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Essential Norms of Composition Followed by Differentiation Operators from Bloch-type Spaces into Bers-type Spaces

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Abstract In this paper we consider the composition followed by differentation operator DC_φ from Bloch-type space B^α into Bers-type space H^∞_β. We give necessary and sufficient conditions for the boundedness and compactness of the above operator. We also give essential norm of such an operator in term of φ.

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1. Introduction

Let $\mathbb{D}$ be the open unit disc in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the space of analytic functions on $\mathbb{D}$. An analytic function f on $\mathbb{D}$ is said to belong to the Bloch-type space(or α-Bloch space) B^α, $(0 < \alpha < \infty)$ if

$$B_\alpha(f) = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f^{'}(z)| < \infty.$$

The expression $B_\alpha(f)$ defines a seminorm on B^α, while the natural norm is given by $||f||_\alpha = B_\alpha(f) + |f(0)|$. This norm makes B^α into a Banach space.
Let B^α_0 denote the subspace of B^α consisting of those $f \in B^\alpha$ for which

$$(1-|z|^2)^\alpha|f^{'}(z)| \to 0 \qquad \text{as} \qquad |z| \to 1.$$

This space is called the little α-Bloch space.
A function $f \in H(\mathbb{D})$ is said to belong to the Bers-type space H^∞_β, $(0 < \beta < \infty)$ if

$$||f||_\beta = \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|f(z)| < \infty.$$

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Let $H^\infty_{\beta,0}$ be the subspace of H^∞_β which consisting of all $f \in H^\infty_\beta$ satisfying

$$(1-|z|^2)^\beta |f(z)| \to 0 \qquad \text{as} \qquad |z| \to 1.$$

This space is called the little Bers-type space.

For an analytic self-map φ of $\mathbb{D}$, we define the composition operator C_φ on $H(\mathbb{D})$ by

$$C_\varphi f(z) = f(\varphi(z)) \quad \text{for} \quad z \in \mathbb{D}.$$

For more information about these types of operators we refer to the books [1, 2]. It is a well known consequence of Littlewoods subordination principle that the composition operator C_φ is bounded on the classical Hardy and Bergman spaces. It is interesting to provide a function theoretic characterization of when φ induces a bounded or compact composition operator on various spaces. For example, C_φ was studied by S. Stevic in [3], where the boundedness and compactness of C_φ between H^∞ and the α-Bloch spaces are investigated.

Let D be the differentiation operator. The composition followed by differentiation operator DC_φ is defined by

$$DC_\varphi(f) = (fo\varphi)^{'} = f^{'}(\varphi)\varphi^{'}, \qquad f \in H(\mathbb{D}).$$

The composition operator is one of the typical bounded operators, while the differentiation operator is typically unbounded on many analytic function spaces. The operator DC_φ was first studied by Hibschweiler and Portnoy in [4], where the boundedness and compactness of DC_φ between Hardy and Bergman spaces are investigated.The operator DC_φ was studied by S. Ohno in [5], where the boundedness and compactness of DC_φ on Bloch spaces are investigated. Boundedness and compactness of the operator DC_φ on the weighted Bergman spaces were described by S. Stevic in [6]. In 2007, boundedness and compactness of the operator DC_φ between Bloch-type spaces were described by S. Li in [7].

Recall that the essential norm $||T||_e$ of a bounded operator T between Banach spaces X and Y is defined as the distance from T to the compact operators, that is

$$||T||_e = \inf\{||T-K|| : K \text{ is compact}\}.$$

Notic that $||T||_e = 0$ if and only if T is compact, so the estimates on $||T||_e$ nead to conditions for T to be compact.

The essential norm of the composition operator on Bloch spases was studied by A. Montes-Rodriguez in [8]. R. Zhao in [9] give estimates for the essential norms of the composition operators between Bloch-type spaces. Essential norms of the weighted composition operators between Bloch-type spaces are investigated by B. D. Macculuer and R. Zhao in [10]. In [11], S. Stevic, estimate essential norms of the weighted composition operators from Bloch-type spaces to a weighted-type space on the unit ball, and A. H. Sanatpour and M. Hassanlou in [12] were proved the lower and upper bound of the essential norms of weighted composition operators between Zygmund-type spaces and Bloch-type spaces. We have characterized boundedness and compactness of weighted composition operators from Bloch-type into Bers-type spaces in [13]. In [14], we gave necessary and sufficient conditions for boundedness and compactness of weighted composition followed and proceeded by differentiation operator from Bloch-type space into Bers-type space, also we obtained the essential norm estimate of that operator.

In this paper, we study the operator DC_φ from Bloch-type space into Bers-type space. We characterize the boundedness and compactness of $DC_\varphi : B^\alpha \to H^\infty_\beta$ in section 2, and boundedness and compactness of $DC_\varphi : B_0^\alpha \to H_{\beta,0}^\infty$ in section 3. Finally we give lower and upper bounds for the essential norm of the operator $DC_\varphi : B^\alpha \to H_\beta^\infty$ in section 4.

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other.

2. Boundedness and Compactness of $DC_\varphi : B^\alpha \to H_\beta^\infty$

In this section, we characterize the boundedness and compactness of $DC_\varphi : B^\alpha \to H_\beta^\infty$.

Theorem 2.1. *Let φ be an analytic self-map of $\mathbb{D}$ and α and β positive real numbers. Then $DC_\varphi : B^\alpha \to H_\beta^\infty$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta |\varphi^{'}(z)|}{(1-|\varphi(z)|^2)^\alpha} < \infty. \tag{2.1}$$

Proof. First, we obtain sufficiency. For a function $f \in B^\alpha$, there exist a constant C such that

$$\begin{aligned}
\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta |DC_\varphi f(z)| &= \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta |f^{'}(\varphi(z))||\varphi^{'}(z)| \\
&= \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta |\varphi^{'}(z)|}{(1-|\varphi(z)|^2)^\alpha}(1-|\varphi(z)|^2)^\alpha |f^{'}(\varphi(z))| \\
&= \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta |\varphi^{'}(z)|}{(1-|\varphi(z)|^2)^\alpha}||f||_\alpha \\
&= C||f||_\alpha.
\end{aligned}$$

Then the operator DC_φ maps B^α boundedly into H_β^∞.
Now, suppose that $DC_\varphi : B^\alpha \to H_\beta^\infty$ is bounded. For fixed $z_o \in \mathbb{D}$, consider the function f_o defined by

$$f_o(z) = \frac{1-|\varphi(z_o)|^2}{\alpha(1-z\overline{\varphi(z_o)})^\alpha}, \tag{2.2}$$

for $z \in \mathbb{D}$. Then

$$f_o^{'}(z) = \frac{(1-|\varphi(z_o)|^2)\overline{\varphi(z_o)}}{(1-z\overline{\varphi(z_o)})^{\alpha+1}},$$

for $z \in \mathbb{D}$. Hence

$$\begin{aligned}
|f_o^{'}(z)| &\le \frac{1-|\varphi(z_o)|^2}{(1-|z\overline{\varphi(z_o)}|)^{\alpha+1}} \\
&\le \frac{1-|\varphi(z_o)|^2}{(1-|z|)^\alpha(1-|\varphi(z_o)|)} \\
&\le \frac{2^{\alpha+1}}{(1-|z|^2)^\alpha},
\end{aligned} \tag{2.3}$$

for all $z \in \mathbb{D}$. So, it follows that $f_o \in B^\alpha$. It is easy to check that

$$f_o^{'}(\varphi(z_o)) = \frac{\overline{\varphi(z_o)}}{(1-|\varphi(z_o)|^2)^\alpha}.$$

Then for $z_o \in \mathbb{D}$

$$\begin{aligned}\frac{(1-|z_o|^2)^\beta|\varphi^{'}(z_o)||\varphi(z_o)|}{(1-|\varphi(z_o)|^2)^\alpha} &= (1-|z_o|^2)^\beta|f_o^{'}(\varphi(z_o))||\varphi^{'}(z_o)| \\ &= (1-|z_o|^2)^\beta|DC_\varphi f_o(z_o)| \\ &\leq ||DC_\varphi f_o||_\beta \\ &\leq C||f_o||_\alpha < \infty.\end{aligned}$$

Since z_o is arbitrary, then

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta|\varphi^{'}(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^\alpha} < \infty.$$

Then (2.1) holds, because of $|\varphi(z)| < 1$ for all $z \in \mathbb{D}$. Therefore the proof of the theorem is completed. ■

Theorem 2.2. *Let φ an analytic self-map of $\mathbb{D}$ and α and β positive real numbers. If $DC_\varphi : B^\alpha \to H_\beta^\infty$ is bounded, then it is compact if and only if*

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta|\varphi^{'}(z)|}{(1-|\varphi(z)|^2)^\alpha} = 0. \tag{2.4}$$

Proof. By the assumption, for every $\varepsilon > 0$, There exist a $\delta \in (0,1)$, such that

$$\frac{(1-|z|^2)^\beta|\varphi^{'}(z)|}{(1-|\varphi(z)|^2)^\alpha} < \varepsilon, \tag{2.5}$$

whenever $\delta < |\varphi(z)| < 1$. To prove the compactness of DC_φ, assume that $(f_k)_{k\in\mathbb{N}}$ is a bounded sequence in B^α such that $||f_k||_\alpha \leq 1$ and converges to zero uniformly on compact subsets of $\mathbb{D}$. From the proof of the Weak Convergence Theorem in [2] , it is sufficient to show that $||DC_\varphi f_k||_\beta \to 0$ as $k \to \infty$.
If $|\varphi(z)| > \delta$, then by (2.5)

$$\begin{aligned}||DC_\varphi f_k||_\beta &= \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|DC_\varphi f_k(z)| \\ &= \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|f^{'}(\varphi(z))||\varphi^{'}(z)| \\ &= \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta|\varphi^{'}(z)|}{(1-|\varphi(z)|^2)^\alpha}||f_k||_\alpha \\ &< \varepsilon||f_k||_\alpha \leq \varepsilon.\end{aligned}$$

Now consider $|\varphi(z)| \leq \delta$. We have

$$||DC_\varphi f_k||_\beta = \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|f_k^{'}(\varphi(z))||\varphi^{'}(z)|.$$

Since $DC_\varphi : B^\alpha \to H_\beta^\infty$ is bounded then $\varphi^{'} = DC_\varphi(z) \in H_\beta^\infty$ and so $(1-|z|^2)^\beta|\varphi^{'}(z)| < \infty$. Since $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$, from Cauchy estimates, it followes that $f_k^{'} \to 0$ uniformly on compact subsets of $\mathbb{D}$, Thus $||DC_\varphi f_k||_\beta \to 0$ as $k \to \infty$.

Now we are going to prove that (2.4) is also necessary condition for compactness of DC_φ. Suppose that $(z_k)_{k\in\mathbb{N}}$ is a sequence in $\mathbb{D}$ such that $|\varphi(z_k)| \to 1$ as $k \to \infty$.

Consider the functions f_k defined by

$$f_k(z) = \frac{1-|\varphi(z_k)|^2}{\alpha(1-z\overline{\varphi(z_k)})^\alpha} \qquad \text{for} \quad z \in \mathbb{D}.$$

Clearly $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$, and

$$|f_k^{'}(z)| \leq \frac{2}{(1-|z|)^\alpha}.$$

So,

$$\begin{aligned}||f_k||_\alpha &= \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f_k^{'}(z)| \\ &\leq \sup_{z\in\mathbb{D}} \frac{2(1-|z|^2)^\alpha}{(1-|z|)^\alpha} \\ &\leq 2^{\alpha+1} < \infty.\end{aligned}$$

Hence, $(||f_k||_\alpha)_{k\in\mathbb{N}}$ is uniformly bounded.

Note that

$$f_k^{'}(\varphi(z_k)) = \frac{(1-|\varphi(z_k)|^2)\overline{\varphi(z_k)}}{(1-\varphi(z_k)\overline{\varphi(z_k)})^{\alpha+1}} = \frac{\overline{\varphi(z_k)}}{(1-|\varphi(z_k)|^2)^\alpha}.$$

Thus

$$\begin{aligned}\frac{(1-|z_k|^2)^\beta|\varphi^{'}(z_k)||\varphi(z_k)|}{(1-|\varphi(z_k)|^2)^\alpha} &= (1-|z_k|^2)^\beta|DC_\varphi f_k(z_k)| \\ &\leq ||DC_\varphi f_k||_\beta.\end{aligned}$$

Since $DC_\varphi : B^\alpha \to H_\beta^\infty$ is compact, then $||DC_\varphi f_k||_\beta \to 0$. Hence

$$\frac{(1-|z_k|^2)^\beta|\varphi^{'}(z_k)||\varphi(z_k)|}{(1-|\varphi(z_k)|^2)^\alpha} \to 0$$

as $k \to \infty$. Since $|\varphi(z_k)| \to 1$ as $k \to \infty$, then

$$\frac{(1-|z_k|^2)^\beta|\varphi^{'}(z_k)|}{(1-|\varphi(z_k)|^2)^\alpha} \to 0$$

as $k \to \infty$. So if DC_φ is compact, then the condition(2.4) holds and the proof is completed.
■

3. Boundedness and Compactness of $DC_\varphi : B_0^\alpha \to H_{\beta,0}^\infty$

In this section, we characterize the boundedness and compactness of $DC_\varphi : B_0^\alpha \to H_{\beta,0}^\infty$.

Theorem 3.1. *Let φ be an analytic self-map of $\mathbb{D}$ and α and β positive real numbers. Then $DC_\varphi : B_0^\alpha \to H_{\beta,0}^\infty$ is bounded if and only if $\varphi^{'} \in H_{\beta,0}^\infty$ and*

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta|\varphi^{'}(z)|}{(1-|\varphi(z)|^2)^\alpha} < \infty. \tag{3.1}$$

Proof. Suppose that DC_φ maps B_0^α boundedly into $H_{\beta,0}^\infty$, then taking $f(z)=z$, we obtain

$$\varphi^{'}=DC_\varphi z\in H_{\beta,0}^\infty.$$

For fixed $z_o\in\mathbb{D}$, the function defined in (2.2) is in fact in B_0^α, so the proof of Theorem 2.1 shows that if DC_φ maps B_0^α boundedly into $H_{\beta,0}^\infty$, then

$$\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta|\varphi^{'}(z)|}{(1-|\varphi(z)|^2)^\alpha}<\infty.$$

Conversely, suppose that φ is such that

$$\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta|\varphi^{'}(z)|}{(1-|\varphi(z)|^2)^\alpha}=C.$$

We will show that DC_φ maps B_0^α boundedly into $H_{\beta,0}^\infty$. By Theorem 2.1, it is clear that, $DC_\varphi:B^\alpha\to H_\beta^\infty$ is bounded, then we only need to prove that $DC_\varphi f\in H_{\beta,0}^\infty$ for any $f\in B_0^\alpha$.
Let $f\in B_0^\alpha$, then there exists $\delta\in(0,1)$, such that

$$(1-|\varphi(z)|^2)^\alpha|f^{'}(\varphi(z))|<\frac{\varepsilon}{C}\qquad\text{as}\qquad\delta<|\varphi(z)|<1.$$

We consider two cases, $\delta<|\varphi(z)|<1$ and $|\varphi(z)|\le\delta$.
First consider $\delta<|\varphi(z)|<1$. Then

$$\begin{aligned}(1-|z|^2)^\beta|DC_\varphi f(z)|&=(1-|z|^2)^\beta|f^{'}(\varphi(z))||\varphi^{'}(z)|\\&=\frac{(1-|z|^2)^\beta|\varphi^{'}(z)|}{(1-|\varphi(z)|^2)^\alpha}(1-|\varphi(z)|^2)^\alpha|f^{'}(\varphi(z))|\\&<C\cdot\frac{\varepsilon}{C}=\varepsilon.\end{aligned}$$

So, $DC_\varphi f\in H_{\beta,0}^\infty$. Next consider $|\varphi(z)|\le\delta$,

$$\begin{aligned}(1-|z|^2)^\beta|DC_\varphi f(z)|&=(1-|z|^2)^\beta|(f^{'}(\varphi(z))||\varphi^{'}(z)|\\&=\frac{(1-|z|^2)^\beta|\varphi^{'}(z)|}{(1-|\varphi(z)|^2)^\alpha}(1-|\varphi(z)|^2)^\alpha|f^{'}(\varphi(z))|\\&\le(1-|z|^2)^\beta|\varphi^{'}(z)|\frac{||f||_\alpha}{(1-\delta^2)^\alpha}\\&=C(1-|z|^2)^\beta|\varphi^{'}(z)|.\end{aligned}$$

Since $\varphi^{'}\in H_{\beta,0}^\infty$, taking the limit from both sides of the above inequality, we obtain

$$\lim_{|z|\to1}(1-|z|^2)^\beta|DC_\varphi f(z)|=0.$$

Thus it follows from the Closed Graph Theorem that, DC_φ maps B_0^α boundedly into $H_{\beta,0}^\infty$. ∎

Next, we characterize the compactness of $DC_\varphi:B_0^\alpha\to H_{\beta,0}^\infty$. For this purpose we need the following Lemma (see [15]).

Lemma 3.2. *Let $\beta > 0$. A closed set K in $H^\infty_{\beta,0}$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1} \sup_{f\in K} (1-|z|^2)^\beta |f(z)| = 0.$$

Theorem 3.3. *Let φ be an analytic self-map of $\mathbb{D}$ and α and β positive real numbers. Then $DC_\varphi : B^\alpha_0 \to H^\infty_{\beta,0}$ is compact if and only if*

$$\lim_{|z|\to 1} \frac{(1-|z|^2)^\beta |\varphi'(z)|}{(1-|\varphi(z)|^2)^\alpha} = 0.$$

Proof. By Lemma 3.2, $DC_\varphi : B^\alpha_0 \to H^\infty_{\beta,0}$ is compact if and only if the set

$$\{DC_\varphi f : f \in B^\alpha_0, ||f||_\alpha \le 1\}$$

has compact closure in $H^\infty_{\beta,0}$ and this follows if and only if

$$\lim_{|z|\to 1} \sup_{||f||_\alpha \le 1} (1-|z|^2)^\beta |DC_\varphi f(z)| = 0$$

where $f \in B^\alpha_0$. Since

$$\begin{aligned} \sup_{||f||_\alpha \le 1} (1-|z|^2)^\beta |DC_\varphi f(z)| &= \sup_{||f||_\alpha \le 1} (1-|z|^2)^\beta |f'(\varphi(z))||\varphi'(z)| \\ &= \sup_{||f||_\alpha \le 1} \frac{(1-|z|^2)^\beta |\varphi'(z)|}{(1-|\varphi(z)|^2)^\alpha} (1-|\varphi(z)|^2)^\alpha |f'(\varphi(z))| \\ &= \frac{(1-|z|^2)^\beta |\varphi'(z)|}{(1-|\varphi(z)|^2)^\alpha} \sup_{||f||_\alpha \le 1} (1-|\varphi(z)|^2)^\alpha |f'(\varphi(z))| \\ &= \frac{(1-|z|^2)^\beta |\varphi'(z)|}{(1-|\varphi(z)|^2)^\alpha} \cdot 1, \end{aligned}$$

and letting $|z| \to 1$, we obtain

$$\lim_{|z|\to 1} \frac{(1-|z|^2)^\alpha |\varphi'(z)|}{(1-|\varphi(z)|^2)^\beta} = 0,$$

so the theorem is proved. ∎

4. Essential Norm of $DC_\varphi : B^\alpha \to H^\infty_\beta$

We begin this section with following two Lemmas (see [9]) and then we give the essential norm of DC_φ from B^α to H^∞_β.

Lemma 4.1. *Let $\alpha > 0$, $n \in \mathbb{N}$, $0 \le x \le 1$ and $H_{n,\alpha}(x) = x^{n-1}(1-x^2)^\alpha$. Then $H_{n,\alpha}$ has the following properties.*

(i)

$$\max_{0\le x\le 1} H_{n,\alpha}(x) = H_{n,\alpha}(r_n) = \begin{cases} 1\ , & as\ n = 1 \\ \left(\dfrac{2\alpha}{n-1+2\alpha}\right)^\alpha \left(\dfrac{n-1}{n-1+2\alpha}\right)^{\frac{n-1}{2}}, & as\ n \ge 2 \end{cases}$$

where

$$r_n = \begin{cases} 0\ , & as\ n = 1 \\ \left(\dfrac{n-1}{n-1+2\alpha}\right)^{\frac{1}{2}}, & as\ n \ge 2. \end{cases}$$

(ii) *For* $n \geq 1$, $H_{n,\alpha}$ *is increasing on* $[0, r_n]$ *and decreasing on* $[r_n, 1]$.

(iii) *For* $n \geq 1$, $H_{n,\alpha}$ *is decreasing on* $[r_n, r_{n+1}]$ *and so*

$$\min_{x\in[r_n,r_{n+1}]} H_{n,\alpha}(x) = H_{n,\alpha}(r_{n+1}) = \Big(\frac{2\alpha}{n+2\alpha}\Big)^{\alpha}\Big(\frac{n}{n+2\alpha}\Big)^{\frac{(n-1)}{2}}.$$

Consequently,

$$\lim_{n\to\infty} n^{\alpha} \min_{x\in[r_n,r_{n+1}]} H_{n,\alpha}(x) = \Big(\frac{2\alpha}{e}\Big)^{\alpha}.$$

We need the following Lemma to obtain the upper estimates of essential norm. For $r \in (0,1)$, let $K_r f(z) = f(rz)$. Then K_r is a compact operator on the space B^{α} or B_0^{α} for any positive number α, with $||K_r|| \leq 1$.

Lemma 4.2. *Let* $0 < \alpha < \infty$. *Then there is a sequence* $\{r_k\}$, $0 < r_k < 1$, *tending to 1, such that the compact operator* $L_n = \frac{1}{n}\sum_{k=1}^{n} K_{r_k}$ *on* B_0^{α} *satisfies*

(i) *For any* $t \in [0,1)$, $\lim_{n\to\infty} \sup_{||f||_{\alpha}\leq 1} \sup_{|z|\leq t} |\big((I-L_n)f\big)'(z)| = 0$.

(ii) $\lim_{n\to\infty} \sup ||I - L_n|| \leq 1$.

Theorem 4.3. *Let* φ *be an analytic self-map of* $\mathbb{D}$, α *and* β *positive real numbers and suppose that* DC_{φ} *is bounded from* B^{α} *to* H_{β}^{∞}. *Then*

$$||DC_{\varphi}||_e = \lim_{t\to 1} \sup_{|\varphi(z)|>t} \frac{(1-|z|^2)^{\beta}|\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha}}.$$

Proof. We first give the lower estimate. Let $n \in \mathbb{N}$. Consider the function z^n. By Lemma 4.1,

$$||z^n||_{\alpha} = \max_{z\in\mathbb{D}} n|z|^{n-1}(1-|z|^2)^{\alpha} = n\Big(\frac{2\alpha}{n-1+2\alpha}\Big)^{\alpha}\Big(\frac{n-1}{n-1+2\alpha}\Big)^{\frac{n-1}{2}},$$

where the maximum is attained at any point on the circle with radius

$$r_n = \Big(\frac{n-1}{n-1+2\alpha}\Big)^{\frac{1}{2}}.$$

Let $f_n(z) = \frac{z^n}{||z^n||_{\alpha}}$. Then $||f_n||_{\alpha} = 1$ and $f_n \to 0$ weakly in B^{α}. This follows since a bounded sequence contained in B_0^{α} which tends to 0 uniformly on compact subsets of $\mathbb{D}$ converges weakly to 0 in B^{α}. In particular, if K is any compact operator from B^{α} to H_{β}^{∞}, then $\lim_{n\to\infty} ||Kf_n||_{\beta} = 0$.

Let $A_n = \{z \in \mathbb{D} : r_n \leq |z| \leq r_{n+1}\}$. Then

$$\begin{aligned}\min_{z\in A_n} |f_n'(z)|(1-|z|^2)^{\alpha} &= \min_{z\in A_n} \frac{n|z|^{n-1}}{||z^n||_{\alpha}}(1-|z|^2)^{\alpha}\\ &= \Big(\frac{n-1+2\alpha}{n+2\alpha}\Big)^{\alpha}\Big(\frac{n^2+(2\alpha-1)n}{n^2+(2\alpha-1)n-2\alpha}\Big)^{\frac{n-1}{2}}.\end{aligned}$$

Simple calculation shows that this minimum tends to 1 as $n \to \infty$. For any compact operator K from B^{α} to H_{β}^{∞},

$$||DC_{\varphi} - K|| \geq \lim_{n\to\infty} \sup ||(DC_{\varphi} - K)f_n||_{\beta} \geq \lim_{n\to\infty} \sup ||DC_{\varphi} f_n||_{\beta}.$$

Thus, for $DC_\varphi : B^\alpha \to H^\infty_\beta$,

$$\begin{aligned}||DC_\varphi||_e &\geq \limsup_{n\to\infty} ||DC_\varphi f_n||_\beta \\ &\geq \limsup_{n\to\infty} \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |DC_\varphi f_n(z)| \\ &\geq \lim_{n\to\infty} \sup_{\varphi(z)\in A_n} (1-|z|^2)^\beta |f_n'(\varphi(z))||\varphi'(z)| \\ &\geq \lim_{n\to\infty} \sup_{\varphi(z)\in A_n} \frac{(1-|z|^2)^\beta |\varphi'(z)|}{(1-|\varphi(z)|^2)^\alpha} \min_{\varphi(z)\in A_n} (1-|\varphi(z)|^2)^\alpha |f_n'(\varphi(z))|.\end{aligned}$$

Since

$$\lim_{n\to\infty} \sup \min_{\varphi(z)\in A_n} (1-|\varphi(z)|^2)^\alpha |f_n'(\varphi(z))| = 1,$$

thus

$$||DC_\varphi||_e \geq \lim_{n\to\infty} \sup_{\varphi(z)\in A_n} \frac{(1-|z|^2)^\beta |\varphi'(z)|}{(1-|\varphi(z)|^2)^\alpha}.$$

Now we are going to give the upper estimate. Let $\{L_n\}$ be the sequence of operators given in Lemma 4.2. Since each L_n is compact as an operator from B^α to B^α, $DC_\varphi L_n : B^\alpha \to H^\infty_\beta$ is also compact and we have

$$\begin{aligned}||DC_\varphi||_e &\leq ||DC_\varphi - DC_\varphi L_n| = ||DC_\varphi(I-L_n)|| \\ &= \sup_{||f||_\alpha\leq 1} ||DC_\varphi(I-L_n)f||_\beta \\ &= \sup_{||f||_\alpha\leq 1} \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |\big((I-L_n)f\big)'(\varphi(z))||\varphi'(z)|. \end{aligned} \tag{4.1}$$

For $z \in \mathbb{D}$, we consider (4.1) in two cases

$$\sup_{||f||_\alpha\leq 1} \sup_{|\varphi(z)|\leq t} (1-|z|^2)^\beta |\big((I-L_n)f\big)'(\varphi(z))||\varphi'(z)| \tag{4.2}$$

and

$$\sup_{||f||_\alpha\leq 1} \sup_{|\varphi(z)|> t} (1-|z|^2)^\beta |\big((I-L_n)f\big)'(\varphi(z))||\varphi'(z)|, \tag{4.3}$$

where $0 < t < 1$ is arbitrary.
Since DC_φ is bounded from B^α into H^∞_β, by Theorem 2.1,

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta |\varphi'(z)|}{(1-|\varphi(z)|^2)^\alpha} < \infty.$$

Hence

$$\sup_{|\varphi(z)|\leq t} (1-|z|^2)^\beta |\varphi'(z)| < \infty.$$

Thus, from (4.2) and using (i) of Lemma 4.2,

$$\lim_{n\to\infty} \sup_{||f||_\alpha\leq 1} \sup_{|\varphi(z)|\leq t} (1-|z|^2)^\beta |\big((I-L_n)f\big)'(\varphi(z))||\varphi'(z)| = 0. \tag{4.4}$$

From (4.3),

$$\sup_{||f||_\alpha\le 1}\sup_{|\varphi(z)|>t}\frac{(1-|z|^2)^\beta|\varphi^{'}(z)|}{(1-|\varphi(z)|^2)^\alpha}(1-|\varphi(z)|^2)^\alpha|\big((I-L_n)f\big)^{'}(\varphi(z))|$$
$$\le ||I-L_n||\sup_{|\varphi(z)|>t}\frac{(1-|z|^2)^\beta|\varphi^{'}(z)|}{(1-|\varphi(z)|^2)^\alpha}.$$

Thus, by (ii) of Lemma 4.2,

$$\limsup_{n\to\infty}\sup_{||f||_\alpha\le 1}\sup_{|\varphi(z)|>t}(1-|z|^2)^\beta|\big((I-L_n)f\big)^{'}(\varphi(z))||\varphi^{'}(z)|$$
$$\le \sup_{|\varphi(z)|>t}\frac{(1-|z|^2)^\beta|\varphi^{'}(z)|}{(1-|\varphi(z)|^2)^\alpha}. \tag{4.5}$$

From (4.1), by using (4.4) and (4.5) as $n\to\infty$, we obtain

$$||DC_\varphi||_e\le \sup_{|\varphi(z)|>t}\frac{(1-|z|^2)^\beta|\varphi^{'}(z)|}{(1-|\varphi(z)|^2)^\alpha}.$$

Since t was arbitrary, so

$$||DC_\varphi||_e\le \lim_{t\to 1}\sup_{|\varphi(z)|>t}\frac{(1-|z|^2)^\beta|\varphi^{'}(z)|}{(1-|\varphi(z)|^2)^\alpha}.$$

The proof of the theorem is completed. ∎

5. Applications of Results

If we take $\alpha=1$ in Theorems 2.1 and 2.2, one can obtain the necessary and sufficient conditions for boundedness and compactness of the operator $DC_\varphi : B\to H^\infty_\beta$, which was given in [16], Corollaries 2.4 and 2.5.

Putting $\varphi=I$, the identity function on $\mathbb{D}$, the operator DC_φ reduces to the differantiation operator. So, we obtain the results about the boundedness, compactness and essential norm of the differentiation operator from Bloch-type space into Bers-type space.

The following corollaries are consequences of the Theorems 2.1, 2.2, 3.1 and 3.3 in this case.

Corollary 5.1. *The operator $DC_\varphi : B^\alpha\to H^\infty_\beta$ is bounded if and only if $DC_\varphi : B^\alpha_0\to H^\infty_{\beta,0}$ is bounded.*

Corollary 5.2. *The operator $DC_\varphi : B^\alpha\to H^\infty_\beta$ is compact if and only if $DC_\varphi : B^\alpha_0\to H^\infty_{\beta,0}$ is compact.*

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