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On the Recursive Sequence $x_{n+1} = \frac{x_{n-7}}{1+x_{n-1}x_{n-3}x_{n-5}}$

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Abstract In this paper we are going to analyze the following difference equation

$$x_{n+1} = \frac{x_{n-7}}{1+x_{n-1}x_{n-3}x_{n-5}} \quad n = 0, 1, 2, \ldots,$$

where $x_{-7}, x_{-6}, x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0 \in (0, \infty)$.

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1. Introduction

Difference equations appear naturally as discrete analogs and as numerical solutions of differential and delay differential equations, having applications in biology, ecology, physics.

Difference equations are used in a variety of contexts, such as in economics to model the evolution through time of variables such as gross domestic product, the inflation rate, the exchange rate, etc. They are used in modeling such time series because values of these variables are only measured at discrete intervals. In econometric applications, linear difference equations are modeled with stochastic terms in the form of autoregressive (AR) models and in models such as vector autoregression (VAR) and autoregressive moving average (ARMA) models that combine AR with other features.

Recently, a high attention to studying the periodic nature of nonlinear difference equations has been attracted. For some recent results concerning the periodic nature of scalar nonlinear difference equations, among other problems, see the references [1-16].

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Cinar [2, 3] studied the following problems with positive initial values:

$$x_{n+1} = \frac{x_{n-1}}{1 + ax_n x_{n-1}},$$

$$x_{n+1} = \frac{x_{n-1}}{-1 + ax_n x_{n-1}},$$

$$x_{n+1} = \frac{ax_{n-1}}{1 + bx_n x_{n-1}},$$

for $n = 0, 1, 2, \ldots,$ respectively.

De Vault et. al [16] studied the following problems

$$x_{n+1} = \frac{A}{x_n} + \frac{1}{x_{n-2}}$$

for $n = 0, 1, 2, \ldots$ and proved it has positive when $A \in (0, \infty)$

Stevic et. al. [15] studied the solvability of the following product-type system of difference equations of the second order

$$z_{n+1} = \frac{z_n^a}{w_{n-1}^b}, w_{n+1} = \frac{w_n^c}{z_{n-1}^d}, n \in \mathbb{N}_0,$$

where $a, b, c, d \in \mathbb{Z}, z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C}$.

Elsayed in a series of papers (see [4]-[10]) studied the behavior of the solution of the following difference equation:

$$x_{n+1} = ax_{n-1} + \frac{bx_n x_{n-1}}{cx_n dx_{n-2}}, \quad n = 0, 1, \ldots,$$

where the initial conditions $x_{-2}x_{-1}, x_0$ are arbitrary positive real numbers and a, b, c, d are positive constants.

Simsek et. al. in a series of papers (see [11–13]) studied the following problems with positive initial values

$$x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}}$$

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2}}$$

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-1}x_{n-3}}$$

$$x_{n+1} = \frac{x_{n-3}}{1 + x_n x_{n-1} x_{n-2}}$$

for $n = 0, 1, 2, \ldots$ in [5, 6, 7, 8] respectively.

In this paper we are going to study the following nonlinear difference equation

$$x_{n+1} = \frac{x_{n-7}}{1 + x_{n-1}x_{n-3}x_{n-5}}, \quad n = 0, 1, 2, \ldots, \tag{1.1}$$

where $x_{-7}, x_{-6}, x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0 \in (0, \infty)$.

2. Main Results

Theorem 2.1. *Consider the difference equation (1.1). Then the following statements are true.*

(a) *The sequences* $(x_{8n-7}), (x_{8n-6}), (x_{8n-5}), (x_{8n-4}), (x_{8n-3}), (x_{8n-2}), (x_{8n-1}), (x_{8n})$ *are decreasing and there exist* $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8 \geq 0$ *such that*

$$\lim_{n\to\infty} x_{8n-7} = a_1,\ \lim_{n\to\infty} x_{8n-6} = a_2,\ \lim_{n\to\infty} x_{8n-5} = a_3,\ \lim_{n\to\infty} x_{8n-4} = a_4,$$

$$\lim_{n\to\infty} x_{8n-3} = a_5,\ \lim_{n\to\infty} x_{8n-2} = a_6,\ \lim_{n\to\infty} x_{8n-1} = a_7,\ \lim_{n\to\infty} x_{8n} = a_8.$$

(b) $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, \ldots)$ *is a solution of equation (1.1) having period eight.*

(c) $a_1 \cdot a_3 \cdot a_5 \cdot a_7 = 0,\ a_2 \cdot a_4 \cdot a_6 \cdot a_8 = 0.$

(d) $n_0 \in \mathbb{N}$ *such that* $x_{n+1} \leq x_{n-5}$ *for all* $n \geq n_0$*, then*

$$\lim_{n\to\infty} x_n = 0.$$

(e) *The following formulas*

$$x_{8n+1} = x_{-7}\left(1 - \frac{x_{-1}x_{-3}x_{-5}}{1 + x_{-1}x_{-3}x_{-5}} \sum_{j=0}^{n} \prod_{i=1}^{4j} \frac{1}{1 + x_{2i-5}x_{2i-3}x_{2i-1}}\right),$$

$$x_{8n+2} = x_{-6}\left(1 - \frac{x_{0}x_{-2}x_{-4}}{1 + x_{0}x_{-2}x_{-4}} \sum_{j=0}^{n} \prod_{i=1}^{4j} \frac{1}{1 + x_{2i-4}x_{2i-2}x_{2i}}\right),$$

$$x_{8n+3} = x_{-5}\left(1 - \frac{x_{-1}x_{-3}x_{-7}}{1 + x_{-1}x_{-3}x_{-5}} \sum_{j=0}^{n} \prod_{i=1}^{4j+1} \frac{1}{1 + x_{2i-5}x_{2i-3}x_{2i-1}}\right),$$

$$x_{8n+4} = x_{-4}\left(1 - \frac{x_{0}x_{-2}x_{-6}}{1 + x_{0}x_{-2}x_{-4}} \sum_{j=0}^{n} \prod_{i=1}^{4j+1} \frac{1}{1 + x_{2i-4}x_{2i-2}x_{2i}}\right),$$

$$x_{8n+5} = x_{-3}\left(1 - \frac{x_{-1}x_{-5}x_{-7}}{1 + x_{-1}x_{-3}x_{-5}} \sum_{j=0}^{n} \prod_{i=1}^{4j+2} \frac{1}{1 + x_{2i-5}x_{2i-3}x_{2i-1}}\right),$$

$$x_{8n+6} = x_{-2}\left(1 - \frac{x_{0}x_{-4}x_{-6}}{1 + x_{0}x_{-2}x_{-4}} \sum_{j=0}^{n} \prod_{i=1}^{4j+2} \frac{1}{1 + x_{2i-4}x_{2i-2}x_{2i}}\right),$$

$$x_{8n+7} = x_{-1}\left(1 - \frac{x_{-3}x_{-5}x_{-7}}{1 + x_{-1}x_{-3}x_{-5}} \sum_{j=0}^{n} \prod_{i=1}^{4j+3} \frac{1}{1 + x_{2i-5}x_{2i-3}x_{2i-1}}\right),$$

$$x_{8n+8} = x_{0}\left(1 - \frac{x_{-2}x_{-4}x_{-6}}{1 + x_{0}x_{-2}x_{-4}} \sum_{j=0}^{n} \prod_{i=1}^{4j+3} \frac{1}{1 + x_{2i-4}x_{2i-2}x_{2i}}\right),$$

hold.

(f) *If* $x_{8n+1} \to a_1 \neq 0$, $x_{8n+3} \to a_3 \neq 0$, $x_{8n+5} \to a_5 \neq 0$, *then* $x_{8n+7} \to 0$ *as* $n \to \infty$. *If* $x_{8n+2} \to a_2 \neq 0$, $x_{8n+4} \to a_4 \neq 0$, $x_{8n+6} \to a_6 \neq 0$, *then* $x_{8n+8} \to 0$ *as* $n \to \infty$.

Proof. (a) Firstly, we consider the equation (1.1). From this equation, we obtain

$$x_{n+1}\left(1+x_{n-1}x_{n-3}x_{n-5}\right)=x_{n-7}.$$

If $x_{n-1}x_{n-3}x_{n-5}\in(0,\infty)$, then $(1+x_{n-1}x_{n-3}x_{n-5})\in(1,\infty)$. Since $x_{n+1}<x_{n-7}$, $n\in\mathbb{N}$, we obtain that there exist

$$\lim_{n\to\infty}x_{8n-7}=a_1,\ \lim_{n\to\infty}x_{8n-6}=a_2,\ \lim_{n\to\infty}x_{8n-5}=a_3,\ \lim_{n\to\infty}x_{8n-4}=a_4,$$
$$\lim_{n\to\infty}x_{8n-3}=a_5,\ \lim_{n\to\infty}x_{8n-2}=a_6,\ \lim_{n\to\infty}x_{8n-1}=a_7,\ \lim_{n\to\infty}x_{8n}=a_8.$$

(b) $(a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,\ldots)$ is a solution of equation (1.1) having period eight.

(c) In view of the equation (1.1),

$$x_{8n+1}=\frac{x_{8n-7}}{1+x_{8n-1}x_{8n-3}x_{8n-5}}.$$

If the limits are put on both sides of the above equality

$$\lim_{n\to\infty}x_{8n+1}=\frac{x_{8n-7}}{1+x_{8n-1}x_{8n-3}x_{8n-5}}$$

is obtained. Then,

$$a_1=\frac{a_1}{1+a_7a_5a_3}\Rightarrow a_1+a_1\cdot a_3a_5a_7=a_1\Rightarrow a_1\cdot a_3a_5a_7=0.$$

Also, we obtain

$$x_{8n+2}=\frac{x_{8n-6}}{1+x_{8n}x_{8n-2}x_{8n-4}}.$$

If the limits are put on both sides of the above equality

$$\lim_{n\to\infty}x_{8n+2}=\frac{x_{8n-6}}{1+x_{8n}x_{8n-2}x_{8n-4}}$$

is obtained. Then,

$$a_1=\frac{a_2}{1+a_6a_4a_2}\Rightarrow a_2+a_2\cdot a_4a_6a_8=a_2\Rightarrow a_2\cdot a_4a_6a_8=0.$$

(d) $n_0\in\mathbb{N}$ such that $x_{n+1}\leq x_{n-5}$ for all $n\geq n_0$, is existed; then,
$a_1\leq a_3\leq a_5\leq a_7\leq a_1$ since $a_1\cdot a_3\cdot a_5\cdot a_7=0$,
$a_2\leq a_4\leq a_6\leq a_8\leq a_2$ since $a_2\cdot a_4\cdot a_6\cdot a_8=0$, the results are obtained above.

(e) Subracting x_{n-7} from the left and right-hand sides in equation (1.1)

$$x_{n+1}-x_{n-7}=\frac{1}{1+x_{n-1}x_{n-3}x_{n-5}}\left(x_{n-1}-x_{n-9}\right)$$

is obtained and the following formula is produced below, for $n\geq 2$,

$$\begin{aligned}x_{2n-3}-x_{2n-11}&=(x_1-x_{-7})\prod_{i=1}^{n-2}\tfrac{1}{1+x_{2i-5}x_{2i-3}x_{2i-1}},\\ x_{2n-2}-x_{2n-10}&=(x_2-x_{-6})\prod_{i=1}^{n-2}\tfrac{1}{1+x_{2i-4}x_{2i-2}x_{2i}},\end{aligned}\tag{2.1}$$

holds. Replacing n by $4j$ in (2.1) and summing from $j=0$ to $j=n$, we obtain:

$$x_{8n+1}-x_{-7}=(x_1-x_{-7})\sum_{j=0}^{n}\prod_{i=1}^{4j}\frac{1}{1+x_{2i-5}x_{2i-3}x_{2i-1}},$$

$$x_{8n+2}-x_{-6}=(x_2-x_{-6})\sum_{j=0}^{n}\prod_{i=1}^{4j}\frac{1}{1+x_{2i-4}x_{2i-2}x_{2i}},$$

Also, replacing n by $4j+1$ in (2.1) and summing from $j=0$ to $j=n$, we obtain:

$$x_{8n+3}-x_{-5}=(x_3-x_{-5})\sum_{j=0}^{n}\prod_{i=1}^{4j+1}\frac{1}{1+x_{2i-5}x_{2i-3}x_{2i-1}},$$

$$x_{8n+4}-x_{-4}=(x_4-x_{-4})\sum_{j=0}^{n}\prod_{i=1}^{4j+1}\frac{1}{1+x_{2i-4}x_{2i-2}x_{2i}},$$

Also, replacing n by $4j+2$ in (2.1) and summing from $j=0$ to $j=n$, we obtain:

$$x_{8n+5}-x_{-3}=(x_5-x_{-3})\sum_{j=0}^{n}\prod_{i=1}^{4j+2}\frac{1}{1+x_{2i-5}x_{2i-3}x_{2i-1}},$$

$$x_{8n+6}-x_{-2}=(x_6-x_{-2})\sum_{j=0}^{n}\prod_{i=1}^{4j+2}\frac{1}{1+x_{2i-4}x_{2i-2}x_{2i}},$$

Also, replacing n by $4j+3$ in (2.1) and summing from $j=0$ to $j=n$, we obtain:

$$x_{8n+7}-x_{-1}=(x_7-x_{-1})\sum_{j=0}^{n}\prod_{i=1}^{4j+3}\frac{1}{1+x_{2i-5}x_{2i-3}x_{2i-1}},$$

$$x_{8n+8}-x_{0}=(x_8-x_{0})\sum_{j=0}^{n}\prod_{i=1}^{4j+3}\frac{1}{1+x_{2i-4}x_{2i-2}x_{2i}},$$

Now, we obtained of the above formulas:

$$x_{8n+1}=x_{-7}\left(1-\frac{x_{-1}x_{-3}x_{-5}}{1+x_{-1}x_{-3}x_{-5}}\sum_{j=0}^{n}\prod_{i=1}^{4j}\frac{1}{1+x_{2i-5}x_{2i-3}x_{2i-1}}\right),$$

$$x_{8n+2}=x_{-6}\left(1-\frac{x_{0}x_{-2}x_{-4}}{1+x_{0}x_{-2}x_{-4}}\sum_{j=0}^{n}\prod_{i=1}^{4j}\frac{1}{1+x_{2i-4}x_{2i-2}x_{2i}}\right),$$

$$x_{8n+3}=x_{-5}\left(1-\frac{x_{-1}x_{-3}x_{-7}}{1+x_{-1}x_{-3}x_{-5}}\sum_{j=0}^{n}\prod_{i=1}^{4j+1}\frac{1}{1+x_{2i-5}x_{2i-3}x_{2i-1}}\right),$$

$$x_{8n+4}=x_{-4}\left(1-\frac{x_{0}x_{-2}x_{-6}}{1+x_{0}x_{-2}x_{-4}}\sum_{j=0}^{n}\prod_{i=1}^{4j+1}\frac{1}{1+x_{2i-4}x_{2i-2}x_{2i}}\right),$$

$$x_{8n+5}=x_{-3}\left(1-\frac{x_{-1}x_{-5}x_{-7}}{1+x_{-1}x_{-3}x_{-5}}\sum_{j=0}^{n}\prod_{i=1}^{4j+2}\frac{1}{1+x_{2i-5}x_{2i-3}x_{2i-1}}\right),$$

$$x_{8n+6}=x_{-2}\left(1-\frac{x_{0}x_{-4}x_{-6}}{1+x_{0}x_{-2}x_{-4}}\sum_{j=0}^{n}\prod_{i=1}^{4j+2}\frac{1}{1+x_{2i-4}x_{2i-2}x_{2i}}\right),$$

$$x_{8n+7} = x_{-1}\left(1 - \frac{x_{-3}x_{-5}x_{-7}}{1 + x_{-1}x_{-3}x_{-5}}\sum_{j=0}^{n}\prod_{i=1}^{4j+3}\frac{1}{1 + x_{2i-5}x_{2i-3}x_{2i-1}}\right),$$

$$x_{8n+8} = x_{0}\left(1 - \frac{x_{-2}x_{-4}x_{-6}}{1 + x_{0}x_{-2}x_{-4}}\sum_{j=0}^{n}\prod_{i=1}^{4j+3}\frac{1}{1 + x_{2i-4}x_{2i-2}x_{2i}}\right).$$

(f) Suppose that $a_1 = a_3 = a_5 = a_7 = 0$. By (e), we have

$$\lim_{n\to\infty} x_{8n+1} = \lim_{n\to\infty} x_{-7}\left(1 - \frac{x_{-1}x_{-3}x_{-5}}{1 + x_{-1}x_{-3}x_{-5}}\sum_{j=0}^{n}\prod_{i=1}^{4j}\frac{1}{1 + x_{2i-5}x_{2i-3}x_{2i-1}}\right),$$

$$a_1 = x_{-7}\left(1 - \frac{x_{-1}x_{-3}x_{-5}}{1 + x_{-1}x_{-3}x_{-5}}\sum_{j=0}^{\infty}\prod_{i=1}^{4j}\frac{1}{1 + x_{2i-5}x_{2i-3}x_{2i-1}}\right),$$

$$a_1 = 0 \Rightarrow \frac{1 + x_{-1}x_{-3}x_{-5}}{x_{-1}x_{-3}x_{-5}} = \sum_{j=0}^{\infty}\prod_{i=1}^{4j}\frac{1}{1 + x_{2i-5}x_{2i-3}x_{2i-1}}. \tag{2.2}$$

Similarly

$$\lim_{n\to\infty} x_{8n+3} = \lim_{n\to\infty} x_{-5}\left(1 - \frac{x_{-1}x_{-3}x_{-7}}{1 + x_{-1}x_{-3}x_{-5}}\sum_{j=0}^{n}\prod_{i=1}^{4j+1}\frac{1}{1 + x_{2i-5}x_{2i-3}x_{2i-1}}\right),$$

$$a_3 = x_{-5}\left(1 - \frac{x_{-1}x_{-3}x_{-7}}{1 + x_{-1}x_{-3}x_{-5}}\sum_{j=0}^{\infty}\prod_{i=1}^{4j+1}\frac{1}{1 + x_{2i-5}x_{2i-3}x_{2i-1}}\right),$$

$$a_3 = 0 \Rightarrow \frac{1 + x_{-1}x_{-3}x_{-5}}{x_{-1}x_{-3}x_{-7}} = \sum_{j=0}^{\infty}\prod_{i=1}^{4j+1}\frac{1}{1 + x_{2i-5}x_{2i-3}x_{2i-1}}. \tag{2.3}$$

From the equation (2.2) and (2.3);

$$\frac{1 + x_{-1}x_{-3}x_{-5}}{x_{-1}x_{-3}x_{-5}} = \sum_{j=0}^{\infty}\prod_{i=1}^{4j}\frac{1}{1 + x_{2i-5}x_{2i-3}x_{2i-1}} >$$

$$\frac{1 + x_{-1}x_{-3}x_{-5}}{x_{-1}x_{-3}x_{-7}} = \sum_{j=0}^{\infty}\prod_{i=1}^{4j+1}\frac{1}{1 + x_{2i-5}x_{2i-3}x_{2i-1}}$$

$$\frac{1 + x_{-1}x_{-3}x_{-5}}{x_{-1}x_{-3}x_{-5}} > \frac{1 + x_{-1}x_{-3}x_{-5}}{x_{-1}x_{-3}x_{-7}},$$

$$\frac{1}{x_{-1}x_{-3}x_{-5}} > \frac{1}{x_{-1}x_{-3}x_{-7}},$$

$$x_{-1}x_{-3}x_{-7} > x_{-1}x_{-3}x_{-5} \Rightarrow x_{-7} > x_{-5}.$$

$$\lim_{n\to\infty} x_{8n+5} = \lim_{n\to\infty} x_{-3}\left(1 - \frac{x_{-1}x_{-5}x_{-7}}{1 + x_{-1}x_{-3}x_{-5}}\sum_{j=0}^{n}\prod_{i=1}^{4j+2}\frac{1}{1 + x_{2i-5}x_{2i-3}x_{2i-1}}\right),$$

$$a_5 = x_{-3}\left(1 - \frac{x_{-1}x_{-5}x_{-7}}{1+x_{-1}x_{-3}x_{-5}} \sum_{j=0}^{\infty} \prod_{i=1}^{4j+2} \frac{1}{1+x_{2i-5}x_{2i-3}x_{2i-1}}\right),$$

$$a_5 = 0 \Rightarrow \frac{1+x_{-1}x_{-3}x_{-5}}{x_{-1}x_{-5}x_{-7}} = \sum_{j=0}^{\infty} \prod_{i=1}^{4j+2} \frac{1}{1+x_{2i-5}x_{2i-3}x_{2i-1}}. \tag{2.4}$$

From the equation (2.3) and (2.4);

$$\frac{1+x_{-1}x_{-3}x_{-5}}{x_{-1}x_{-3}x_{-7}} = \sum_{j=0}^{\infty} \prod_{i=1}^{4j+1} \frac{1}{1+x_{2i-5}x_{2i-3}x_{2i-1}} >$$
$$\frac{1+x_{-1}x_{-3}x_{-5}}{x_{-1}x_{-5}x_{-7}} = \sum_{j=0}^{\infty} \prod_{i=1}^{4j+2} \frac{1}{1+x_{2i-5}x_{2i-3}x_{2i-1}}$$
$$\frac{1+x_{-1}x_{-3}x_{-5}}{x_{-1}x_{-3}x_{-7}} > \frac{1+x_{-1}x_{-3}x_{-5}}{x_{-1}x_{-5}x_{-7}}$$
$$\frac{1}{x_{-1}x_{-3}x_{-7}} > \frac{1}{x_{-1}x_{-5}x_{-7}}$$
$$x_{-1}x_{-5}x_{-7} > x_{-1}x_{-3}x_{-7} \Rightarrow x_{-5} > x_{-3}.$$

$$\lim_{n\to\infty} x_{8n+7} = \lim_{n\to\infty} x_{-1}\left(1 - \frac{x_{-3}x_{-5}x_{-7}}{1+x_{-1}x_{-3}x_{-5}} \sum_{j=0}^{n} \prod_{i=1}^{4j+3} \frac{1}{1+x_{2i-5}x_{2i-3}x_{2i-1}}\right),$$

$$a_7 = x_{-1}\left(1 - \frac{x_{-3}x_{-5}x_{-7}}{1+x_{-1}x_{-3}x_{-5}} \sum_{j=0}^{\infty} \prod_{i=1}^{4j+3} \frac{1}{1+x_{2i-5}x_{2i-3}x_{2i-1}}\right),$$

$$a_7 = 0 \Rightarrow \frac{1+x_{-1}x_{-3}x_{-5}}{x_{-3}x_{-5}x_{-7}} = \sum_{j=0}^{\infty} \prod_{i=1}^{4j+3} \frac{1}{1+x_{2i-5}x_{2i-3}x_{2i-1}}. \tag{2.5}$$

From the equation (2.4) and (2.5);

$$\frac{1+x_{-1}x_{-3}x_{-5}}{x_{-1}x_{-5}x_{-7}} = \sum_{j=0}^{\infty} \prod_{i=1}^{4j+2} \frac{1}{1+x_{2i-5}x_{2i-3}x_{2i-1}} >$$
$$\frac{1+x_{-1}x_{-3}x_{-5}}{x_{-3}x_{-5}x_{-7}} = \sum_{j=0}^{\infty} \prod_{i=1}^{4j+3} \frac{1}{1+x_{2i-5}x_{2i-3}x_{2i-1}}$$
$$\frac{1+x_{-1}x_{-3}x_{-5}}{x_{-1}x_{-5}x_{-7}} > \frac{1+x_{-1}x_{-3}x_{-5}}{x_{-3}x_{-5}x_{-7}}$$
$$\frac{1}{x_{-1}x_{-5}x_{-7}} > \frac{1}{x_{-3}x_{-5}x_{-7}}$$
$$x_{-3}x_{-5}x_{-7} > x_{-1}x_{-5}x_{-7} \Rightarrow x_{-3} > x_{-1}.$$

We obtain $x_{-7} > x_{-5} > x_{-3} > x_{-1}$. A contradiction supposing that $a_1 = a_3 = a_5 = a_7 = 0$ is produced. Similarly, for $a_2 = a_4 = a_6 = a_8 = 0$ we have

$x_{-6} > x_{-4} > x_{-2} > x_0$. A contradiction completing the proof of theorem is found.

∎

3. Numerical Examples

Example 3.1. Consider the following equation $x_{n+1} = \frac{x_{n-7}}{1+x_{n-1}.x_{n-3}.x_{n-5}}$. If the initial conditions are selected follows:
$x_{-7} = 0.9,\ x_{-6} = 0.8, x_{-5} = 0.7,\ x_{-4} = 0.6,\ x_{-3} = 0.5, x_{-2} = 0.4,\ x_{-1} = 0.3,\ x_0 = 0.2.$
The graph of the solution is given below.

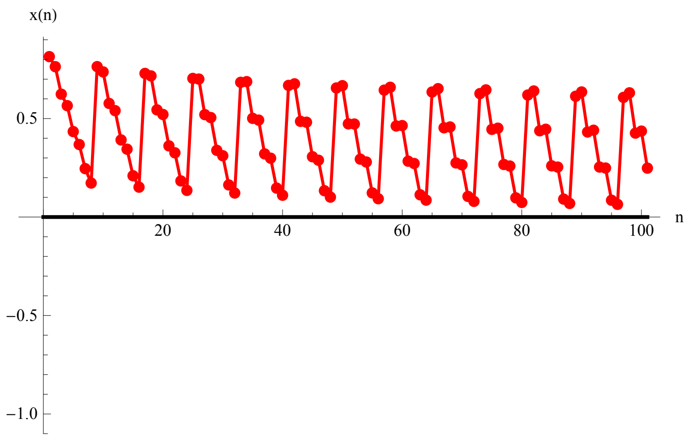

Figure 1. *x_n graph of the solution.*

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