**Thai Journal of Mathematics**
Volume 20 Number 1 (2022)
Pages 79–98

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

# Application of The Combination of Variational Inequalities for Fixed Point Problems and Optimization Problems

**Kanyanee Saechou and Atid Kangtunyakarn***

*Department of Mathematics, School of Science, King Mongkut's Institute of Technology Ladkrabang, Bangkok 10520, Thailand*
*e-mail : kanyanee.my26@gmail.com (K. Saechou); beawrock@hotmail.com (A. Kangtunyakarn)*

**Abstract** In this paper, we obtain a strong convergence theorem for finding a common element of the set of solutions of variational inequality problems and equilibrium problems. Moreover, we apply our main result to obtain a strong convergence theorem for finding a common element of the set of fixed point problems of strictly pseudo-contractive mappings and a convergence theorem involving minimization problems. Furthermore, we utilize our main theorem for numerical examples.

**MSC:** 47J20; 47H09; 46N10; 47H08

**Keywords:** variational inequality; equilibrium problem; optimization problems; $\kappa$-Strict pseudo contraction mapping

Submission date: 14.02.2018 / Acceptance date: 13.01.2022

## 1. Introduction

Let $H$ be a real Hilbert space and $C$ be a nonempty closed convex subset of $H$. A mapping $T$ of $C$ into itself is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \qquad \forall x, y \in C. \tag{1.1}$$

A mapping $T$ is said to be *$\kappa$-strictly pseudo-contractive* if there exists a constant $\kappa \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa\|(I - T)x - (I - T)y\|^2, \qquad \forall x, y \in C.$$

Note that the class of $\kappa$-strictly pseudo-contractions strictly included the class of nonexpansive mappings. It is well-know that (1.1) is equivalent to

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2 - \frac{1 - \kappa}{2}\|(I - T)x - (I - T)y\|^2, \ \ \forall x, y \in D(T).$$

*Corresponding author.

Published by The Mathematical Association of Thailand.

Throughout this paper we denote $F(T)$ is the set of fixed points of $T$ (i.e., $F(T) = \{x \in H : Tx = x\}$).

A mapping $A$ of $C$ into $H$ is called *$\alpha$-inverse strongly monotone* (see [1]), if there exists a positive real number $\alpha$ such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

If $\langle x - y, Ax - Ay\rangle \geq 0$, a mapping $A$ is called monotone operator.

Let $F : C \times C \to \mathbb{R}$ be a bifunction. The *equilibrium problems* of $F$ is to find $x \in C$, such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{1.2}$$

The set of solutions of the equilibrium problems is denoted by $EP(F)$. Many physic, optimization, and economic problems seek some element of $EP(F)$; see more detail in [2, 3]. Over decades ago, there are many researchers studied the equilibrium problems, see, for instance [4–6]

Let $B : C \to H$. The *variational inequality problems* is to find a point $u \in C$ such that

$$\langle Bu, v - u\rangle \geq 0, \quad \forall v \in C. \tag{1.3}$$

The set of solutions of the variational inequality is denoted by $VI(C, B)$. Numerous problems in physics, optimization, minimax problems, game theory, the Nash equilibrium problems in noncooperative games reduce to find an element of (1.3), see more detail in [6, 7].

By modification of (1.3), Kangtunyakarn [4] introduce the combination of variational inequality problems which is to find a point $x^* \in C$ such that

$$\langle y - x^*, (aA + (1 - a)B)x^*\rangle \geq 0, \quad \forall y \in C, \ a \in (0, 1). \tag{1.4}$$

The set of all solution of (1.4) is denoted by $VI(C, aA + (1-a)B)$. If $A = B$, $VI(C, aA + (1 - a)B)$ reduce to $VI(C, B)$.

So, He proved a strong convergence theorem for finding a common element of the set of fixed point problems of infinite family of strictly pseudo-contractive mappings and the set of equilibrium problems and two set of variational inequality problems as follows;

**Theorem 1.1.** *Let $C$ be a closed convex subset of Hilbert space $H$ and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A_1) - (A_4)$, let $A, B : C \to H$ be $\alpha$ and $\beta$-inverse strongly monotone, respectively. Let $\{T_i\}_{i=1}^{\infty}$ be $\kappa_i$-strict pseudo-contractive mappings of $C$ into itself with $\kappa = sup_{i\in\mathbb{N}}\kappa_i$ and let $\rho_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j + \alpha_2^j \leq b < 1$, and $\alpha_1^j, \alpha_2^j, \alpha_3^j \in (\kappa, 1)$ for all $j = 1, 2, ....$ For every $n \in \mathbb{N}$, let $S_n$ and $S$-mapping generated by $T_n, ..., T_1$ and $\rho_n, \rho_{n-1}, ..., \rho_1$ and $T_n, T_{n-1}, ...,$ and $\rho_n, \rho_{n-1}, ...,$ respectively. Assume that $\mathcal{F} = \bigcap_{i=1}^{\infty} F(T_i) \bigcap EP(F) \bigcap VI(C, A) \bigcap VI(C, B) \neq \emptyset$ and let $\{x_n\}$ and $\{u_n\}$ be generated by $x_1, u \in C$ and*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)S_n P_C\Big(I - \gamma(aA + (1 - a)B)\Big)u_n, \quad \forall n \geq 1, \end{cases} \tag{1.5}$$

*where $\alpha_n, a \in (0, 1), 0 < \gamma < \min\{2\alpha, 2\beta\}$ and $\{r_n\} \subset [b, c] \subset (0, \min\{2\alpha, 2\beta\})$, satisfy the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty,$

(ii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \ \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty,$

(iii) $\sum_{n=1}^{\infty} \alpha_1^n < \infty.$

*Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in \mathcal{F}$ where $z = P_{\mathcal{F}}u$.*

In this paper, we prove a strong convergence theorem for finding a common element of the set of solutions of variational inequality problems and equilibrium problems. Moreover, we apply our main result to obtain a strong convergence theorem for finding a common element of the set of fixed point problems of strictly pseudo-contractive mappings and a convergence theorem involving minimization problems. Finally, we give three numerical examples for our results to compare their converge.

## 2. Preliminaries

This section needs the following lemmas to prove our main result. Let $C$ be a closed convex subset of a real Hilbert space $H$, let $P_C$ be a metric projection of $H$ onto $C$ i.e., for $x \in H$, $P_C x$ satisfies the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

**Lemma 2.1.** [8] *Given $x \in H$ and $y \in C$. Then $P_C x = y$ if and only if there holds the inequality*

$$\langle x - y, y - z \rangle \geq 0, \qquad \forall z \in C.$$

**Lemma 2.2.** [9] *Let $\{s_n\}$ be a sequence of nonnegative real number satisfying*

$$s_{n+1} = (1 - \alpha_n)s_n + \alpha_n \beta_n, \qquad \forall n \geq 0,$$

*where $\{\alpha_n\}, \{\beta_n\}$ satisfy the conditions*

(1) $\{\alpha_n\} \subset [0,1], \ \sum_{n=1}^{\infty} \alpha_n = \infty,$

(2) $\limsup_{n\to\infty} \beta_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\alpha_n \beta_n| < \infty.$

*Then $\lim_{n\to\infty} s_n = 0$.*

**Lemma 2.3.** [? ] *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} = (1 - \alpha_n)s_n + \delta_n, \qquad \forall n \geq 0,$$

*where $\{\alpha_n\}$ is a sequence in (0,1) and $\{\delta_n\}$ is a sequence such that*

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty,$

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty.$

*Then $\lim_{n\to\infty} s_n = 0$.*

**Lemma 2.4.** [8] *Let $H$ be a Hilbert space, let $C$ be a nonempty closed convex subset of $H$ and let $A$ be a mapping of $C$ into $H$. Let $u \in C$. Then for $\lambda > 0$,*

$$u = P_C(I - \lambda A)u \;\Leftrightarrow\; u \in VI(C, A),$$

*where $P_C$ is the metric projection of $H$ onto $C$.*

**Lemma 2.5.** [10] *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$ and $S : C \to C$ be a self-mapping of $C$. If $S$ is a $\kappa$-strict pseudo-contractive mapping, then $S$ satisfies the Lipschitz condition*

$$\|Sx - Sy\| \leq \frac{1+\kappa}{1-\kappa}\|x - y\|, \qquad \forall x, y \in C.$$

*For solving the equilibrium problems for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that satisfies the following conditions:*

(A1) $F(x, x) = 0, \qquad \forall x \in C,$
(A2) *$F$ is monotone, i.e.,* $F(x, y) + F(y, x) \leq 0, \qquad \forall x, y \in C,$
(A3) $\forall x, y, z \in C,\ \lim_{t\to 0^+} F(tz + (1-t)x, y) \leq F(x, y),$
(A4) $\forall x \in C, y \mapsto F(x, y)$ *is convex and lower semicontinuous.*

**Lemma 2.6.** [2] *Let $C$ be a nonempty closed convex subset of $H$, and let $F$ be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \qquad \forall x \in C. \tag{2.1}$$

**Lemma 2.7.** [3] *Assume that $F : C \times C \to \mathbb{R}$ satisfies $(A1)-(A4)$. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_z(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \qquad \forall y \in C\}. \tag{2.2}$$

*for all $z \in H$. Then, the following hold:*

(1) *$T_r$ is single-valued,*
(2) *$T_r$ is firmly nonexpansive i.e.,*

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle, \qquad \forall x, y \in H,$$

(3) $F(T_r) = EP(F)$,
(4) *$EP(F)$ is closed and convex.*

**Lemma 2.8.** [4] *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$ and let $A, B : C \to H$ be $\alpha$ and $\beta$-inverse strongly monotone mappings, respectively, with $\alpha, \beta > 0$ and $VI(C, A) \bigcap VI(C, B) \neq \emptyset$. Then*

$$VI(C, aA + (1-a)B) = VI(C, A) \bigcap VI(C, B), \qquad \forall a \in (0, 1). \tag{2.3}$$

*Furthermore if $0 < \gamma < \min\{2\alpha, 2\beta\}$, we have $I - \gamma(aA + (1-a)B)$ is a nonexpansive mapping.*

## 3. Main Results

This section proves a strong convergence theorem for finding a common element of the set of solutions of variational inequality problems and equilibrium problems.

**Theorem 3.1.** *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. For every $i = 1, 2, ..., n$, let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A_1) - (A_4)$. Let $A, B : C \to H$ be $\alpha$ and $\beta$-inverse strongly monotone respectively with $\mathbb{F} = VI(C, A) \cap VI(C, B) \cap \bigcap_{i=1}^{N} EP(F_i) \neq \emptyset$. Let sequence $\{x_n\}$ and $\{u_n^i\}$ generated by $u, x_1 \in C$ and*

$$\begin{cases} F_i(u_n^i, v) + \dfrac{1}{r_n}\langle v - u_n^i, u_n^i - x_n\rangle \geq 0, \; for\ all\ v \in C\ and\ \ i = 1, 2, ..., N, \\ x_{n+1} = \alpha_n u + \beta_n P_C(I - \lambda(aA + (1-a)B))x_n + \gamma_n \displaystyle\sum_{i=1}^{N} a_i u_n^i, \; for\ all\ n \geq 1, \end{cases} \tag{3.1}$$

*where $\{\alpha_n\}$,$\{\beta_n\}$,$\{\gamma_n\} \subseteq [0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$ and $a \in (0, 1)$. Suppose that the following conditions hold :*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(ii) $\beta_n, \gamma_n \in [c, d] \subset (0, 1), \ \forall n \in \mathbb{N}$,
(iii) $\sum_{i=1}^{N} a_i = 1$, *where* $a_i > 0$ *for all* $i = 1, 2, ..., N$,
(iv) $0 < a < r_n < b$ *for all* $n \in \mathbb{N}$,
(v) $\lambda \in (0, 2\eta)$, *where* $\eta = min\{\alpha, \beta\}$,
(vi) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty.$

*Then $\{x_n\}$ converges strongly to $z_0 = P_{\mathbb{F}} u$.*

*Proof.* We will divide our prove into 5 steps.

Step 1. We will show that the sequence $\{x_n\}$ is bounded. Since

$$F_i(u_n^i, v) + \frac{1}{r_n}\langle v - u_n^i, u_n^i - x_n\rangle \geq 0, \text{ for all } v \in C \text{ and } \ i = 1, 2, ..., N.$$

By Lemma 2.7, we have $u_n^i = T_{r_n}^i(x_n)$ and $EP(F_i) = F(T_{r_n}^i)$, for all $i = 1, 2, ..., N$.

Let $z \in \mathbb{F} = VI(C, A) \cap VI(C, B) \cap \bigcap_{i=1}^{N} EP(F_i)$.

By Lemma 2.8 and Lemma 2.4, we have

$$z \in VI\Big(C, aA + (1-a)B\Big) = F\Big(P_C(I - \lambda(aA + (1-a)B))\Big).$$

From Lemma 2.8 and nonexpansiveness of $T_{r_n}^i$, we have

$$\|x_{n+1} - z\|$$

$$
\begin{aligned}
&=\|\alpha_n u+\beta_n P_C(I-\lambda(aA+(1-a)B))x_n+\gamma_n\sum_{i=1}^N a_i u_n^i - z\| \\
&\le \alpha_n\|u-z\|+\beta_n\|P_C(I-\lambda(aA+(1-a)B))x_n - z\|+\gamma_n\sum_{i=1}^N a_i\|T_{r_n}^i(x_n)-z\| \\
&\le \alpha_n\|u-z\|+\beta_n\|x_n-z\|+\gamma_n\|x_n-z\| \\
&=\alpha_n\|u-z\|+(1-\alpha_n)\|x_n-z\|. \qquad (3.2)
\end{aligned}
$$

Putting $M = max\{\|u-z\|, \|x_1 - z\|\}$. From (3.2), we can show by induction that $\|x_n - z\| \le M,\ \forall n \in \mathbb{N}$. It implies that $\{x_n\}$ is bounded and so is $\{u_n^i\}$ for all $i = 1, 2, ..., N$.

Step 2. We will show that $\lim\limits_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

Putting $D = aA + (1-a)B,\ \forall a \in (0,1)$. From difinition of $x_n$, we have

$$
\begin{aligned}
&\|x_{n+1}-x_n\| \\
&=\|\alpha_n u+\beta_n P_C(I-\lambda D)x_n+\gamma_n\sum_{i=1}^N a_i u_n^i-\alpha_{n-1}u-\beta_{n-1}P_C(I-\lambda D)x_{n-1} \\
&\quad -\gamma_{n-1}\sum_{i=1}^N a_i u_{n-1}^i\| \\
&\le\|\alpha_n-\alpha_{n-1}|\|u\|+\beta_n\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\|P_C(I-\lambda D)x_{n-1}\| \\
&\quad +|\gamma_n-\gamma_{n-1}|\|\sum_{i=1}^N a_i u_{n-1}^i\|+\gamma_n\sum_{i=1}^N a_i\|u_n^i-u_{n-1}^i\|. \qquad (3.3)
\end{aligned}
$$

Since $u_n^i = T_{r_n}^i x_n$. By difinition of $T_{r_n}^i$, we have

$$F_i(T_{r_n}^i x_n, v) + \frac{1}{r_n}\langle v - T_{r_n}^i x_n, T_{r_n}^i x_n - x_n\rangle \ \ge\ 0,\ \forall v \in C. \qquad (3.4)$$

Similarly

$$F_i(T_{r_{n+1}}^i x_{n+1}, v) + \frac{1}{r_{n+1}}\langle v - T_{r_{n+1}}^i x_{n+1}, T_{r_{n+1}}^i x_{n+1} - x_{n+1}\rangle \ \ge\ 0,\ \forall v \in C. \qquad (3.5)$$

From (3.4) and (3.5), we obtain

$$F_i(T_{r_n}^i x_n, T_{r_{n+1}}^i x_{n+1}) + \frac{1}{r_n}\langle T_{r_{n+1}}^i x_{n+1} - T_{r_n}^i x_n, T_{r_n}^i x_n - x_n\rangle \ \ge\ 0, \qquad (3.6)$$

and

$$F_i(T_{r_{n+1}}^i x_{n+1}, T_{r_n}^i x_n) + \frac{1}{r_{n+1}}\langle T_{r_n}^i x_n - T_{r_{n+1}}^i x_{n+1}, T_{r_{n+1}}^i x_{n+1} - x_{n+1}\rangle \ \ge\ 0. \qquad (3.7)$$

By (3.6) and (3.7), we have

$$
\begin{aligned}
&\frac{1}{r_n}\langle T_{r_{n+1}}^i x_{n+1} - T_{r_n}^i x_n, T_{r_n}^i x_n - x_n\rangle + \frac{1}{r_{n+1}}\langle T_{r_n}^i x_n - T_{r_{n+1}}^i x_{n+1}, T_{r_{n+1}}^i x_{n+1} - x_{n+1}\rangle \\
&\ge\ 0,
\end{aligned}
$$

it follows that

$$\langle T_{r_n}^i x_n - T_{r_{n+1}}^i x_{n+1}, \frac{T_{r_{n+1}}^i x_{n+1} - x_{n+1}}{r_{n+1}} - \frac{T_{r_n}^i x_n - x_n}{r_n}\rangle \ \ge\ 0.$$

This implies that

$$\begin{aligned} 0 \quad \leq \quad & \langle T^i_{r_{n+1}} x_{n+1} - T^i_{r_n} x_n, T^i_{r_n} x_n - T^i_{r_{n+1}} x_{n+1} + T^i_{r_{n+1}} x_{n+1} - x_n \\ & - \frac{r_n}{r_{n+1}} (T^i_{r_{n+1}} x_{n+1} - x_{n+1}) \rangle. \end{aligned} \tag{3.8}$$

It follows that

$$\begin{aligned} & \|T^i_{r_{n+1}} x_{n+1} - T^i_{r_n} x_n\|^2 \\ \leq & \langle T^i_{r_{n+1}} x_{n+1} - T^i_{r_n} x_n, T^i_{r_{n+1}} x_{n+1} - x_n - \frac{r_n}{r_{n+1}} (T^i_{r_{n+1}} x_{n+1} - x_{n+1}) \rangle \\ = & \langle T^i_{r_{n+1}} x_{n+1} - T^i_{r_n} x_n, x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right) (T^i_{r_{n+1}} x_{n+1} - x_{n+1}) \rangle \\ \leq & \|T^i_{r_{n+1}} x_{n+1} - T^i_{r_n} x_n\| \|x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right) (T^i_{r_{n+1}} x_{n+1} - x_{n+1})\| \\ \leq & \|T^i_{r_{n+1}} x_{n+1} - T^i_{r_n} x_n\| \left( \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}} |r_{n+1} - r_n| \|T^i_{r_{n+1}} x_{n+1} - x_{n+1}\| \right) \\ \leq & \|T^i_{r_{n+1}} x_{n+1} - T^i_{r_n} x_n\| \left( \|x_{n+1} - x_n\| + \frac{1}{a} |r_{n+1} - r_n| \|T^i_{r_{n+1}} x_{n+1} - x_{n+1}\| \right). \end{aligned} \tag{3.9}$$

It follows that

$$\|u^i_{n+1} - u^i_n\| \leq \|x_{n+1} - x_n\| + \frac{1}{a} |r_{n+1} - r_n| \|u^i_{n+1} - x_{n+1}\|. \tag{3.10}$$

Substituting (3.10) into (3.3), we have

$$\begin{aligned} & \|x_{n+1} - x_n\| \\ \leq & \|\alpha_n - \alpha_{n-1}\| \|u\| + \beta_n \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \|P_C(I - \lambda D) x_{n-1}\| \\ & + |\gamma_n - \gamma_{n-1}| \left\| \sum_{i=1}^N a_i u^i_{n-1} \right\| + \gamma_n \sum_{i=1}^N a_i \|u^i_n - u^i_{n-1}\| \\ \leq & \|\alpha_n - \alpha_{n-1}\| \|u\| + \beta_n \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \|P_C(I - \lambda D) x_{n-1}\| \\ & + |\gamma_n - \gamma_{n-1}| \left\| \sum_{i=1}^N a_i u^i_{n-1} \right\| \\ & + \gamma_n \sum_{i=1}^N a_i \left( \|x_n - x_{n-1}\| + \frac{1}{a} |r_n - r_{n-1}| \|u^i_n - x_n\| \right) \\ = & \|\alpha_n - \alpha_{n-1}\| \|u\| + (1 - \alpha_n) \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \|P_C(I - \lambda D) x_{n-1}\| \\ & + |\gamma_n - \gamma_{n-1}| \left\| \sum_{i=1}^N a_i u^i_{n-1} \right\| + \frac{\gamma_n}{a} |r_n - r_{n-1}| \sum_{i=1}^N a_i \|u^i_n - x_n\| \\ \leq & |\alpha_n - \alpha_{n-1}| M_1 + (1 - \alpha_n) \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| M_1 \\ & + |\gamma_n - \gamma_{n-1}| M_1 + \frac{\gamma_n}{a} |r_n - r_{n-1}| M_1, \end{aligned} \tag{3.11}$$

where $M_1 = \max_{n\in\mathbb{N}}\{\|u\|, \|x_n\|, \Big\|\sum_{i=1}^{N} a_i u_n^i\Big\|, \sum_{i=1}^{N} a_i\|u_n^i - x_n\|\}$.

By (3.11), Lemma 2.3, and conditions (i)-(vi), we obtain

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.12}$$

Step 3. We will show that $\lim_{n\to\infty} \|u_n^i - x_n\| = 0$ and $\lim_{n\to\infty} \|P_C(I - \lambda D)x_n - x_n\| = 0$, where $D = aA + (1-a)B$, $\forall a \in (0,1)$. Since $u_n^i = T_{r_n}^i x_n$ and $T_{r_n}^i$ is a firmly nonexpansive mapping, we have

$$\begin{aligned}\|z - T_{r_n}^i x_n\|^2 &= \|T_{r_n}^i z - T_{r_n}^i x_n\|^2\\ &\leq \langle T_{r_n}^i z - T_{r_n}^i x_n, z - x_n\rangle\\ &= \frac{1}{2}(\|T_{r_n}^i x_n - z\|^2 + \|x_n - z\|^2 - \|T_{r_n}^i x_n - x_n\|^2).\end{aligned}$$

Hence

$$\|u_n^i - z\|^2 \leq \|x_n - z\|^2 - \|u_n^i - x_n\|^2. \tag{3.13}$$

From Lemma 2.8, (3.13) and definition of $x_n$, we have

$$\begin{aligned}\|x_{n+1} - z\|^2 =&\Big\|\alpha_n(u - z) + \beta_n\Big(P_C(I - \lambda D)x_n - z\Big) + \gamma_n\Big(\sum_{i=1}^{N} a_i u_n^i - z\Big)\Big\|^2\\ \leq&\alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 + \gamma_n\Big\|\sum_{i=1}^{N} a_i u_n^i - z\Big\|^2\\ \leq&\alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 + \gamma_n\sum_{i=1}^{N} a_i\|u_n^i - z\|^2\\ \leq&\alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 + \gamma_n\sum_{i=1}^{N} a_i(\|x_n - z\|^2 - \|u_n^i - x_n\|^2)\\ \leq&\alpha_n\|u - z\|^2 + \|x_n - z\|^2 - \gamma_n\sum_{i=1}^{N} a_i\|u_n^i - x_n\|^2.\end{aligned}$$

It implies that

$$\begin{aligned}&\gamma_n\sum_{i=1}^{N} a_i\|u_n^i - x_n\|^2\\ &\quad\leq\alpha_n\|u - z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2\\ &\quad\leq\alpha_n\|u - z\|^2 + (\|x_n - z\| - \|x_{n+1} - z\|)(\|x_n - z\| + \|x_{n+1} - z\|)\\ &\quad\leq\alpha_n\|u - z\|^2 + \|x_n - x_{n+1}\|(\|x_n - z\| + \|x_{n+1} - z\|).\end{aligned} \tag{3.14}$$

By (3.12) and condition (i), we have

$$\lim_{n\to\infty} \|u_n^i - x_n\| = 0,\ \text{for all } i = 1, 2, ..., N. \tag{3.15}$$

From nonexpansiveness of $P_C$, we have

$$\|P_C(I - \lambda D)x_n - z\|^2$$

$$
\begin{aligned}
&= \|P_C(I-\lambda D)x_n - P_C(I-\lambda D)z\|^2 \\
&\le \|(I-\lambda D)x_n - (I-\lambda D)z\|^2 \\
&\le \|x_n - z\|^2 - 2\lambda\langle x_n - z, Dx_n - Dz\rangle + \|\lambda(Dx_n - Dz)\|^2. \qquad (3.16)
\end{aligned}
$$

For every $x, y \in C$, we have

$$
\begin{aligned}
\langle Dx - Dy, x - y\rangle &= \langle aAx + (1-a)Bx - aAy - (1-a)By, x - y\rangle \\
&= a\langle Ax - Ay, x - y\rangle + (1-a)\langle Bx - By, x - y\rangle \\
&\ge a\alpha\|Ax - Ay\|^2 + (1-a)\beta\|Bx - By\|^2 \\
&\ge \eta(a\|Ax - Ay\|^2 + (1-a)\|Bx - By\|^2) \\
&\ge \eta\|Dx - Dy\|^2. \qquad (3.17)
\end{aligned}
$$

Then $D$ is $\eta$-inverse strongly monotone.
From (3.16)and (3.17), we have

$$
\begin{aligned}
\|P_C(I-\lambda D)x_n - z\|^2 &\le \|x_n - z\|^2 - 2\lambda\langle x_n - z, Dx_n - Dz\rangle + \|\lambda(Dx_n - Dz)\|^2 \\
&\le \|x_n - z\|^2 - 2\lambda\eta\|Dx_n - Dz\|^2 + \lambda^2\|Dx_n - Dz\|^2 \\
&= \|x_n - z\|^2 - \lambda(2\eta - \lambda)\|Dx_n - Dz\|^2. \qquad (3.18)
\end{aligned}
$$

By definition of $x_n$ and (3.18), we have

$$
\begin{aligned}
\|x_{n+1} - z\|^2 =& \Big\|\alpha_n(u - z) + \beta_n\Big(P_C(I-\lambda D)x_n - z\Big) + \gamma_n\Big(\sum_{i=1}^{N} a_i u_n^i - z\Big)\Big\|^2 \\
&\le \alpha_n\|u - z\|^2 + \beta_n\Big[\|x_n - z\|^2 - \lambda(2\eta - \lambda)\|Dx_n - Dz\|^2\Big] \\
&\quad + \gamma_n\|x_n - z\|^2 \\
&\le \alpha_n\|u - z\|^2 + \|x_n - z\|^2 - \lambda\beta_n(2\eta - \lambda)\|Dx_n - Dz\|^2.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
\lambda\beta_n(2\eta - \lambda)\|Dx_n - Dz\|^2 &\le \alpha_n\|u - z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\
&\le \alpha_n\|u - z\|^2 + \|x_n - x_{n+1}\|(\|x_n - z\| + \|x_{n+1} - z\|). \qquad (3.19)
\end{aligned}
$$

From (3.12), (3.19) and condition(i)

$$
\lim_{n\to\infty}\|Dx_n - Dz\|^2 = 0. \qquad (3.20)
$$

From definition of $P_C(I-\lambda D)$ and Lemma 2.8, we have

$$
\begin{aligned}
&\|P_C(I-\lambda D)x_n - z\|^2 \\
=& \|P_C(I-\lambda D)x_n - P_C(I-\lambda D)z\|^2 \\
\le& \langle (I-\lambda D)x_n - (I-\lambda D)z, P_C(I-\lambda D)x_n - z\rangle \\
=& \frac{1}{2}\Big(\|(I-\lambda D)x_n - (I-\lambda D)z\|^2 + \|P_C(I-\lambda D)x_n - z\|^2 \\
&- \|(I-\lambda D)x_n - (I-\lambda D)z - (P_C(I-\lambda D)x_n - z)\|^2\Big) \\
\le& \frac{1}{2}\Big(\|x_n - z\|^2 + \|P_C(I-\lambda D)x_n - z\|^2
\end{aligned}
$$

$$
\begin{aligned}
&- \|x_n - P_C(I-\lambda D)x_n - \lambda(Dx_n - Dz)\|^2\Big) \\
\leq &\frac{1}{2}\Big(\|x_n - z\|^2 + \|P_C(I-\lambda D)x_n - z\|^2 - \|x_n - P_C(I-\lambda D)x_n\|^2 \\
&- \|\lambda(Dx_n - Dz)\|^2 + 2\lambda\langle x_n - P_C(I-\lambda D)x_n, Dx_n - Dz\rangle\Big) \\
\leq &\frac{1}{2}\Big(\|x_n - z\|^2 + \|P_C(I-\lambda D)x_n - z\|^2 - \|x_n - P_C(I-\lambda D)x_n\|^2 \\
&+ 2\lambda\|x_n - P_C(I-\lambda D)x_n\|\|Dx_n - Dz\|\Big).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|P_C(I-\lambda D)x_n - z\|^2 \leq \|x_n - z\|^2 &- \|x_n - P_C(I-\lambda D)x_n\|^2 \\
&+ 2\lambda\|x_n - P_C(I-\lambda D)x_n\|\|Dx_n - Dz\|. \qquad (3.21)
\end{aligned}
$$

By definition of $x_n$, (3.21) and nonexpansiveness of $T^i_{r_n}$, we have

$$
\begin{aligned}
&\|x_{n+1} - z\|^2 \\
&= \Big\|\alpha_n(u - z) + \beta_n\Big(P_C(I-\lambda D)x_n - z\Big) + \gamma_n\Big(\sum_{i=1}^{N} a_i u_n^i - z\Big)\Big\|^2 \\
&\leq \alpha_n\|u - z\|^2 + \beta_n\Big\|P_C\Big(I-\lambda D\Big)x_n - z\Big\|^2 + \gamma_n\|x_n - z\|^2 \\
&\leq \alpha_n\|u - z\|^2 + \beta_n\Big[\|x_n - z\|^2 - \|x_n - P_C(I-\lambda D)x_n\|^2 \\
&\quad + 2\lambda\|x_n - P_C(I-\lambda D)x_n\|\|Dx_n - Dz\|\Big] + \gamma_n\|x_n - z\|^2 \\
&\leq \alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 - \beta_n\|x_n - P_C(I-\lambda D)x_n\|^2 \\
&\quad + 2\lambda\|x_n - P_C(I-\lambda D)x_n\|\|Dx_n - Dz\| + \gamma_n\|x_n - z\|^2 \\
&\leq \alpha_n\|u - z\|^2 + \|x_n - z\|^2 - \beta_n\|x_n - P_C(I-\lambda D)x_n\|^2 \\
&\quad + 2\lambda\|x_n - P_C(I-\lambda D)x_n\|\|Dx_n - Dz\|.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\beta_n\|x_n - P_C(I-\lambda D)x_n\|^2 \leq &\alpha_n\|u - z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\
&+ 2\lambda\|x_n - P_C(I-\lambda D)x_n\|\|Dx_n - Dz\| \\
\leq &\alpha_n\|u - z\|^2 + (\|x_n - z\| + \|x_{n+1} - z\|)\|x_{n+1} - x_n\| \\
&+ 2\lambda\|x_n - P_C(I-\lambda D)x_n\|\|Dx_n - Dz\|. \qquad (3.22)
\end{aligned}
$$

From condition(i), (3.22), (3.12) and (3.20), we have

$$
\lim_{n\to\infty}\|x_n - P_C(I-\lambda D)x_n\| = 0. \qquad (3.23)
$$

Step 4. We will show that $\limsup_{n\to\infty}\langle u - z_0, x_n - z_0\rangle \leq 0$, where $z_0 = P_{\mathbb{F}}u$. To show this inequality, take a subsequence$\{x_{n_k}\}$ of $\{x_n\}$, such that

$$
\limsup_{n\to\infty}\langle u - z_0, x_n - z_0\rangle = \lim_{k\to\infty}\langle u - z_0, x_{n_k} - z_0\rangle. \qquad (3.24)
$$

Without loss of generality, we may assume that $x_{n_k} \rightharpoonup \omega$ as $k \to \infty$ where $\omega \in C$. From (3.15), we have $u^i_{n_k} \rightharpoonup \omega$ as $k \to \infty$, for all $i = 1, 2, ..., N$.

Assume that $\omega \neq P_C(I-\lambda D)\omega$, where $D = aA + (1-a)B$.
By nonexpansiveness of $P_C(I-\lambda D)$, (3.23) and Opial's property, we have

$$\begin{aligned}
\liminf_{k\to\infty} \|x_{n_k} - \omega\| \\
&< \liminf_{k\to\infty} \|x_{n_k} - P_C(I-\lambda D)\omega\| \\
&\leq \liminf_{k\to\infty} \Big( \|x_{n_k} - P_C(I-\lambda D)x_{n_k}\| + \|P_C(I-\lambda D)x_{n_k} - P_C(I-\lambda D)\omega\| \Big) \\
&\leq \liminf_{k\to\infty} \|x_{n_k} - \omega\|.
\end{aligned}$$

This is a contradiction, then we have

$$\omega \in F\Big(P_C(I-\lambda D)\Big) = F\bigg(P_C\Big(I - \lambda(aA + (1-a)B)\Big)\bigg). \tag{3.25}$$

From Lemma 2.4 and Lemma 2.8, we have

$$F\bigg(P_C\Big(I - \lambda(aA + (1-a)B)\Big)\bigg) = VI(C,A)\bigcap VI(C,B). \tag{3.26}$$

From (3.25) and (3.26), we have

$$\omega \in VI(C,A)\bigcap VI(C,B).$$

Since

$$F_i(u_n^i, v) + \frac{1}{r_n}\langle v - u_n^i, u_n^i - x_n\rangle \geq 0,$$

for all $v \in C$ and $i = 1, 2, ..., N$.
By (A2), we have

$$\frac{1}{r_n}\langle v - u_n^i, u_n^i - x_n\rangle \geq F_i(v, u_n^i), \qquad \forall v \in C.$$

In particular

$$\Big\langle v - u_{n_k}^i, \frac{1}{r_{n_k}}(u_{n_k}^i - x_{n_k})\Big\rangle \geq F_i(v, u_{n_k}^i),$$

for all $v \in C$ and $i = 1, 2, ..., N$.
From (A4) and (3.15), we have

$$F_i(v, \omega) \leq 0, \tag{3.27}$$

for all $v \in C$ and $i = 1, 2, ..., N$.
Let $u_t := tv + (1-t)\omega, \quad \forall t \in (0,1]$, we have $u_t \in C$ and from (A1), (A4) and (3.27), we obtain

$$0 = F_i(u_t, u_t) \leq tF_i(u_t, v) + (1-t)F_i(u_t, \omega) \leq tF_i(u_t, v),$$

for all $i = 1, 2, ..., N$.
Hence $F_i(tv + (1-t)\omega, v) \geq 0, \forall t \in (0,1]$ and $\forall v \in C$.
Letting $t \to 0^+$ and using assumption (A3), we can conclude that

$$F_i(\omega, v) \geq 0, \qquad \forall v \in C \text{ and } i = 1, 2, ..., N.$$

Therefore, $\omega \in \bigcap_{i=1}^{N} EP(F_i)$. Hence $\omega \in \mathbb{F}$.
Since $x_{n_k} \rightharpoonup \omega$ and $\omega \in \mathbb{F}$, we have

$$\limsup_{n\to\infty}\langle u - z_0, x_n - z_0\rangle = \lim_{k\to\infty}\langle u - z_0, x_{n_k} - z_0\rangle = \langle u - z_0, \omega - z_0\rangle \leq 0. \tag{3.28}$$

Step 5. Finally, we show that $\lim_{n\to\infty} x_n = z_0$ where $z_0 = P_{\mathbb{F}}u$.
By nonexpansive of $P_C\Big(I - \lambda(aA + (1-a)B)\Big)$, we have

$$\begin{aligned}
&\|x_{n+1} - z_0\|^2 \\
=&\Bigg\|\alpha_n(u - z_0) + \beta_n\Bigg(P_C\Big(I - \lambda(aA + (1-a)B)\Big)x_n - z_0\Bigg) \\
&+ \gamma_n\Bigg(\sum_{i=1}^{N} a_i u_n^i - z_0\Bigg)\Bigg\|^2 \\
\leq&\Bigg\|\beta_n\Bigg(P_C\Big(I - \lambda(aA + (1-a)B)\Big)x_n - z_0\Bigg) + \gamma_n\Bigg(\sum_{i=1}^{N} a_i u_n^i - z_0\Bigg)\Bigg\|^2 \\
&+ 2\alpha_n\langle u - z_0, x_{n+1} - z_0\rangle \\
\leq&\beta_n\Big\|P_C\Big(I - \lambda(aA + (1-a)B)\Big)x_n - z_0\Big\|^2 + \gamma_n\Bigg\|\sum_{i=1}^{N} a_i u_n^i - z_0\Bigg\|^2 \\
&+ 2\alpha_n\langle u - z_0, x_{n+1} - z_0\rangle \\
\leq&\beta_n\|x_n - z_0\|^2 + \gamma_n\sum_{i=1}^{N} a_i\|u_n^i - z_0\|^2 + 2\alpha_n\langle u - z_0, x_{n+1} - z_0\rangle \\
\leq&\beta_n\|x_n - z_0\|^2 + \gamma_n\|x_n - z_0\|^2 + 2\alpha_n\langle u - z_0, x_{n+1} - z_0\rangle \\
=&(1-\alpha_n)\|x_n - z_0\|^2 + 2\alpha_n\langle u - z_0, x_{n+1} - z_0\rangle.
\end{aligned} \tag{3.29}$$

From(3.28) and Lemma 2.2, we obtain that $\{x_n\}$ converge strongly to $z_0 = P_{\mathbb{F}}u$.
This completes the proof of Theorem 3.1. ■

**Corollary 3.2.** *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. For every $i = 1, 2, ..., n$, let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A_1) - (A_4)$. Let $A : C \to H$ be $\alpha$-inverse strongly monotone mapping.*
*Assume that $\mathbb{F} = VI(C, A) \cap \bigcap_{i=1}^{N} EP(F_i) \neq \emptyset$. Let sequence $\{x_n\}$ and $\{u_n^i\}$ generated by $u, x_1 \in C$ and*

$$\begin{cases} F_i(u_n^i, v) + \dfrac{1}{r_n}\langle v - u_n^i, u_n^i - x_n\rangle \geq 0, \ \textit{for all } v \in C \textit{ and } \ i = 1, 2, ..., N, \\ x_{n+1} = \alpha_n u + \beta_n P_C(I - \lambda A)x_n + \gamma_n \displaystyle\sum_{i=1}^{N} a_i u_n^i, \ \textit{for all } n \geq 1, \end{cases} \tag{3.30}$$

*where* $\{\alpha_n\}$,$\{\beta_n\}$,$\{\gamma_n\} \subseteq [0,1]$ *with* $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \in \mathbb{N}$ *and* $a \in (0,1)$. *Suppose that the following conditions hold :*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(ii) $\beta_n, \gamma_n \in [c,d] \subset (0,1),\ \forall n \in \mathbb{N}$,
(iii) $\sum_{i=1}^{N} a_i = 1$, *where* $a_i > 0$ *for all* $i = 1,2,...,N$,
(iv) $0 < a < r_n < b$ *for all* $n \in \mathbb{N}$,
(v) $\lambda \in (0,2\alpha)$,
(vi) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty.$

*Then* $\{x_n\}$ *converges strongly to* $z_0 = P_{\mathbb{F}} u$.

*Proof.* Put $A \equiv B$ in Theorem 3.1. The conclusion of Corollary 3.2 can be obtained from Theorem 3.1. ∎

**Corollary 3.3.** *Let* $C$ *be a nonempty closed convex subset of a real Hilbert space* $H$. *Let* $A, B : C \to H$ *be* $\alpha$ *and* $\beta$*-inverse strongly monotone respectively with* $\mathbb{F} = VI(C,A) \cap VI(C,B) \neq \emptyset$. *Let sequence* $\{x_n\}$ *generated by* $x_1 \in C$ *and*

$$x_{n+1} = \alpha_n u + \beta_n P_C (I - \lambda(aA + (1-a)B))x_n + \gamma_n x_n,\ \textit{for all}\ n \geq 1, \tag{3.31}$$

*where* $\{\alpha_n\}$,$\{\beta_n\}$,$\{\gamma_n\} \subseteq [0,1]$ *with* $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \in \mathbb{N}$ *and* $a \in (0,1)$. *Suppose that the following conditions hold :*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(ii) $\beta_n, \gamma_n \in [c,d] \subset (0,1),\ \forall n \in \mathbb{N}$,
(iii) $\sum_{i=1}^{N} a_i = 1$, *where* $a_i > 0$ *for all* $i = 1,2,...,N$,
(iv) $\lambda \in (0,2\eta)$, *where* $\eta = min\{\alpha,\beta\}$,
(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty.$

*Then* $\{x_n\}$ *converges strongly to* $z_0 = P_{\mathbb{F}} u$.

*Proof.* Put $F_i \equiv 0$ for all $i = 1,2,...,N$ in Theorem 3.1, we have

$$\langle u - u_n^i, u_n^i - x_n \rangle \geq 0,\ \forall u \in C\ \textit{and}\ i = 1,2,...,N. \tag{3.32}$$

From Lemma 2.1 and (3.32), we have $x_n = P_C x_n = u_n^i$, for all $i = 1,2,...,N$. From Theorem 3.1 we can conclude the desired conclusion. ∎

## 4. Application

### 4.1. Fixed point problems of strictly pseudo-contractive mapping

Next, we prove a strong convergence theorem involving fixed point problems of $\kappa$-strict pseudo-contractive mapping.

A mapping $T : C \to C$ is said to be *$\kappa$-strictly pseudo-contractive* if there exists a constant $\kappa \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa\|(I - T)x - (I - T)y\|^2, \ \forall x, y \in C.$$

**Remark 4.1.**

(i) It is well-know that $I - T$ is $\frac{1-\kappa}{2}$-inverse strongly monotone mapping.

(ii) If $T : C \to C$ be $\kappa$-strictly pseudo-contractive with $F(T) \neq \emptyset$, then $F(T) = VI(C, I - T)$, see more detail in [5].

**Theorem 4.2.** *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. For every $i = 1, 2, ..., N$, let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A_1) - (A_4)$. Let $T, S$ be $\kappa$ and $\overline{\kappa}$-strict pseudo-contractive mapping of $C$ into itself, with $\mathbb{F} = F(T) \cap F(S) \cap \bigcap_{i=1}^{N} EP(F_i) \neq \emptyset$. Let sequence $\{x_n\}$ and $\{u_n^i\}$ generated by $u, x_1 \in C$ and*

$$\begin{cases} F_i(u_n^i, v) + \frac{1}{r_n}\langle v - u_n^i, u_n^i - x_n\rangle \geq 0, \ \text{for all } v \in C \text{ and } \ i = 1, 2, ..., N, \\ x_{n+1} = \alpha_n u + \beta_n P_C(I - \lambda(a(I - T) + (1 - a)(I - S)))x_n + \gamma_n \sum_{i=1}^{N} a_i u_n^i, \\ \hfill \text{for all } n \geq 1, \end{cases} \tag{4.1}$$

*where $\{\alpha_n\}$,$\{\beta_n\}$,$\{\gamma_n\} \subseteq [0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$ and $a \in (0, 1)$. Suppose that the following conditions hold :*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\beta_n, \gamma_n \in [c, d] \subset (0, 1), \ \forall n \in \mathbb{N}$,

(iii) $\sum_{i=1}^{N} a_i = 1$, *where* $a_i > 0$ *for all* $i = 1, 2, ..., N$,

(iv) $0 < a < r_n < b$ *for all* $n \in \mathbb{N}$,

(v) $\lambda \in (0, 2\eta)$, *where* $\eta = min\{\frac{1-\kappa}{2}, \frac{1-\overline{\kappa}}{2}\}$,

(vi) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty.$

*Then $\{x_n\}$ converges strongly to $z_0 = P_{\mathbb{F}}u$.*

*Proof.* The conclusion of Theorem 4.2 can be obtained from Theorem 3.1 and Remark 4.1. ∎

## 4.2. Constrained convex optimization problems

Let $f : C \to \mathbb{R}$ be a convex, Fréchet differentiable function, where $C$ is a closed convex subset of a real Hilbert space of $H$. The *constrain convex optimization problem* is to find $x^* \in C$, such that

$$f(x^*) = \min_{x\in C} f(x), \tag{4.2}$$

we use the symbol $\Omega_f$ is the set of all solution of (4.2).

Before prove Theorem 4.4, we need the following lemma.

**Lemma 4.3.** [11] *(Optimality condition) A necessary condition of optimality for a point $x^* \in C$ to be a solution of the minimization problem (4.2) is that $x^*$ solves the variational inequality*

$$\langle \nabla f(x^*), x - x^* \rangle \geq 0, \ \forall x \in C. \tag{4.3}$$

*Equivalently, $x^* \in C$ solves the fixed point equation*

$$x^* = P_C(x^* - \lambda \nabla f(x^*)),$$

*for every constant $\lambda > 0$. If, in addition, $f$ is convex, then the optimality condition (4.3) is also sufficient.*

**Theorem 4.4.** *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. For every $i = 1, 2, ..., N$, let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A_1) - (A_4)$. Let $f, g : C \to \mathbb{R}$ be convex functions with gradient $\nabla f$ is $\frac{1}{L_f}$-inverse strongly monotone and continuous function for all $L_f > 0$ and $\nabla g$ is $\frac{1}{L_g}$-inverse strongly monotone and continuous function for all $\overline{L_g} > 0$ with $\mathbb{F} = \Omega_f \cap \Omega_g \cap \bigcap_{i=1}^{N} EP(F_i) \neq \emptyset$. Let sequence $\{x_n\}$ and $\{u_n^i\}$ generated by $u, x_1 \in C$ and*

$$\begin{cases} F_i(u_n^i, v) + \dfrac{1}{r_n}\langle v - u_n^i, u_n^i - x_n \rangle \geq 0, \ \textit{for all } v \in C \textit{ and } \ i = 1, 2, ..., N, \\ x_{n+1} = \alpha_n u + \beta_n P_C(I - \lambda(a\nabla f + (1-a)\nabla g))x_n + \gamma_n \sum_{i=1}^{N} a_i u_n^i, \ \forall \ n \geq 1, \end{cases} \tag{4.4}$$

*where $\{\alpha_n\}$,$\{\beta_n\}$,$\{\gamma_n\} \subseteq [0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$ and $a \in (0, 1)$. Suppose that the following conditions hold :*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\beta_n, \gamma_n \in [c, d] \subset (0, 1), \ \forall n \in \mathbb{N}$,

(iii) $\sum_{i=1}^{N} a_i = 1$, *where* $a_i > 0$ *for all* $i = 1, 2, ..., N$,

(iv) $0 < a < r_n < b$ *for all* $n \in \mathbb{N}$,

(v) $\lambda \in (0, 2\eta)$, *where* $\eta = min\{\frac{1}{L_f}, \frac{1}{L_g}\}$.

*Then $\{x_n\}$ converges strongly to $z_0 = P_{\mathbb{F}}u$.*

*Proof.* The conclusion of Theorem 4.4 can be obtained from Theorem 3.1 and Lemma 4.3. ∎

## 5. Example and Numerical Results

In this section, three examples are given to support our results.

**Example 5.1.** Let $\mathbb{R}$ be the set of real numbers. For every $i = 1, 2, ..., N$, let $F_i : [0, 100] \times [0, 100] \to [0, 100]$ defined by $F_i(x, y) = i(2x^2 + xy + y^2)$, $\forall x, y \in \mathbb{R}$. Let $A, B : [0, 100] \to [0, 100]$ defined by $Ax = 3x$ and $Bx = \frac{6x}{7}$, $\forall x \in \mathbb{R}$. Then $VI(C, A) \cap$

$VI(C,B)\cap\bigcap_{i=1}^{N}EP(F_i)=\{0\}$. Let $u\in C$ and $\{x_n\}$,$\{u_n^i\}$ be the sequences generated by (3.1), for all $i=1,2,...,N$. By the definition of $F_i$, we have

$$0\leq F_i(u_n^i,v)+\frac{1}{r_n}\langle v-u_n^i,u_n^i-x_n\rangle,$$

for all $i=1,2,...,N$ and $n\in\mathbb{N}$.
Choose $r_n=1$,

$$\begin{aligned}0&\leq F_i(u_n^i,v)+\langle v-u_n^i,u_n^i-x_n\rangle\\&=i(-2{u_n^i}^2+u_n^iv+v^2)+(v-u_n^i)(u_n^i-x_n)\\&=i(-2{u_n^i}^2+u_n^iv+v^2)+vu_n^i-vx_n-{u_n^i}^2+u_n^ix_n\\&=(-2i-1){u_n^i}^2+(i+1)u_n^iv+iv^2-vx_n+u_n^ix_n\\&=iv^2+((i+1)u_n^i-x_n)v+(-2i-1){u_n^i}^2+u_n^ix_n.\end{aligned}$$

Let $G(v)=iv^2+((i+1)u_n^i-x_n)v+(-2i-1){u_n^i}^2+u_n^ix_n$.
$G(v)$ is a quadratic function of u with coefficient $a=i,\ b=(i+1)u_n^i-x_n$, $c=(-2i-1){u_n^i}^2+u_n^ix_n$. Determine the discriminant $\Delta$ of G as follows:

$$\begin{aligned}\Delta&=b^2-4ac\\&=((i+1)u_n^i-x_n)^2-4(i)((-2i-1){u_n^i}^2+u_n^ix_n)\\&=(i+1)^2{u_n^i}^2-2(i+1)u_n^ix_n+{x_n}^2-4i(-2i-1){u_n^i}^2-4iu_n^ix_n\\&=((i+1)^2-4i(-2i-1)){u_n^i}^2-2(3i+1)u_n^ix_n+{x_n}^2\\&=(9i^2+6i+1){u_n^i}^2-2(3i+1)u_n^ix_n+{x_n}^2\\&=((3i+1)u_n^i-x_n)^2,\end{aligned}$$

we know that $G(v)\geq 0,\ \forall v\in\mathbb{R}$. If it has at most one solution in $\mathbb{R}$, then $\Delta\leq 0$, so we obtain

$$u_n^i=\frac{x_n}{3i+1},\tag{5.1}$$

for all $i=1,2,...,N$.
Put $\alpha_n=\frac{1}{3n},\beta_n=\frac{2n+1}{9n},\gamma_n=\frac{14n-8}{18n},\lambda=\frac{1}{3},a=\frac{1}{2}$.
From (5.1) we rewrite (3.1) as follows:

$$\begin{aligned}x_{n+1}=&\frac{1}{3n}u+\frac{2n+1}{9n}P_{[0,100]}\Big(I-\frac{1}{3}(\frac{1}{2}A+(1-\frac{1}{2})B)\Big)x_n\\&+\frac{14n-8}{18n}\sum_{i=1}^{N}\Big(\frac{1}{3^i}+\frac{1}{N3^N}\Big)\frac{x_n}{3i+1}.\end{aligned}\tag{5.2}$$

It is clear that the sequences $\{\alpha_n\},\{\beta_n\}$ *and* $\{\gamma_n\}$ satisfy all the conditions of theorem 3.1, we can conclude that the sequences $\{x_n\}$ and $\{u_n^i\}$ converge strongly to 0. The table 1 shows the values of sequences $\{u_n^i\}$ and $\{x_n\}$, where $u=x_1=1$ and $u=x_1=-1$.

TABLE 1. The values of $\{u_n^i\}$ and $\{x_n\}$ with $n = N = 30$

| n | $u = x_1 = 1$ | | $u = x_1 = -1$ | |
|---|---|---|---|---|
| | $u_n^i$ | $x_n$ | $u_n^i$ | $x_n$ |
| 1 | 0.010989 | 1.000000 | -0.010989 | -1.000000 |
| 2 | 0.005353 | 0.487120 | -0.010063 | -0.915692 |
| 3 | 0.003163 | 0.287809 | -0.006855 | -0.623774 |
| 4 | 0.002097 | 0.190808 | -0.004161 | -0.378611 |
| 5 | 0.001521 | 0.138441 | -0.002498 | -0.227329 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 15 | 0.000388 | 0.035350 | -0.000396 | -0.036052 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 26 | 0.000214 | 0.019487 | -0.000216 | -0.019670 |
| 27 | 0.000206 | 0.018724 | -0.000208 | -0.018891 |
| 28 | 0.000198 | 0.018018 | -0.000200 | -0.018172 |
| 29 | 0.000191 | 0.017364 | -0.000192 | -0.017506 |
| 30 | 0.000184 | 0.016755 | -0.000186 | -0.016887 |

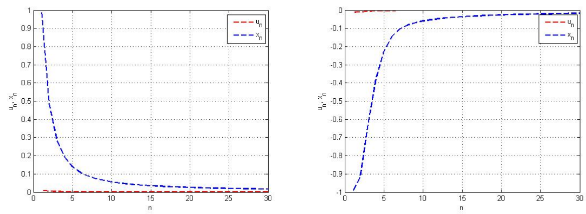

FIGURE 1. The convergence comparison with different initial values $u$ and $x_1$

**Example 5.2.** Let $F_i, \alpha_n, \beta_n, \gamma_n, \lambda$ *and* $a$ defined as in Example 5.1. Let $T, S : [0, 100] \to [0, 100]$ defined by $Tx = \frac{x}{4}$ and $Sx = \frac{x^2}{3}$, $\forall x \in \mathbb{R}$. Then $F(T) \cap F(S) \cap \bigcap_{i=1}^{N} EP(F_i) = \{0\}$. Let $u \in C$ and $\{x_n\}$,$\{u_n^i\}$ be the sequences generated by (4.1), for all $i = 1, 2, ..., N$. Then, we have

$$x_{n+1} = \frac{1}{3n}u + \frac{2n+1}{9n}P_{[0,100]}\Big(I - \frac{1}{3}(\frac{1}{2}(I-T) + (1-\frac{1}{2})(I-S))\Big)x_n + \frac{14n-8}{18n}\sum_{i=1}^{N}\Big(\frac{1}{3^i} + \frac{1}{N3^N}\Big)\frac{x_n}{3i+1}. \quad (5.3)$$

From Theorem 4.2, we can conclude that the sequences $\{x_n\}$ and $\{u_n^i\}$ converge strongly to 0. The table 2 shows the values of sequences $\{u_n^i\}$ and $\{x_n\}$, where $u = x_1 = 1$ and $u = x_1 = -1$.

TABLE 2. The values of $\{u_n^i\}$ and $\{x_n\}$ with $n = N = 30$

| n | $u = x_1 = 1$ | | $u = x_1 = -1$ | |
|---|---|---|---|---|
| | $u_n^i$ | $x_n$ | $u_n^i$ | $x_n$ |
| 1 | 0.010989 | 1.000000 | -0.010989 | -1.000000 |
| 2 | 0.006843 | 0.622702 | -0.006538 | -0.594925 |
| 3 | 0.003585 | 0.326250 | -0.003411 | -0.310437 |
| 4 | 0.002094 | 0.190558 | -0.002028 | -0.184527 |
| 5 | 0.001415 | 0.128807 | -0.001392 | -0.126641 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 15 | 0.000349 | 0.031744 | -0.000348 | -0.031713 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 26 | 0.000193 | 0.017550 | -0.000193 | -0.017541 |
| 27 | 0.000185 | 0.016865 | -0.000185 | -0.016857 |
| 28 | 0.000178 | 0.016231 | -0.000178 | -0.016224 |
| 29 | 0.000172 | 0.015644 | -0.000172 | -0.015637 |
| 30 | 0.000166 | 0.015097 | -0.000166 | -0.015091 |

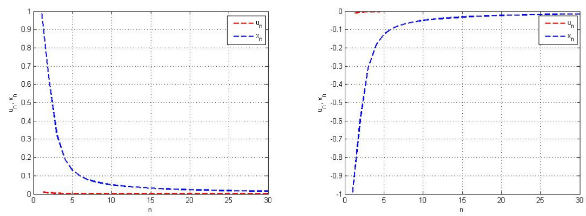

FIGURE 2. The convergence comparison with different initial values $u$ and $x_1$

**Example 5.3.** Let $F_i, \alpha_n, \beta_n, \gamma_n, \lambda$ *and* $a$ defined as in Example 5.1. Let $f, g : [0, 100] \to [0, 100]$ defined by $f(x) = 3x^2$ and $g(x) = \frac{2x^2}{3}$, $\forall x \in \mathbb{R}$ respectively. Then $\Omega_f \cap \Omega_g \cap \bigcap_{i=1}^{N} EP(F_i) = \{0\}$. Let $u \in C$ and $\{x_n\}$,$\{u_n^i\}$ be the sequences generated by (4.4), for all $i = 1, 2, ..., N$. Then, we have

$$x_{n+1} = \frac{1}{3n}u + \frac{2n+1}{9n}P_{[0,100]}\Big(I - \frac{1}{3}(\frac{1}{2}\nabla f + (1-\frac{1}{2})\nabla g)\Big)x_n + \frac{14n-8}{18n}\sum_{i=1}^{N}\Big(\frac{1}{3^i} + \frac{1}{N3^N}\Big)\frac{x_n}{3i+1}. \tag{5.4}$$

From theorem 4.4, we can conclude that the sequences $\{x_n\}$ and $\{u_n^i\}$ converge strongly to 0. The table 3 shows the values of sequences $\{u_n^i\}$ and $\{x_n\}$, where $u = x_1 = 1$ and $u = x_1 = -1$.

TABLE 3. The values of $\{u_n^i\}$ and $\{x_n\}$ with $n = N = 30$

| n | $u = x_1 = 1$ | | $u = x_1 = -1$ | |
|---|---|---|---|---|
| | $u_n^i$ | $x_n$ | $u_n^i$ | $x_n$ |
| 1 | 0.010989 | 1.000000 | -0.010989 | -1.000000 |
| 2 | 0.003231 | 0.293999 | -0.012185 | -1.108814 |
| 3 | 0.002594 | 0.236010 | -0.010509 | -0.956282 |
| 4 | 0.001870 | 0.170136 | -0.007819 | -0.711558 |
| 5 | 0.001416 | 0.128843 | -0.005114 | -0.465369 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 15 | 0.000385 | 0.035055 | -0.000400 | -0.036405 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 26 | 0.000213 | 0.019407 | -0.000217 | -0.019754 |
| 27 | 0.000205 | 0.018650 | -0.000208 | -0.018969 |
| 28 | 0.000197 | 0.017951 | -0.000200 | -0.018244 |
| 29 | 0.000190 | 0.017301 | -0.000193 | -0.017572 |
| 30 | 0.000183 | 0.016697 | -0.000186 | -0.016948 |

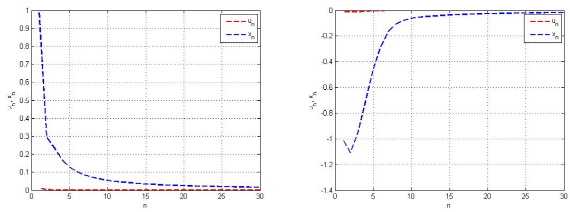

FIGURE 3. The convergence comparison with different initial values $u$ and $x_1$

**Conclusion**

1. By comparing the convergence of $\{x_n\}$ in three examples, we can conclude that the convergence of $\{x_n\}$ in an Example 5.2 is faster than the convergence of $\{x_n\}$ in Example 5.1 and 5.3.
2. Theorem 3.1, 4.2 and 4.4 guarantees the convergence of $\{x_n\}$ and $\{u_n^i\}$, for all $i = 1, 2, ..., N$, in Example 5.1, 5.2 and 5.3, respectively.

## ACKNOWLEDGEMENTS

This research is supported by the Research Administration Division of King Mongkut s Institute of Technology Ladkrabang.

## REFERENCES

[1] H. Iiduka, W. Takahashi, Weak convergence theorem by Ces'aro means for non-expansive mappings and inverse-strongly monotone mappings, J. Nonlinear Convex Anal. 7 (1) (2006) 105–113.

[2] E. Blum, E. Oettli, From optimization and variational inequalities to equilibrium problems, Math. Stud. 63 (1994), 123–145.

[3] P.L. Combettes, S.A. Hirstoaga, Equilibrium programming in Hilbert spaces, J. Nonlinear Convex Anal. 6 (1) (2005), 117-136.

[4] A. Kangtunyakarn, A new iterative scheme for fixed point problems of infinite family of $\kappa_i$-pseudo contractive mappings, equilibrium problem, variational inequality problems, J. Glob. Optim. 56 (4) (2013) 1543–1562.

[5] A. Kangtunyakarn, Convergence theorem of $\kappa$-strictly pseudo-contractive mapping and a modification of generalized equilibrium problems, Fixed Point Theory and Applications. Article Number: 89 (2012) 17 pages.

[6] A. Kangtunyakarn, A new iterative algorithm for the set of fixed-point problems of nonexpansive mappings and the set of equilibrium problem and variational inequalities problem, Abstract and Applied Analysis, Article ID 562689 (2011) 24 pages.

[7] J.L. Lions, G. Stampacchia, Variational inequalities, Comm. Pure Appl. Math. 20 (1967) 493–517.

[8] W. Takahashi, Nonlinear Functional Analysis, Yokohama Publishers, Yokohama, 2000.

[9] H.K. Xu, An iterative approach to quadratic optimization, J. Optim. Theory Appl. 116 (3) (2003) 659–678.

[10] G. Marino, H.K. Xu, Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces, J. Math. Anal. Appl. 329 (2007) 336–346.

[11] M. Su, H.K. Xu, Remarks on the gradient-projection algorithm, J. Nonlinear Anal. Optim. 1 (2010) 35–43.