implies that x_v is a minimizer to the scalar optimization problem

$$\min_{x\in\Omega} \varphi(x),$$

where

$$\varphi(x) := \psi(x) + \frac{\max_{k\in K}\{\epsilon_k\}}{v}\|x - x_v\|, \ x \in U \cap \Omega. \tag{3.8}$$

This means that x_v is a minimizer to an unconstrained (scalar) optimization problem

$$\min_{x\in X} \varphi(x) + \delta(x;\Omega). \tag{3.9}$$

Combine the property of Fermat's rule (2.7) with the above problem (3.9), we obtain that

$$0 \in \partial(\varphi + \delta(\cdot\,;\Omega))(x_v). \tag{3.10}$$

Since φ is Lipschitz continuous at x_v and lower semicontinuous of $\delta(x_v;\ \Omega)$, by the sum rule (2.9) adapted to (3.10) and from the relation in (2.6) that

$$0 \in \partial\varphi(x_v) + N(x_v;\Omega). \tag{3.11}$$

Note that (see [17] in Theorem 3.4)

$$\partial\|\cdot - x_v\|(x_v) = \mathbb{B}_{\mathbb{R}^n}.$$

Since Lemma 2.1 to the function φ is defined in (3.8) and (3.11), we get

$$0 \in \partial\psi(x_v) + \frac{\max_{k\in K}\{\epsilon_k\}}{v}\mathbb{B}_{\mathbb{R}^n} + N(x_v;\Omega). \tag{3.12}$$

Using the Mordukhovich/limiting subdiffrential of maximum functions (see [23], Theorem 3.46(ii)). Moreover according to (2.9) and applied to ψ in (3.2), we get

$$\begin{aligned}\partial\psi(x_v) \subseteq \Big\{ &\sum_{k=1}^{m}\lambda_k\partial f_k(x_v) + \sum_{i=1}^{l}\mu_i\partial_x G_i(x_v) \mid \lambda_k \in K,\ \mu_i \ge 0,\\ &\sum_{k=1}^{m}\lambda_k + \sum_{i=1}^{p}\mu_i = 1,\ \lambda_k[f_k(x_v) - f_k(\bar{x}) + \epsilon_k - \psi(x_v)] = 0,\ k \in K,\\ &\mu_i[\sup_{w_i\in\Omega_i} G_i(x_v) - \psi(x_v)] = 0,\ i \in I\Big\}.\end{aligned} \tag{3.13}$$

It remains to combine (3.12) and (3.13) with condition (3.7), thus (3.4) holds. ∎

In the theorem above, we now derive a Fritz-John necessary condition for local robust weakly Pareto solutions of the consider problem (UP). * Let $\bar{x} \in \operatorname{loc} S^w(RP)$. Then there are $\lambda_k \ge 0,\ k \in K = \{1,\dots,m\}$, and $\mu_i \ge 0,\ i \in I = \{1,\dots,p\}$ with $\sum_{k\in K}\lambda_k + \sum_{i\in I}\mu_i = 1$ such that

$$0 \in \sum_{k\in K}\lambda_k\partial f_k(\bar{x}) + \sum_{i\in I}\mu_i \mathrm{co}\Big[\partial_x g_i(\bar{x},\omega_i)\Big] + N(\bar{x}\,;\Omega),$$

$$\mu_i \sup_{w_i\in\Omega_i} g_i(\bar{x}, w_i) = 0,\ i \in I,$$

where the function ψ was defined in (3.3).

Proof. Let $\epsilon := (\epsilon_1, ..., \epsilon_m) \in \mathbb{R}^m_+ \backslash \{0\}$. Then, we have $\bar{x} \in \epsilon\text{-}S^w(RP)$. For $v = \sqrt{\max_{k\in K}\{\epsilon_k\}}$. This is the aim of the following corollary. However in order to use Theorem 3.2, we find $x_\epsilon \in \Omega$ and $\lambda_k^\epsilon \geq 0,\ k \in K,\ \mu_i^\epsilon \geq 0,\ i \in I$ with $\sum_{k\in K}\lambda_k^\epsilon + \sum_{i\in I}\mu_i^\epsilon = 1$, such that $\|x_\epsilon - \bar{x}\| \leq \sqrt{\max_{k\in K}\{\epsilon_k\}}$ and

$$0 \in \sum_{k\in K}\lambda_k^\epsilon \partial f_k(x_\epsilon, u_k) + \sum_{i\in I}\mu_i^\epsilon \mathrm{co}\Big[\bigcup_{w_i\in\Omega_i(x_v)}\partial_x g_i(x_v, w_i)\Big]$$

$$+ \sqrt{\max_{k\in K}\{\epsilon_k\}}\mathbb{B}_{\mathbb{R}^n} + N(x_\epsilon; \Omega), \tag{3.14}$$

$$\mu_i^\epsilon[\max_{i\in I} g_i(x_\epsilon, w_i) - \psi^\epsilon(x_\epsilon)] = 0,\ i \in I, \tag{3.15}$$

where ψ^ϵ is defined by

$$\psi^\epsilon = \max_{k\in K, i\in I}\{f_k(x) - f_k(\bar{x}) + \epsilon_k,\ G_i(x)\},\ x \in \mathbb{R}^n.$$

Since (3.14), we obtain that there exist $z_k^{*\epsilon} \in \partial f_k(x_\epsilon),\ k \in K,\ \{w_\epsilon\} \subseteq \Omega_i(x_\epsilon)$ such that $x_i^{*\epsilon} \in \partial g_i(x_\epsilon, w_\epsilon),\ i \in I$, and $x^{*\epsilon} \in \mathbb{B}_{\mathbb{R}^n}$ such that

$$-(\sum_{k\in K}\lambda_k^\epsilon z_k^{*\epsilon} + \sum_{i\in I}\mu_i^\epsilon x_i^{*\epsilon} + \sqrt{\max_{k\in K}\{\epsilon_k\}}x^{*\epsilon}) \in N(x_\epsilon; \Omega) \tag{3.16}$$

By taking $\epsilon \to 0$, the definition of locally Lipschitz functions $f_k,\ k \in K,\ g_i, i \in I$, and their gradients are bounded in this inequality (2.8). So, we have $\lambda_k^\epsilon \to \lambda_k \geq 0,\ z_k^{*\epsilon} \to z_k^* \in \mathbb{R}^n, k \in K, \mu_i^\epsilon \to \mu_i \geq 0, x_i^{*\epsilon} \to x_i^* \in \mathbb{R}^n, i \in I$, with $\sum_{k\in K}\lambda_k + \sum_{i\in I}\mu_i = 1$, as well as $x^{*\epsilon} \to x^* \in \mathbb{B}_{\mathbb{R}^n}$. By (2.5), the relation $z_k^{*\epsilon} \in \partial f_k(x_\epsilon)$ is equivalent to

$$(z_k^*, -1) \in N((x_\epsilon, f_k(x_\epsilon)); \mathrm{epi}\, f_k), \tag{3.17}$$

Let $\epsilon \to 0$, for all $k \in K$ in (3.17) and (2.3), we obtain that

$$(z_k^*, -1) \in N((\bar{x}, f_k(\bar{x})); \mathrm{epi}\, f_k),$$

It means that $z_k^* \in \partial f_k(\bar{x}),\ k \in K$. In the same way, we have $x_i^* \in \partial g_i(\bar{x}, w_i), i \in I$. Taking $\epsilon \to 0$ in (3.16) and the robustness property in (2.4), we observe

$$-(\sum_{k\in K}\lambda_k z_k^* + \sum_{i\in I}\mu_i x_i^*) \in N(\bar{x}\,; \Omega). \tag{3.18}$$

Thus (3.19) holds.

The equation (3.15) has become the form (3.20) by passing to a subsequence, taking $\epsilon \to 0$ in $\psi^\epsilon(x_\epsilon)$, that is $\psi^\epsilon(x_\epsilon) \to 0$. This proof is completely. ■

When nonempty compact Ω_i are singleton sets, we propose necessary condition for approximate (weakly) Pareto solution in [17].

Corollary 3.3. Suppose that $\bar{x} \in \mathrm{loc}\,\epsilon\text{-}S^w(RP)$. For any $v > 0$, there exist $x_v \in \Omega$ and $\lambda_k \geq 0,\ k \in K = \{1, \ldots, m\},\ \mu_i \geq 0,\ i \in I = \{1, \ldots, p\}$ with $\sum_{k\in K}\lambda_k + \sum_{i\in I}\mu_i =$

$1, \|x_v - \bar{x}\| \leq v$ and

$$0 \in \sum_{k\in K} \lambda_k \partial f_k(x_v) + \sum_{i\in l} \mu_i \partial g_i(x_v) + \frac{\max_{k\in K}\{\epsilon_k\}}{v}\mathbb{B}_{\mathbb{R}^n} + N(x_v;\, \Omega),$$
$$\lambda_k[f_k(x_v) - f_k(\bar{x}) + \epsilon_k - \psi(x_v)] = 0,\ k \in K,$$
$$\mu_i[\sup_{w_i\in\Omega_i} g_i(x_v) - \psi(x_v)] = 0,\ i \in I,$$

where the function ψ was defined in (3.3).

In the special case when taking arbitrarily $\epsilon := (\epsilon_1, \cdots, \epsilon_m) \in \mathbb{R}^m_+\backslash\{0\}$, and nonempty compact $\Omega_i,\ \forall i \in I$ is a singleton sets, we derive necessary condition for weakly Pareto solution in [17].

Corollary 3.4. Let $\bar{x} \in \mathrm{loc}\, S^w(RP)$. Then there exist $\lambda_k \geq 0,\ k \in K$, and $\mu_i \geq 0,\ i \in I$ with $\sum_{k\in K} \lambda_k + \sum_{i\in I} \mu_i = 1$ such that

$$0 \in \sum_{k\in K} \lambda_k \partial f_k(\bar{x}) + \sum_{i\in l} \mu_i \partial g_i(\bar{x}) + N(\bar{x}\,;\, \Omega), \tag{3.19}$$
$$\mu_i g_i(\bar{x}) = 0,\ i \in I. \tag{3.20}$$

We now establish some Fritz-John necessary condition for local robust quasi approximate (weakly) Pareto solution of this considered problem.

Theorem 3.5. Let $\bar{x} \in$ local quasi-ϵ-S^w(RP). Then there exist $\lambda_k \geq 0,\ k \in K$, and $\mu_i \geq 0,\ i \in I$ with $\sum_{k\in K} \lambda_k + \sum_{i\in I} \mu_i = 1$ such that

$$0 \in \sum_{k\in K} \lambda_k \partial f_k(\bar{x}) + \sum_{i\in I} \mu_i \mathrm{co}\Big[\bigcup_{w_i\in\Omega_i(\bar{x})} \partial_x g_i(\bar{x}, w_i)\Big]$$
$$+ \sum_{k\in K} \lambda_k \epsilon_k \mathbb{B}_{\mathbb{R}^n} + N(\bar{x}\,;\, \Omega),$$
$$\mu_i \max_{i\in I} g_i(\bar{x}, w_i) = 0,\ i \in I. \tag{3.21}$$

Proof. Let $\bar{x} \in$ local quasi-ϵ-S^w(RP). Then $\bar{x} \in A$, there exist neighborhood $x \in U$ and there is no $x \in A \cap U$ such that

$$f_k(x) + \epsilon_k\|x - \bar{x}\| \leq f_k(\bar{x}),\ \forall k \in K.$$

Let the function Φ is defined by

$$\Phi(x) := \max_{k\in K, i\in I}\{f_k(x) - f_k(\bar{x}) + \epsilon_k\|x - \bar{x}\|,\, G_i(x)\},\ x \in U \cap \Omega$$

Similarly to the second part of the proof of Theorem 3.2, we have

$$0 \in \partial\Phi(\bar{x}) + N(\bar{x}; \Omega), \tag{3.22}$$

and

$$\partial\Phi(\bar{x}) \subset \{\sum_{k\in K}[\partial f_k(\bar{x}) + \epsilon_k \mathbb{B}_{\mathbb{R}^n}] + \sum_{i\in I} \mu_i \mathrm{co}\Big[\bigcup_{w_i\in\Omega_i(\bar{x})} \partial_x g_i(\bar{x}, w_i)\Big]$$
$$\lambda_k \geq 0,\ k \in K,\ \mu_i \geq 0,\ i \in I,\ \sum_{k\in K} \lambda_k + \sum_{i\in I} \mu_i = 1,$$
$$\mu_i \sup_{i\in I} g_i(\bar{x}, v_i) = 0,\ i \in I\}. \tag{3.23}$$

The proof is completed by combine (3.22) and (3.23). ∎

The next corollary can be omitted by a very similar proof in Corollary 3.4

Corollary 3.6. Let $\bar{x}$ is a local quasi-ϵ-$S^w(RP)$. Then there exist $\lambda_k \geq 0,\ k \in K, \mu_i \geq 0, i \in I$ with $\sum_{k\in K} \lambda_k^\epsilon + \sum_{i\in I} \mu_i^\epsilon = 1$ such that

$$\begin{aligned} &0 \in \sum_{k\in K} \lambda_k \partial f_k(\bar{x}) + \sum_{i\in I} \mu_i \text{co}\Big[\bigcup_{w_i\in\Omega_i(\bar{x})} \partial_x g_i(\bar{x}, w_i)\Big] + N(\bar{x}; \Omega), \\ &\mu_i \sup_{i\in I} g_i(\bar{x}, w_i) = 0,\ i \in I. \end{aligned} \tag{3.24}$$

Obviously, the conditions to (3.21) hints us to stae the generalized robust approximate Karush-Kuhn-Tucker (KKT) type condition for studied problem.

4. Sufficient Approximate Optimality Theorem

Now, we purpose sufficient conditions for local robust approximate quasi (weakly) Pareto solutions of the considered problem (UP). Firstly, we introduce the generalized robust approximate KKT condition for (RP) and properties of (strictly) generalized convexity type before.

Definition 4.1. Let $\epsilon \geq 0$. A point $(\bar{x}, \lambda_k, \mu_i, w_i) \in C \times \mathbb{R}^n \times \mathbb{R}^n \times \Omega$ is said to satisfy the generalized robust approximate (KKT) condition for (UP), if

$$\begin{aligned} &0 \in \sum_{k\in K} \lambda_k \partial f_k(\bar{x}) + \sum_{i\in I} \mu_i \text{co}\Big[\bigcup_{w_i\in\Omega_i(\bar{x})} \partial_x g_i(\bar{x}, w_i)\Big] \\ &\quad + \sum_{k\in K} \lambda_k \epsilon_k \mathbb{B}_{\mathbb{R}^n} + N(\bar{x}; \Omega), \\ &\mu_i \max_{i\in I} g_i(\bar{x}, w_i) = 0,\ i \in I. \end{aligned} \tag{4.1}$$

Remark 4.2. From Theorem 3.5, if $\bar{x}$ is a quasi-ϵ-(weakly)-Pareto solution of problem (RP), then generalized robust approximate (KKT) condition defined by the following constrained qualification (CQ) : we called condition (CQ) is satisfied at $\bar{x} \in C$ if there do not exist $\mu_i \geq 0,\ i \in I(\bar{x})$ not all zero, such that

$$0 \in \sum_{i\in I(\bar{x})} \mu_i \text{co}\Big[\bigcup_{w_i\in\Omega_i(\bar{x})} \partial_x g_i(\bar{x}, w_i)\Big] + N(\bar{x}; \Omega), \tag{CQ}$$

where $I(\bar{x}) := \{i \in I \mid g_i(\bar{x}, w_i) = 0\}$.

In this case of $\Omega := \mathbb{R}^n$ and $g_i,\ i \in I$ being a continuously differentiable function at the referenced point, we call this above inequality as (CQ) has become Mangasarian-Fromovitz constraint qualification (cf. [8]).

In order to formulate sufficient condition for local robust approximate quasi (weakly) Pareto solution of (UP) in the next theorem, we first introduce a concept of (strictly) generalized convexity type at given point for locally Lipchitz functions.

Definition 4.3. (i) We say that $(f; g)$ is generalized convex on Ω at $\bar{x} \in \Omega$ if for any $x \in \Omega,\ z_k^* \in \partial f_k(\bar{x}),\ k \in K$ and $x_w^* \in \partial g_i(\bar{x}, w), w \in \Omega_i(\bar{x}),\ i \in I$, there exists $v \in N(\bar{x}; \Omega)^\circ$ such that

$$\begin{aligned} f_k(x) - f_k(\bar{x}) &\geq \langle z_k^*, v\rangle,\ \forall k \in K, \\ g_i(x, w) - g_i(\bar{x}, w) &\geq \langle x_w^*, v\rangle,\ w \in \Omega_i(\bar{x}),\ i \in I,\ \text{and} \\ \langle u^*, v\rangle &\leq \|x - \bar{x}\|,\ \forall u^* \in \mathbb{B}_{\mathbb{R}^n}. \end{aligned}$$

(ii) We say that $(f;g)$ is strictly generalized convex on Ω at $\bar{x} \in \Omega$ if for any $x \in \Omega \backslash \{\bar{x}\}$, $z_k^* \in \partial f_k(\bar{x}, u_k)$, $k \in K$ and $x_w^* \in \partial g_i(\bar{x}, w), w \in \Omega_i(\bar{x})$, $i \in I$, there exists $v \in N(\bar{x};\Omega)^\circ$ such that

$$\begin{aligned} f_k(x) - f_k(\bar{x}) &> \langle z_k^*, v\rangle, \ \forall k \in K, \\ g_i(x, w) - g_i(\bar{x}, w) &\geq \langle x_w^*, v\rangle, \ i \in I, \text{ and} \\ \langle u^*, v\rangle &\leq \|x - \bar{x}\|, \ \forall u^* \in \mathbb{B}_{\mathbb{R}^n}. \end{aligned}$$

Theorem 4.4. (Robust KKT Sufficient Optimality Condition). Assume that f_k and $g_i(\cdot, w_i)$ for all $k \in K$ and $i \in I$ are locally Lipschitz and satisfy the approximate (KKT) condition.

(i) If $(f;g)$ is a generalized convex on Ω at $\bar{x}$, then $\bar{x} \in \text{loc quasi}\,\epsilon\text{-}S^w(\text{RP})$.
(ii) If $(f;g)$ is a strictly generalized convex on Ω at $\bar{x}$, then $\bar{x} \in \text{loc quasi}\,\epsilon\text{-S}(\text{RP})$.

Proof. Let $\bar{x} \in C$ satisfies approximate (KKT) condition. Then there are $\lambda_k \geq 0$, $z_k^* \in \partial f_k(\bar{x})$, $k \in K$ with $\sum_{k\in K} \lambda_k \neq 0$, and $\mu_i \geq 0, i \in I$, $\mu_{ij} \geq 0$, $x_{ij}^* \in \partial_x g_i(\bar{x}, w_{ij}), w_{ij} \in \Omega_i(\bar{x})$ $j = 1, \ldots, j_i$, $j_i \in \mathbb{N}$, $\sum_{j=1}^{j_i} \mu_{ij} = 1$ such that

$$-\Big(\sum_{k\in K} \lambda_k z_k^* + \sum_{i\in I} \mu_i (\sum_{j=1}^{j_i} \mu_{ij} x_{ij}^*) + \sum_{k\in K} \lambda_k \epsilon_k u^*\Big) \in N(\bar{x};\Omega), \tag{4.2}$$

$$\mu_i \sup_{w_i \in \Omega_i(\bar{x})} g_i(\bar{x}, w_i) = 0, \ i \in I. \tag{4.3}$$

(i) Assume that $\bar{x}$ is not a local quasi-ϵ-weakly Pareto solution of (RP). Then for any neighborhood U of $\bar{x} \in A$, there is $\hat{x} \in A \cap U$ such that

$f_k(\hat{x}) - f_k(\bar{x}) + \epsilon_k \|\hat{x} - \bar{x}\| < 0, \ \forall k \in K,$

where $\sum_{k\in K} \lambda_k \neq 0$. On the other hand, using the definition of polar cone and the generalized convexity of $(f;g)$ on Ω at $\bar{x}$, we deduce from (4.2) that for such $\hat{x}$, there exists $w \in N(\bar{x};\Omega)^\circ$ such that

$$\begin{aligned} 0 &\leq \sum_{k\in K} \lambda_k \langle z_k^*, v\rangle + \sum_{i\in I} \mu_i (\sum_{j=1}^{j_i} \mu_{ij} \langle x_{ij}^*, v\rangle) + \sum_{k\in K} \lambda_k \epsilon_k \langle u^*, w\rangle \\ &\leq \sum_{k\in K} \lambda_k [f_k(\hat{x}) - f_k(\bar{x})] + \sum_{i\in I} \mu_i \Big(\sum_{j=1}^{j_i} \mu_{ij} [g_i(\hat{x}, w_{ij}) - g_i(\bar{x}, w_{ij})]\Big) \\ &+ \sum_{k\in K} \lambda_k \epsilon_k \|\hat{x} - \bar{x}\|. \end{aligned} \tag{4.4}$$

Hence

$$\begin{aligned} &\sum_{k\in K} \lambda_k f_k(\bar{x}) + \sum_{i\in I} \mu_i \Big(\sum_{j=1}^{j_i} \mu_{ij} g_i(\bar{x}, w_{ij})\Big) \\ &\leq \sum_{k\in K} f_k(\hat{x}) + \sum_{i\in I} \mu_i \Big(\sum_{j=1}^{j_i} \mu_{ij} g_i(\hat{x}, w_{ij})\Big) + \sum_{k\in K} \lambda_k \epsilon_k \|\hat{x} - \bar{x}\|. \end{aligned} \tag{4.5}$$

By $w_{ij} \in \Omega_i(\bar{x})$, we have

$g_i(\bar{x}, w_{ij}) = \sup_{w_i \in \Omega_i(\bar{x})} g_i(\bar{x}, w_i), \ \forall i \in I, \forall j = 1, \ldots, j_i.$

Note that $\mu_i g_i(\bar{x}, w_{ij}) = 0$, $\forall i \in I$, $\forall j = 1, \ldots, j_i$. Since $\hat{x} \in A$, we have $\mu_i g_i(\hat{x}, w_{ij}) \leq 0$, $i \in I$ and $j = 1, \ldots, j_i$. By accent inequality (4.5), we actually

$$\begin{aligned}
\sum_{k\in K} \lambda_k f_k(\bar{x}) &= \sum_{k\in K} \lambda_k f_k(\bar{x}) + \sum_{i\in I} \Big(\sum_{j=1}^{ji} \mu_{ij}\mu_i g_i(\bar{x}, w_{ij}) \Big) \\
&\leq \sum_{k\in K} \lambda_k f_k(\bar{x}) + \sum_{i\in I} \Big(\sum_{j=1}^{ji} \mu_{ij}\mu_i g_i(\hat{x}, w_{ij}) \Big) \\
&+ \sum_{k\in K} \lambda_k \epsilon_k \|\hat{x} - \bar{x}\| \\
&\leq \sum_{k\in K} \lambda_k f_k(\bar{x}) + \sum_{k\in K} \lambda_k \epsilon_k \|\hat{x} - \bar{x}\|, \qquad (4.6)
\end{aligned}$$

which is contradict to (4.5), thus the proof (i) is complete.

(ii) Suppose that $\bar{x}$ is not a local quasi-ϵ-Pareto solution of (RP). Then for every neighborhood U of $\bar{x}$, there is $\tilde{x} \in A \cap U$ such that

$$f_k(\tilde{x}) - f_k(\bar{x}) + \epsilon_k \|\tilde{x} - \bar{x}\| \leq 0, \ \forall k \in K, \qquad (4.7)$$

where at least one inequality is strict. Suppose that $\tilde{x} \neq \bar{x}$ and the inequality

$$\sum_{k\in K} \lambda_k f_k(\bar{x}, u_k) + \sum_{k\in K} \lambda_k \epsilon_k \|\tilde{x} - \bar{x}\| \leq \sum_{k\in K} \lambda_k f_k(\bar{x}, u_k). \qquad (4.8)$$

On the other side, due to definition of polar cone (2.1), $\sum_{k\in K} \lambda_k \neq 0$ and the strictly generalized convexity of $(f\,;\, g)$ on Ω at $\bar{x}$. We deduce from (4.2) that for each $\tilde{x}$, there is $w \in N(\bar{x}\,;\,\Omega)^\circ$ such that

$$\begin{aligned}
0 &\leq \sum_{k\in K} \langle z_k^*, v\rangle + \sum_{i\in I} \mu_i \Big(\sum_{j=1}^{j_i} \mu_{ij} \langle x_i^*, v\rangle \Big) + \sum_{k\in K} \lambda_k \epsilon_k \langle u^*, v\rangle \\
&< \sum_{k\in K} [f_k(\tilde{x}) - f_k(\bar{x})] + \sum_{i\in I} \mu_i \Big(\sum_{j=1}^{j_i} \mu_{ij} [g_i(\tilde{x}, w_{ij}) - g_i(\bar{x}, w_{ij})] \Big) \\
&+ \sum_{k\in K} \lambda_k \epsilon_k \|\tilde{x} - \bar{x}\|. \qquad (4.9)
\end{aligned}$$

Similarly to the part of proof (i), we obtain that

$$\sum_{k\in K} \lambda_k f_k(\bar{x}) < \sum_{k\in K} \lambda_k f_k(\tilde{x}) + \sum_{k\in K} \lambda_k \epsilon_k \|\tilde{x} - \bar{x}\|,$$

which contradicts (4.7). This completes the proof of Theorem.

■

5. Duality in Robust Multiobjective Optimization for Approximate Pareto Solutions

In this section, we design robust dual problem (stated in an approximate form) for multiobjective optimization problem to the primal one and establish (converse) duality relations between them.

Let us first define

$$\mathbb{R}_+^{\mathbb{N}} := \Big\{\mu := (\mu_i,\ \mu_{ij}),\ i \in I,\ j = 1,\dots,j_i \mid j_i \in \mathbb{N},\ \mu_i \geq 0,\ \sum_{j=1}^{j_i} \mu_{ij} = 1\Big\}.$$

Assume that $z \in \mathbb{R}^n$, $\lambda := (\lambda_1,\dots,\lambda_m) \in \mathbb{R}_+^m\backslash\{0\}$, and $\mu \in \mathbb{R}_+^{\mathbb{N}}$, we denote a vector Lagrangian function L by

$$\begin{aligned} L(z,\lambda,\mu) &= L_k(z,\lambda,\mu) \\ &= f_k(z) + \sum_{i\in I} \mu_i\Big(\sum_{j=1}^{j_i} \mu_{ij} g_i(z, w_{ij})\Big)e, \end{aligned}$$

where $e := (1,\dots,1) \in \mathbb{R}^m$, and $w_{ij} \in \Omega_i$, $i = 1,\dots p$, $j = 1,\dots,j_i$. Here, for the sake of unifying variables in the objective function and constraints of the dual problem(given below).

In connection with the robust multiobjective problem (RP), we address a dual robust multiobjective optimization problem in a dual form of:

$$\max_{\mathbb{R}_+^m}\{L_k(z,\lambda,\mu) \mid (z,\lambda,\mu) \in C_D\}. \tag{RD}$$

Here the feasible set C_D is given by

$$\begin{aligned} C_D \quad := \Big\{&(z,\lambda,\mu) \in \Omega \times (\mathbb{R}_+^m\backslash\{0\}) \times \mathbb{R}_+^p \mid 0 \in \textstyle\sum_{k\in k} \lambda_k \partial f_k(z) \\ &+ \textstyle\sum_{i\in I} \mu_i\Big(\sum_{j=1}^{j_i} \mu_{ij} x_{ij}^*\Big) + \sum_{k\in K} \lambda_k \epsilon_k \mathbb{B}_{\mathbb{R}^{n^*}} + N(z;\,\Omega)\ , \\ &\textstyle\sum_{k\in K} \lambda_k = 1\ , x_{ij}^* \in \{\cup \partial_x g_i(z, w_{ij}) | w_{ij} \in \Omega_i(z)\}\Big\}, \end{aligned} \tag{5.1}$$

where $\Omega_i(z)$ is defined as in (3.2) by replacing $\bar{x}$ with z.

Now we define robust quasi-ϵ-(weakly) Pareto solutions of the considered problem (RD) similarly to the statement of problem (RP).

Definition 5.1. Let $L := (L_1,\dots,L_m)$, and $\epsilon := (\epsilon_1,\dots,\epsilon_m) \in \mathbb{R}_+^m\backslash\{0\}$. We say that $(\bar{x},\bar{\lambda},\bar{\mu})$ is a local quasi-ϵ-Pareto solution of problem (RD) if and only if there exist neighborhood U of (z,λ,μ) and there is no $(z,\lambda,\mu) \in C_D \cap U$ such that

$$L_k(z,\lambda,\mu) \geq L_k(\bar{z},\bar{\lambda},\bar{\mu}) + \epsilon_k\|(\bar{z},\bar{\lambda},\bar{\mu}) - (z,\lambda,\mu)\|,\ \forall k \in K, \tag{5.2}$$

with a least one strict inequality.

Now we establish duality theorem for quasi-ϵ-Pareto solution between the problem (RP) and the dual problem (RD).

Theorem 5.2. (Duality) Let $\bar{x} \in$ quasi-ϵ-$S^w(RP)$ be such that the (CQ) defined in (CQ) is satisfied at this point. Then there exist $\bar{\lambda} := (\bar{\lambda}_k)\ \bar{\lambda}_k \geq 0$, $k \in K$, not all zero simultaneously, and $\bar{\mu} := (\bar{\mu}_i)\ ,\bar{\mu}_i \geq 0$, $i \in I$ such that $(\bar{x},\bar{\lambda},\bar{\mu}) \in C_D$ and $f(\bar{x}) = L(\bar{z},\bar{\lambda},\bar{\mu})$. In addition,

(i) If $(f;g)$ is a generalized convex on Ω at $z \in \Omega$, then $(\bar{z},\bar{\lambda},\bar{\mu}) \in$ loc quasi ϵ-S^w(RD).

(ii) If $(f;g)$ is a strictly generalized convex on Ω at $z \in \Omega$, then $(\bar{z},\bar{\lambda},\bar{\mu}) \in$ loc quasi ϵ-S(RD).

Proof. By applying Theorem 3.5 and the concept of (CQ), thus $\bar{x}$ satisfies the approximate (KKT) condition. It means that there exist $\tilde{\lambda}_k \geq 0,\ \tilde{z}_k^* \in \partial f_k(\bar{x})\ k \in K$ with $\sum_{k\in K} \tilde{\lambda}_k \neq 0$, and $\tilde{\mu}_i \geq 0,\ i \in I,\ \tilde{\mu}_{ij} \geq 0,\ \tilde{x}_{ij}^* \in \partial_x g_i(\bar{x}, w_{ij}),\ w_{ij} \in \Omega_i(\bar{x}),\ j = 1, \ldots, j_i, j_i \in \mathbb{N},\ \sum_{j=1}^{j_i} \tilde{\mu}_{ij} = 1$, whatever $\tilde{\mu}^* \in B_{\mathbb{R}^{n*}}$ such that

$$-\Big(\sum_{k\in K} \tilde{\lambda}_k \tilde{z}_k^* + \sum_{i\in I} \tilde{\mu}_i \Big(\sum_{j=1}^{j_i} \tilde{\mu}_{ij} \tilde{x}_{ij}^*\Big) + \sum_{k\in K} \tilde{\lambda}_k \epsilon_k \tilde{u}^*\Big) \in N(\bar{x};\, \Omega), \tag{5.3}$$
$$\tilde{\mu}_i \sup_{w_i\in\Omega_i} g_i(\bar{x}, w_i) = 0,\ i \in I.$$

Letting

$$\bar{\lambda}_k := \frac{\tilde{\lambda}_k}{\sum_{k\in K} \tilde{\lambda}_k},\ k \in K \ \text{ and } \ \bar{\mu}_i := \frac{\tilde{\mu}_i \Big(\sum_{j=1}^{j_i} \tilde{\mu}_{ij}\Big)}{\sum_{k\in K} \tilde{\lambda}_k},\ i \in I,$$

then, we have $\bar{\lambda} := (\bar{\lambda}_k),\ \bar{\lambda}_k \geq 0,\ k \in K$ with $\sum_{k\in K} \bar{\lambda}_k = 1$, and $\bar{\mu} := (\bar{\mu}_i),\ \bar{\mu}_i \geq 0,\ i \in I$. In additional, if $\tilde{\lambda}_k$'s, and $\tilde{\mu}_i$'s are replaced by $\bar{\lambda}_k$'s, and $\bar{\mu}_i$'s, respectively, thus the relation in (5.3) is also valid. So, $(\bar{x}, \bar{\lambda}, \bar{\mu}) \in C_D$.

Since

$$\langle \bar{x}, g(\bar{x}, w_{ij})\rangle = \sum_{i\in I} \bar{\mu}_i g_i(\bar{x}, w_{ij}) = \frac{1}{\sum_{k\in K} \tilde{\lambda}_k} \sum_{i\in I} \tilde{\mu}_i \Big(\sum_{j=1}^{j_i} \tilde{\mu}_{ij}\Big) g_i(\bar{x}, w_{ij}) = 0,$$

we obtain that

$$f(\bar{x}) = f(\bar{x}) + \langle \bar{\mu}, g(\bar{x}, w_{ij})\rangle e = L(\bar{x}, \bar{\lambda}, \bar{\mu}). \tag{5.4}$$

(i) Assume to contrary that $(\bar{x}, \bar{\lambda}, \bar{\mu}) \notin$ loc quasi- ϵ-S^w(RD). Then for any neighborhood U of (z, λ, μ) there is $(z, \lambda, \mu) \in C_D \cap U$ such that

$$L_k(z, \lambda, \mu) > L_k(\bar{x}, \bar{\lambda}, \bar{\mu}) + \epsilon_k \|(\bar{x}, \bar{\lambda}, \bar{\mu}) - (z, \lambda, \mu)\|,\ \forall k \in K, \tag{5.5}$$

where $L_k(z, \lambda, \mu) := f_k(z) + \sum_{i\in I} \tilde{\mu}_i \Big(\sum_{j=1}^{j_i} \tilde{\mu}_{ij}\Big) g(z, w_{ij}),\ \ w_{ij} \in \Omega_i(z)$, $i \in I,\ j = 1, \ldots, j_i$ for each $k \in K$. Since $(z, \lambda, \mu) \in C_D \cap U$, there exist $\lambda_k \geq 0,\ z_k^* \in \partial f_k(z),\ k \in K$ with $\sum_{k\in K} \lambda_k = 1,\ \mu_i \geq 0,\ i \in I,\ \mu_{ij} \geq 0,\ x_{ij}^* \in \partial_x g_i(z, w_{ij}),\ w_{ij} \in \Omega_i(z),\ j = 1, \ldots, j_i$, and $u^* \in B_{\mathbb{R}^{n*}}$ such that

$$-\Big(\sum_{k\in K} \lambda_k z_k^* + \sum_{i\in I} \mu_i \Big(\sum_{j=1}^{j_i} \mu_{ij} x_{ij}^*\Big) + \sum_{k\in K} \lambda_k \epsilon_k u^*\Big) \in N(z; \Omega). \tag{5.6}$$

By definition of (2.1) and the generalized convexity of $(f;\, g)$ on Ω at z, we infer from (5.6) that for such $\bar{x}$ there exists $v \in N(z;\, \Omega)^\circ$ such that

$$\begin{aligned} 0 \quad &\leq \sum_{k\in K} \lambda_k \langle z_k^*, v\rangle + \sum_{i\in I} \mu_i \Big(\sum_{j=1}^{j_i} \mu_{ij} \langle x_{ij}^*, v\rangle\Big) + \sum_{k\in K} \lambda_k \epsilon_k \langle u^*, v\rangle \\ &\leq \sum_{k\in K} \lambda_k [f_k(\bar{x}) - f_k(z)] \\ &+ \sum_{i\in I} \mu_i \Big(\sum_{j=1}^{j_i} \mu_{ij} [g_i(\bar{x}, w_{ij}) - g_i(z, w_{ij})]\Big) + \sum_{k\in K} \lambda_k \epsilon_k \|\bar{x} - z\|. \end{aligned}$$

Moreover, due to $\bar{x} \in C$, we have

$$\sum_{i\in I}\mu_i\Big(\sum_{j=1}^{j_i}\mu_{ij}g_i(\bar{x}, w_{ij})\Big) \leq 0,\ i \in I.$$

and then

$$\begin{aligned}&\sum_{k\in K}\lambda_k f_k(z) + \sum_{i\in I}\mu_i\Big(\sum_{j=1}^{j_i}\mu_{ij}g_i(z, w_{ij})\Big)\\ &\leq \sum_{k\in K}\lambda_k f_k(\bar{x}) + \sum_{i\in I}\mu_i\Big(\sum_{j=1}^{j_i}\mu_{ij}g_i(\bar{x}, w_{ij})\Big) + \sum_{k\in K}\lambda_k\epsilon_k\|\bar{x} - z\|.\end{aligned}$$

From the above inequality, thus

$$\begin{aligned}&\sum_{k\in K}\lambda_k f_k(z) + \sum_{i\in I}\mu_i\Big(\sum_{j=1}^{j_i}\mu_{ij}g_i(z, w_{ij})\Big)\\ &\leq \sum_{k\in K}\lambda_k f_k(\bar{x}) + \sum_{k\in K}\lambda_k\epsilon_k\|\bar{x} - z\|.\end{aligned} \tag{5.7}$$

By (5.5) and definition of $L_k(z, \lambda, \mu)$, we obtain that

$$f_k(z) + \sum_{i\in I}\mu_i\Big(\sum_{j=1}^{j_i}\mu_{ij}g_i(z, w_{ij})\Big) > f_k(\bar{x}) + \epsilon_k\|(\bar{x}, \bar{\lambda}, \bar{\mu}) - (z, \lambda, \mu)\|,\ \forall k \in K.$$

Since $\sum_{k\in K}\lambda_k = 1$, so

$$\begin{aligned}\sum_{k\in K}\lambda_k f_k(z) + \sum_{i\in I}\mu_i\Big(\sum_{j=1}^{j_i}\mu_{ij}g_i(z, w_{ij})\Big) &> \sum_{k\in K}\lambda_k f_k(\bar{x})\\ &+ \sum_{k\in K}\lambda_k\epsilon_k\|(\bar{x}, \bar{\lambda}, \bar{\mu}) - (z, \lambda, \mu)\|,\end{aligned}$$

a contradiction due to the fact that (5.7), so the proof of (i) is completed.

(ii) Suppose that $(\bar{x}, \bar{\lambda}, \bar{\mu}) \notin$ loc quasi- ϵ-$S(RD)$. Then for any neighborhood U of (z, λ, μ), there exists $(z, \lambda, \mu) \in C_D \cap U$ such that

$$L_k(z, \lambda, \mu) \geq L_k(\bar{x}, \bar{\lambda}, \bar{\mu}) + \epsilon_k\|(\bar{x}, \bar{\lambda}, \bar{\mu}) - (z, \lambda, \mu)\|,\ \forall k \in K, \tag{5.8}$$

where at least one inequality is strict. Let $\bar{x} \neq z$. Since $(z, \lambda, \mu) \in C_D \cap U$, by employing inequality (5.6) holds true. Again, by using definition of polar cone (2.1) together the generalized convexity of $(f; g)$ on Ω at z, we also have (5.6) that for such $\bar{x}$, there is $v \in N(z; \Omega)^\circ$ such that

$$\begin{aligned}0 &\leq \sum_{k\in K}\lambda_k\langle z_k^*, v\rangle + \sum_{i\in I}\mu_i\Big(\sum_{j=1}^{j_i}\mu_{ij}\langle x_{ij}^*, v\rangle\Big) + \sum_{k\in K}\lambda_k\epsilon_k\langle u^*, v\rangle\\ &< \sum_{k\in K}\lambda_k[f_k(\bar{x}) - f_k(z)] + \sum_{i\in I}\mu_i\Big(\sum_{j=1}^{j_i}\mu_{ij}[g_i(\bar{x}, w_{ij}) - g_i(z, w_{ij})]\Big)\\ &+ \sum_{k\in K}\lambda_k\epsilon_k\|\bar{x} - z\|.\end{aligned}$$

Similarly, if you see (i) will find this below inequality is completed, that is,

$$\sum_{k\in K}\lambda_k f_k(z)+\sum_{i\in I}\mu_i\Big(\sum_{j=1}^{j_i}\mu_{ij}g_i(z,w_{ij})\Big)<\sum_{k\in K}\lambda_k f_k(\bar{x})+\sum_{k\in K}\lambda_k\epsilon_k\|\bar{x}-z\| \quad (5.9)$$

and

$$\sum_{k\in K}\lambda_k f_k(z)+\sum_{i\in I}\mu_i\Big(\sum_{j=1}^{j_i}\mu_{ij}g_i(z,w_{ij})\Big)>\sum_{k\in K}\lambda_k f_k(\bar{x}) + \sum_{k\in K}\lambda_k\epsilon_k\|(\bar{x},\bar{\lambda},\bar{\mu})-(z,\lambda,\mu)\|,$$

which is ridiculous, and complete the proof.

■

Finally, in this section. we will present converse-like duality theorem for quasi ϵ-(weakly) Pareto solutions between the primal problem (RD) and the dual problem (RD).

Theorem 5.3. (Converse Duality) Let $(\bar{x},\bar{\lambda},\bar{\mu})\in C_D$ such that $f(\bar{x})=L(\bar{z},\bar{\lambda},\bar{\mu})$.

(i) If $\bar{x}\in C$ and $(f;g)$ is a generalized convex on Ω at $\bar{x}$, then $\bar{x}\in$ loc quasi-ϵ-S^w(RP).

(ii) If $\bar{x}\in C$ and $(f;g)$ is a strictly generalized convex on Ω at $\bar{x}$, then $\bar{x}\in$ loc quasi-ϵ-S(RP).

Proof. Assume that $(\bar{x},\bar{\lambda},\bar{\mu})\in C_D$. Then, there exist $\bar{\lambda}_k\geq 0$, $z_k^*\in\partial f_k(\bar{x})$, $k\in K$ with $\sum_{k\in K}\bar{\lambda}_k=1$, $\bar{\mu}_i\geq 0$, $i\in I$, $\bar{\mu}_{ij}\geq 0$, there exist $x_{ij}^*\in\partial_x g_i(\bar{x},w_{ij})$, $w_{ij}\in\Omega_i(\bar{x})$, $j=1,\ldots,j_i$, $j_i\in\mathbb{N}$, $\sum_{j=1}^{j_i}\bar{\mu}_{ij}=1$ and $u^*\in\mathbb{B}_{\mathbb{R}^{n*}}$ such that

$$\begin{aligned}&-\Big(\sum_{k\in K}\bar{\lambda}_k z_k^*+\sum_{i\in I}\bar{\mu}_i\Big(\sum_{j=1}^{j_i}\bar{\mu}_{ij}x_{ij}^*\Big)+\sum_{k\in K}\bar{\lambda}_k\epsilon_k u^*\Big)\in N(\bar{x};\Omega),\\ &\mu_i\sup_{w_i\in\Omega_i}g_i(\bar{x},w_{ij})=0,\ i\in I.\end{aligned} \quad (5.10)$$

(i) Assume on the contrary that $\bar{x}\notin$ loc quasi-ϵ-S^w(*RP*). Then for any neighborhood U of x, there is $\hat{x}\in C\cap U$ such that

$$f_k(\hat{x})+\epsilon_k\|\hat{x}-\bar{x}\|<f_k(\bar{x}),\ k\in K,$$

where at least one inequality is strict. This follow that $\hat{x}\neq\bar{x}$ and therefore

$$\sum_{k\in K}\bar{\lambda}_k f_k(\hat{x})+\sum_{k\in K}\bar{\lambda}_k\epsilon_k\|\hat{x}-\bar{x}\|<\sum_{k\in K}\bar{\lambda}_k f_k(\bar{x}),\ k\in K, \quad (5.11)$$

where $\sum_{k\in K}\bar{\lambda}_k\neq 0$. By definition of polar cone (2.1) and the generalized convexity of $(f;g)$ on Ω at $\bar{x}$, we obtain (5.10) that for $\hat{x}$ above as there exists

$v \in N(\bar{x};\Omega)^\circ$,

$$0 \leq \sum_{k\in K} \bar{\lambda}_k \langle z_k^*, v\rangle + \sum_{i\in I} \bar{\mu}_i \Big(\sum_{j=1}^{j_i} \bar{\mu}_{ij} \langle x_{ij}^*, v\rangle \Big) + \sum_{k\in K} \lambda_k \epsilon_k \langle u^*, v\rangle$$

$$< \sum_{k\in K} \bar{\lambda}_k [f_k(\hat{x}) - f_k(\bar{x})] + \sum_{i\in I} \bar{\mu}_i \Big(\sum_{j=1}^{j_i} \bar{\mu}_{ij} [g_i(\hat{x}, w_{ij}) - g_i(\bar{x}, w_{ij})] \Big)$$

$$+ \sum_{k\in K} \bar{\lambda}_k \epsilon_k \|\hat{x} - \bar{x}\|.$$

Since $f(\bar{x}) = L(\bar{x}, \bar{\lambda}, \bar{\mu})$, $\sum_{i\in I} \bar{\mu}_i \Big(\sum_{j=1}^{j_i} \bar{\mu}_{ij} g_i(\bar{x}, w_{ij}) \Big) = 0$, $i \in I$, and due to $\hat{x} \in C$, we have $\sum_{i\in I} \bar{\mu}_i \Big(\sum_{j=1}^{j_i} \bar{\mu}_{ij} g_i(\hat{x}, w_{ij}) \Big) \leq 0$. From the above inequality, we obtain that

$$\begin{aligned} \sum_{k\in K} \bar{\lambda}_k f_k(\bar{x}) &= \sum_{k\in K} \bar{\lambda}_k f_k(\bar{x}) + \sum_{i\in I} \bar{\mu}_i \Big(\sum_{j=1}^{j_i} \bar{\mu}_{ij} g_i(\bar{x}, w_{ij}) \Big) \\ &\leq \sum_{k\in K} \bar{\lambda}_k f_k(\hat{x}) + \sum_{i\in I} \bar{\mu}_i \Big(\sum_{j=1}^{j_i} \bar{\mu}_{ij} g_i(\hat{x}, w_{ij}) \Big) \\ &+ \sum_{k\in K} \bar{\lambda}_k \epsilon_k \|\hat{x} - \bar{x}\| \\ &\leq \sum_{k\in K} \bar{\lambda}_k f_k(\hat{x}) + \sum_{k\in K} \bar{\lambda}_k \epsilon_k \|\hat{x} - \bar{x}\|, \end{aligned}$$

which a contradiction, and so the proof of (i) has been established.

(ii) In the same way, if you see proof of (i) then (ii) is completed by using the one has the generalized convexity of $(f;\, g)$ is strict on Ω at $\bar{x}$ instead of the generalized convexity of $(f;\, g)$ on Ω at this considered point.

■

We finish this section by the interesting in (CQ) imposed in Theorem 5.2 plays an important role. That is, it proves that if $\bar{x}$ is a quasi-ϵ-(weakly) Pareto solution of the problem at which the condition (CQ) is not satisfied, then we might not find out a pair $(\bar{\lambda}, \bar{\mu})$ in Theorem 5.2 such that $(\bar{x}, \bar{\lambda}, \bar{\mu})$ belongs to the feasible set of the corresponding dual problem. even in the case convex. wait the example.

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