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A Fixed Point Approach to Non-Archimedean Stabilities of IQD and IQA Functional Equations

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Abstract In the present work, an inverse-quadratic difference functional equation and an inverse-quadratic adjoint functional equation are solved. The non-Archimedean stabilities of these equations are proved via fixed point method. Suitable counter-examples are presented to justify the failure of stability of these equations for singular cases. As an application, two new relationships are induced through the solution of these equations using inverse square law in physics.

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1. Introduction

The issue raised in [1] is the source for the speculation of stability of functional equations. The question devised by Ulam was responded by Hyers [2] which made a ground breaking idea in the conjecture of stability of functional equation. The outcome attained by Hyers is termed as Hyers–Ulam stability or ϵ-stability of functional equation. Then, Hyers result was simplified by Aoki [3]. Also, Hyers result was modified by Rassias [4] considering the upper bound as sum of powers of norms (Hyers-Ulam-T. Rassias stability). Later, Rassias [5] established Hyers result by taking the upper bound as product of powers of norms (Ulam-Gavruta-J. Rassias stability). In 1994, to promote the stability result into simple form, Gavruta [6] reinstated the upper bound by a common governing function (generalized Hyers-Ulam stability).

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In [7], for the first time, Bodaghi and Kim introduced and studied the Ulam-Gavruta-Rassias stability for the quadratic reciprocal functional equation

$$f(2x+y)+f(2x-y)=\frac{2f(x)f(y)[4f(y)+f(x)]}{(4f(y)-f(x))^2} \tag{1.1}$$

After that, this equation (1.2) is generalized in [8] as

$$f((m+1)x+my)+f((m+1)x-my)=\frac{2f(x)f(y)[(m+1)^2f(y)+m^2f(x)]}{((m+1)^2f(y)-m^2f(x))^2} \tag{1.2}$$

where $m\in\mathbb{Z}$ with $m\neq 0,-1$. In [8], the authors established the generalized Hyers-Ulam-Rassias stability for the functional equation (1.2) in non-Archimedean fields. Other form of a quadratic reciprocal functional equation can be found in [9].

The generalized Hyers-Ulam stability of inverse-quadratic functional equation in two variables of the form

$$I_q(x+y)=\frac{I_q(x)I_q(y)}{\left(\sqrt{I_q(x)}+\sqrt{I_q(y)}\right)^2} \tag{1.3}$$

is investigated in [10] in the setting of real numbers. It is easy to verify that the inverse-quadratic function $I_q(x)=\frac{1}{x^2}$ is a solution of equation (1.3). For further stability results using fixed point method concerning different types of functional equations and rational functional equations, one may refer to ([11–17]).

Here, we bringout a few fundamental perceptions of non-Archimedean field and fixed point alternative theorem in non-Archimedean settings.

Definition 1.1. Let $\mathbb{U}$ be a field with a mapping (valuation) $|\cdot|$ from $\mathbb{U}$ into $[0,\infty)$. Then $\mathbb{U}$ is said to be a *non-Archimedean field* if the upcoming requirements exist:

(i) $|\ell|=0$ if and only if $\ell=0$;
(ii) $|\ell_1\ell_2|=|\ell_1||\ell_2|$;
(iii) $|\ell_1+\ell_2|\leq max\{|\ell_1|,|\ell_2|\}$

for all $\ell_1,\ell_2\in\mathbb{U}$.

It is evident that $|1|=|-1|=1$ and $|\ell|\leq 1$ for all $\ell\in\mathbb{N}$. Furthermore, we presume that $|\cdot|$ is non-trivial, that is, there exists an $\alpha_0\in\mathbb{U}$ such that $|\alpha_0|\neq 0,1$.

Suppose V is a vector space over a scalar field $\mathbb{U}$ with a non-Archimedean non-trivial valuation $|\cdot|$. A function $\|\cdot\|:V\longrightarrow\mathbb{R}$ is a *non-Archimedean norm* (*valuation*) if it satisfies the ensuing requirements:

(i) $\|x\|=0$ if and only if $x=0$;
(ii) $\|\rho x\|=|\rho|\,\|x\|\quad(\rho\in\mathbb{U},x\in V)$;
(iii) the strong triangle inequality (Ultrametric); namely,

$$\|x+y\|\leq\max\{\|x\|,\|y\|\}\quad(x,y\in V).$$

Then $(V,||\cdot||)$ is known as a non-Archimedean space. By virtue of the inequality

$$\|x_\ell-x_k\|\leq\max\{\|x_{j+1}-x_j\|:k\leq j\leq\ell-1\}\quad(\ell>k),$$

a sequence $\{x_k\}$ is Cauchy if and only if $\{x_{k+1}-x_k\}$ converges to 0 in a non-Archimedean space. If every Cauchy sequence is convergent in the space, then it is called as complete non-Archimedean space.

Definition 1.2. Assume B is a nonempty set. Suppose $d : B \times B \longrightarrow [0, \infty]$ satisfies the ensuing properties:

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ (symmetry);
(iii) $d(x, z) \leq \max\{d(x, y), d(y, z)\}$ (strong triangle inequality)

for all $x, y, z \in B$. Then (B, d) is called a *generalized non-Archimedean metric space*. Suppose every Cauchy sequence in B is convergent, then (B, d) is called complete.

Example 1.3. *Let $\mathbb{U}$ be a non-Archimedean field. Assume X and Y are two non-Archimedean spaces over $\mathbb{U}$. If Y is complete and $\psi : X \longrightarrow [0, \infty)$, for every $p, q : X \longrightarrow Y$, define $d(p, q) = \inf\{\epsilon > 0 : |p(x) - q(x)| \leq \epsilon\psi(x),\ \forall x \in X\}$.*

Applying Theorem 2.5 [18], a new version of the alternative fixed point principle in the setting of non-Archimedean space is proposed in [19] as follows:

Theorem 1.4 ([19])**.** (Non-Archimedean version of alternative fixed point principle) *Suppose (B, d) is a non-Archimedean generalized metric space. Let a mapping $\Gamma : B \longrightarrow B$ be a strictly contractive, (that is $d(\Gamma(x), \Gamma(y)) \leq Kd(y, x)$, for all $x, y \in B$ and a Lipschitz constant $K < 1$), then either*

(i) *$d\left(\Gamma^k(x), \Gamma^{k+1}(x)\right) = \infty$ for all $k \geq 0$, or;*
(ii) *there exists some $k_0 \geq 0$ such that $d\left(\Gamma^k(x), \Gamma^{k+1}(x)\right) < \infty$ for all $k \geq k_0$;*

the sequence $\left\{\Gamma^k(x)\right\}$ is convergent to a fixed point $x^\star$ of Γ; $x^\star$ is the unique invariant point of Γ in the set $H = \{y \in X : d\left(\Gamma^{k_0}(x), y\right) < \infty\}$ and $d\left(y, x^\star\right) \leq d(y, \Gamma(y))$ for all y in this set.

In this study, we concentrate on the following functional equations

$$I_q\left(\frac{x+y}{2}\right) - I_q(x+y) = \frac{3I_q(x)I_q(y)}{\left(\sqrt{I_q(x)} + \sqrt{I_q(y)}\right)^2} \tag{1.4}$$

and

$$I_q\left(\frac{x+y}{2}\right) + I_q(x+y) = \frac{5I_q(x)I_q(y)}{\left(\sqrt{I_q(x)} + \sqrt{I_q(y)}\right)^2}. \tag{1.5}$$

We observe that the inverse-quadratic function $I_q(x) = \frac{1}{x^2}$ fulfills equations (1.4) and (1.5). For this reason, we name the equations (1.4) and (1.5) as Inverse-Quadratic Difference (IQD) functional equation and Inverse-Quadratic Adjoint (IQA) functional equation, respectively. We attain the primary stabilities of the above equations (1.4) and (1.5) in non-Archimedean fileds by means of fixed point scheme.

Let us presume that $\mathbb{U}$ and $\mathbb{V}$ are a non-Archimedean field and a complete non-Archimedean field, respectively, in our entire study. In the sequel, we represent $\mathbb{U}^* = \mathbb{U}\backslash\{0\}$, where $\mathbb{U}$ is a non-Archimedean field. For the sake of easy computation, we describe the difference operators $\Delta_1 I_q, \Delta_2 I_q : \mathbb{U}^* \times \mathbb{U}^* \longrightarrow \mathbb{V}$ by

$$\Delta_1 I_q(x, y) = I_q\left(\frac{x+y}{2}\right) - I_q(x+y) - \frac{3I_q(x)I_q(y)}{\left(\sqrt{I_q(x)} + \sqrt{I_q(y)}\right)^2}$$

and

$$\Delta_2 I_q(x,y) = I_q\left(\frac{x+y}{2}\right) + I_q(x+y) - \frac{5I_q(x)I_q(y)}{\left(\sqrt{I_q(x)}+\sqrt{I_q(y)}\right)^2}$$

for all $x, y \in \mathbb{U}^*$.

2. Solution of Equations (1.4) and (1.5)

In this section, we solve the equations (1.4) and (1.5) for their solution. In the following theorem, let $x, y \in \mathbb{U}^*$.

Theorem 2.1. *Let $I_q : \mathbb{U}^* \longrightarrow \mathbb{V}$ be a mapping. Then the following statements are equivalent.*

(i) *I_q satisfies (1.3).*
(ii) *I_q satisfies (1.4).*
(iii) *I_q satisfies (1.5).*

Hence, the solution of (1.4) and (1.5) is also an inverse-quadratic mapping.

Proof. Firstly, let us prove that if I_q satisfies (1.3), then it satisfies (1.4). For this, let us switch y into x in (1.3), to get

$$I_q(2x) = \frac{1}{4}I_q(x). \tag{2.1}$$

Now, let us consider $\frac{x}{2}$ in place of x in (2.1), to find

$$I_q\left(\frac{x}{2}\right) = 4I_q(x). \tag{2.2}$$

Again, substitute (x, y) by $(\frac{x}{2}, \frac{y}{2})$ in (1.3) and apply (2.2), to acquire

$$I_q\left(\frac{x+y}{2}\right) = \frac{4I_q(x)I_q(y)}{\left(\sqrt{I_q(x)}+\sqrt{I_q(y)}\right)^2}. \tag{2.3}$$

Subtract (1.3) from (2.3), to arrive at (1.5).

Next, we prove that if I_q satisfies (1.4), then it satisfies (1.5). For this, let us plug y by x in (1.5), to get

$$I_q(2x) = \frac{1}{4}I_q(x). \tag{2.4}$$

Now, replace x by $\frac{x}{2}$ in (2.4), to obtain

$$I_q\left(\frac{x}{2}\right) = 4I_q(x). \tag{2.5}$$

Let us use (2.5) in (1.5), to get

$$I_q(x+y) = \frac{I_q(x)I_q(y)}{\left(\sqrt{I_q(x)}+\sqrt{I_q(y)}\right)^2}. \tag{2.6}$$

Now, the summation of (2.6) and (2.3) leads to (1.5).

Finally, we prove that if I_q satisfies (1.5), then it satisfies (1.3). To prove this, take $y = x$ in (1.5), to get

$$I_q(2x) = \frac{1}{4}I_q(x). \tag{2.7}$$

Now, replacing x by $\frac{x}{2}$ in (2.7), to obtain

$$I_q\left(\frac{x}{2}\right) = 4I_q(x). \tag{2.8}$$

Apply (2.8) in (1.5), to arrive at (1.3), which completes the proof. ∎

3. Non-Archimedean Stabilities of Equations (1.4) and (1.5)

In this section, we prove the existence of non-Archimedean stabilities of equations (1.4) and (1.5) through fixed point method.

Theorem 3.1. *Let $j \in \{1,2\}$. Assume a mapping $I_q : \mathbb{U}^* \longrightarrow \mathbb{V}$ satisfies the inequality*

$$|\Delta_j I_q(x,y)| \leq \xi(x,y) \tag{3.1}$$

for all $x,y \in \mathbb{U}^*$, where $\xi : \mathbb{U}^* \times \mathbb{U}^* \longrightarrow \mathbb{V}$ is a given function. If $0 < L < 1$,

$$|2|^{-2}\xi\left(2^{-1}x, 2^{-1}y\right) \leq L\xi(x,y) \tag{3.2}$$

for all $x,y \in \mathbb{U}^*$, then there exists a unique invers-quadratic mapping $r_q : \mathbb{U}^* \longrightarrow \mathbb{V}$ satisfying the equations (1.4) and (1.5), respectively for $j = 1,2$ and

$$|I_q(x) - r_q(x)| \leq L|2|^2\xi(x,x) \tag{3.3}$$

for all $x \in \mathbb{U}^*$.

Proof. Firstly, let us prove this theorem for the case when $j = 1$. Plugging (x,y) by $\left(\frac{x}{2}, \frac{y}{2}\right)$ in (3.1), we obtain

$$\left|I_q(x) - 2^{-2}I_q\left(2^{-1}x\right)\right| \leq \xi\left(2^{-1}x, 2^{-1}x\right) \tag{3.4}$$

for all $x \in \mathbb{U}^*$. Let $\mathcal{A} = \{p | p : \mathbb{U}^* \longrightarrow \mathbb{V}\}$, and define $d(p,q) = \inf\{\gamma > 0 : |p(x) - q(x)| \leq \gamma\xi(x,x),\ \text{for all } x \in \mathbb{U}^*\}$. In view of Example 1.3, we find that d turns into a complete generalized non-Archimedean complete metric on $\mathcal{A}$. Let $\Gamma : \mathcal{A} \longrightarrow \mathcal{A}$ be a mapping defined by $\Gamma(p)(x) = 2^{-2}p\left(2^{-1}x\right)$ for all $x \in \mathbb{U}^*$ and $p \in \mathcal{A}$. Then Γ is strictly contractive on $\mathcal{A}$, in fact if $|p(x) - q(x)| \leq \gamma\xi(x,x)$, $(x \in \mathbb{U}^*)$, then by (3.2), we obtain

$$\begin{aligned}|\Gamma(p)(x) - \Gamma(q)(x)| = |2|^{-2}\left|p\left(2^{-1}x\right) - q\left(2^{-1}x\right)\right| &\leq \gamma|2|^{-2}\xi\left(2^{-1}x, 2^{-1}x\right)\\ &\leq \gamma L\xi(x,x) \quad (x \in \mathbb{U}^*).\end{aligned}$$

From the above, we conclude that $(\Gamma(p), \Gamma(q)) \leq Ld(p,q)$ $(p,q \in \mathcal{A})$. Consequently, the mapping d is strictly contractive with Lipschitiz constant L. Using (3.4), we have

$$|\Gamma(p)(x) - p(x)| = \left|2^{-2}p\left(2^{-1}x\right) - p(x)\right| \leq \xi\left(\frac{x}{2}, \frac{x}{2}\right) \leq |2|^2L\xi(x,x) \quad (u \in \mathbb{U}^*).$$

This indicates that $d(\Gamma(p), p) \leq L|2|^2$. Due to Theorem 1.4 (ii), Γ has a distinct invariant point $r_q : \mathbb{U}^* \longrightarrow \mathbb{V}$ in the set $G = \{g \in F : d(x,g) < \infty\}$ and for each $x \in \mathbb{U}^*$, $r_q(x) = \lim_{s\to\infty} \Gamma^s I_q(x) = \lim_{s\to\infty} 2^{-2s}I_q\left(2^{-s}x\right)$ $(x \in \mathbb{U}^*)$. Therefore, for all $x,y \in \mathbb{U}^*$,

$$\begin{aligned}|\Delta_1 r_q(x,y)| = \lim_{s\to\infty} |2|^{-2s}\left|\Delta_1 I_q\left(2^{-s}x, 2^{-s}y\right)\right| &\leq \lim_{s\to\infty} |2|^{-2s}\xi\left(2^{-s}x, 2^{-s}y\right)\\ &\leq \lim_{s\to\infty} L^s\xi(x,y) = 0\end{aligned}$$

which shows that r_q is inverse-quadratic. Theorem 1.4 (ii) implies $d(I_q, r_q(x)) \leq d(\Gamma(I_q), I_q)$, that is, $|I_q(x) - r_q(x)| \leq |2|^2L\xi(x,x)$ $(x \in \mathbb{U}^*)$. Let $r_q' : \mathbb{U}^* \longrightarrow \mathbb{V}$ be a inverse-quadratic mapping which satisfies (3.3), then r_q' is a fixed point of Γ in $\mathcal{A}$. However, by Theorem

1.4, Γ has only one fixed point in G. This completes the uniqueness assertion of the theorem. ∎

For the case $j = 2$, we can achieve the stability results concerning equation (1.5) through similar arguments as in the case of $j = 1$. Hence we omit the proof of the stability results of equation (1.5). The ensuing theorem is dual of Theorem 3.1. Hence, we omit the proof as it is analogous to Theorem 3.1.

Theorem 3.2. *Suppose the mapping $I_q : \mathbb{U}^\star \longrightarrow \mathbb{V}$ satisfies the inequality (3.2). If $0 < L < 1$, $|2|^2\xi(2x, 2y) \leq L\xi(x, y)$, for all $x, y \in \mathbb{U}^\star$, then there exists a unique inverse-quadratic mapping $r_q : \mathbb{U}^\star \longrightarrow \mathbb{V}$ satisfying the equations (1.4) and (1.5), respetively for $j = 1, 2$ and $|I_q(x) - r_q(x)| \leq L\xi\left(\frac{x}{2}.\frac{x}{2}\right)$, for all $x \in \mathbb{U}^\star$.*

The following corollaries are immediate consequences of Theorems 3.1 and 3.2. In the following corollaries, we assume that $|2| < 1$ for a non-Archimdean field $\mathbb{U}$. In the subsequent outcomes, let us assume that $I_q : \mathbb{U}^\star \longrightarrow \mathbb{V}$ to be a mapping. Also, let $j \in \{1, 2\}$ in the following results.

Corollary 3.3. *Let ξ (independent of x, y)> 0 be a constant. Suppose the mapping I_q satisfies the inequality $|\Delta_j I_q(x, y)| \leq \epsilon$, for all $x, y \in \mathbb{U}^\star$. Then there exists a unique inverse-quadratic mapping $r_q : \mathbb{U}^\star \longrightarrow \mathbb{V}$ satisfying the equations (1.4) and (1.5), respectively for $j = 1, 2$ with the result $|I_q(x) - r_q(x)| \leq \epsilon$, for all $x \in \mathbb{U}^\star$*

Proof. Replacing $\xi(x, y)$ by ϵ and then choosing $L = |2|$ in Theorem 3.1, we get

$$|I_q(x) - r_q(x)| \leq |2|^3\epsilon \leq \epsilon$$

for all $x \in \mathbb{U}^\star$. ∎

Corollary 3.4. *Let $\beta \neq -2$ and $k_1 \geq 0$ be real numbers exists for a mapping I_q such that $|\Delta_j I_q(x, y)| \leq k_1(|x|^\beta + |y|^\beta)$, for all $x, y \in \mathbb{U}^\star$. Then there exists a unique inverse-quadratic mapping $r_q : \mathbb{U}^\star \longrightarrow \mathbb{V}$ satisfying the equations (1.4) and (1.5), respectively for $j = 1, 2$ with the result*

$$|I_q(x) - r_q(x)| \leq \begin{cases} \frac{|2|k_1}{|2|^\beta}|x|^\beta, & \beta < -2 \\ |2|^3k_1|x|^\beta, & \beta > -2 \end{cases}$$

for all $x \in \mathbb{U}^\star$.

Proof. Assuming $\xi(x, y) = k_1\left(|x|^\beta + |y|^\beta\right)$ in Theorems 3.1 and 3.2 and then selecting $L = |2|^{-\beta-2}$, $\beta < -2$ and $L = |2|^{\beta+2}$, $\beta > -2$, respectively, the proof follows. ∎

Corollary 3.5. *Let $k_2 \geq 0$ and $\beta \neq -2$ be real numbers, and the mapping I_q satisfies the inequality $|\Delta_j I_q(x, y)| \leq k_2|x|^{\beta/2}|y|^{\beta/2}$, for all $x, y \in \mathbb{U}^*$. Then there exists a unique inverse-quadratic mapping $r_q : \mathbb{U}^* \longrightarrow \mathbb{V}$ satisfying the equations (1.4) and (1.5), respectively for $j = 1, 2$ with the result*

$$|I_q(x) - r_q(x)| \leq \begin{cases} \frac{k_2}{|2|^\beta}|x|^\beta, & \beta < -2 \\ |2|^2k_2|x|^\beta, & \beta > -2 \end{cases}$$

for all $x \in \mathbb{U}^*$.

Proof. It is easy to prove this corollary, by taking $\xi(x, y)$ into $k_2|x|^{\beta/2}|y|^{\beta/2}$ and then choosing $L = |2|^{-\beta-2}$, $\beta < -2$ and $L = |2|^{\beta+2}$, $\beta > -2$, respectively in Theorems 3.1 and 3.2. ∎

4. Counter-Examples

In the sequel, We illustrate some counter-examples that the functional equations (1.4) and (1.5) fail to stable for the critical case when $\beta = -2$ in Corollary 3.4.
Let us consider a function $\xi : \mathbb{R}^\star \longrightarrow \mathbb{R}$ defined as follows:

$$\xi(x) = \begin{cases} \frac{\delta}{x^2}, & \text{for } x \in (1, \infty) \\ \delta, & \text{otherwise} \end{cases} \tag{4.1}$$

Let $I_q : \mathbb{R}^\star \longrightarrow \mathbb{R}$ be a mapping defined via

$$I_q(x) = \sum_{n=0}^{\infty} 4^{-n}\xi(2^{-n}x) \tag{4.2}$$

for all $x \in \mathbb{R}^\star$. By the definition of I_q, it gets transformed into counter-example for the fact that the equation (1.4) fails to be stable for the singular case $\beta = -2$ in Corollary 3.4 in the subsequent theorem.

Theorem 4.1. *Let $j \in \{1, 2\}$. If the function I_q defined in (4.2) satisfies the following approximation*

$$|\Delta_j I_q(x, y)| \leq \frac{44\delta}{3}(|x|^{-2} + |y|^{-2}) \tag{4.3}$$

for all $x, y \in \mathbb{R}^\star$, then there do not exist a inverse-reciprocal quadratic mapping $r_q : \mathbb{R}^\star \longrightarrow \mathbb{R}$ and a constant $\mu > 0$ such that

$$|I_q(x) - r_q(x)| \leq \mu|x|^{-2} \tag{4.4}$$

for all $x \in \mathbb{R}^\star$.

Proof. Firstly, let us assume $j = 1$. Then, let us show that I_q satisfies (4.3). By simple derivation, we have

$$|I_q(x)| = \left|\sum_{n=0}^{\infty} 4^{-n}\xi(2^{-n}x)\right| \leq \sum_{n=0}^{\infty} \frac{\delta}{4^n} = \frac{4\delta}{3}$$

Therefore, we see that I_q is bounded by $\frac{4\delta}{3}$ on $\mathbb{R}$. If $|x|^{-2} + |y|^{-2} \geq 1$, then the left-hand side of (4.3) is less than $\frac{44\delta}{3}$. Now, suppose that $0 < |x|^{-2} + |y|^{-2} < 1$. Hence, there exists a positive integer k with the following relation:

$$\frac{1}{4^{k+1}} \leq |x|^{-2} + |y|^{-2} < \frac{1}{4^k} \tag{4.5}$$

From the above relation (4.5), we have $4^k(|x|^{-2} + |y|^{-2}) < 1$ which gives $\frac{4^k}{x^2} < 1, \frac{4^k}{y^2} < 1$. Therefore, $\frac{x^2}{4^k} > 1, \frac{y^2}{4^k} > 1$. The last two inequalities lead to $\frac{x^2}{4^{k-1}} > 4 > 1$, $\frac{y^2}{4^{k-1}} > 4 > 1$ and as a result, we have

$$\frac{1}{2^{k-1}}(x) > 1, \frac{1}{2^{k-1}}(y) > 1, \frac{1}{2^{k-1}}(x + y) > 1, \frac{1}{2^{k-1}}\left(\frac{x + y}{2}\right) > 1$$

Hence, for each value of $n \in \{0, 1, 2, ..., k - 1\}$, we obtain

$$\frac{1}{2^n}(x) > 1, \frac{1}{2^n}(y) > 1, \frac{1}{2^n}(x + y) > 1, \frac{1}{2^n}\left(\frac{x + y}{2}\right) > 1$$

and $\Delta_1\xi(2^{-n}x, 2^{-n}y) = 0$ for $n \in \{0, 1, 2, ..., k-1\}$. Utilizing (4.1) and by the definition of I_q, we obtain

$$|\Delta_1 I_q(x,y)| \leq \sum_{n=k}^{\infty} \frac{\delta}{4^n} + \sum_{n=k}^{\infty} \frac{\delta}{4^n} + \frac{3}{4}\sum_{n=k}^{\infty} \frac{\delta}{4^n} \leq \frac{11\delta}{4}\sum_{n=k}^{\infty}\frac{1}{4^n}$$
$$\leq \frac{11\delta}{4}\frac{1}{4^k}\left(1-\frac{1}{4}\right)^{-1} \leq \frac{11\delta}{3}\frac{1}{4^k} \leq \frac{44\delta}{3}\frac{1}{4^{k+1}} \leq \frac{44\delta}{3}(|x|^{-2}+|y|^{-2})$$

for all $x, y \in \mathbb{R}^\star$. Therefore, the inequality (4.3) holds. We claim that the equation (1.4) fails to be stable for $\beta = -2$ in Corollary 3.4. For this, let us assume to the contrary that there exists an inverse-quadratic mapping $r_q : \mathbb{R}^\star \longrightarrow \mathbb{R}$ satisfying (4.4). Therefore, we have

$$|r_q(x)| \leq (\mu+1)|x|^{-2} \tag{4.6}$$

However, we can choose a positive integer m with $m\delta > \mu + 1$. If $x \in (1, 2^{m-1})$ then $2^{-n}x \in (1, \infty)$ for all $n = 0, 1, 2, ..., m-1$ and thus

$$|r_q(x)| = \sum_{n=0}^{\infty}\frac{\xi(2^{-n}x)}{4^n} \geq \sum_{n=0}^{m-1}\frac{\frac{4^n\delta}{x^2}}{4^n} = \frac{m\delta}{x^2} > (\mu+1)x^{-2}$$

which contradicts (4.6). Therefore, the equation (1.4) fails to be stable for $\beta = -2$ in Corollary 3.4.

We can also illustrate a similar counter-example for the instability of equation (1.5) when $\beta = -2$ in Corollary 3.4. ■

5. Elucidation of Equations (1.4) and (1.5) via Inverse Square Law

In Physics, the inverse square law states that the intensity of light is inversely proportional to the square of its distance from a light source. This implies that when the distance increases from a light source, then the intensity of light is proportional to $\frac{1}{x^2}$, where x is the distance of the light source. The inverse square law is used to determine astronomical distances. A light source of known intrinsic brightness can be used to measure its distance from the Earth using inverse square law. Owing to the solution of equations (1.4) and (1.5), we can produce two new relations using inverse square law.

(1) We have $I_q\left(\frac{x+y}{2}\right) = \frac{4}{(x+y)^2}$ and $I_q(x+y) = \frac{1}{(x+y)^2}$. Therefore, the left-hand side of (1.4) indicates that it is the difference between the intensities of the light source at distances $\frac{x+y}{2}$ and $x+y$. Also, we have $I_q(x) = \frac{1}{x^2}$ and $I_q(y) = \frac{1}{y^2}$. The right-hand side of (1.4) specifies that it is the ratio of 3 times multiplied with the product of the intensities of light source at distances x and y to the square of sum of square of the distances x and y.

(2) By a similar reason, the left-hand side of (1.5) signifies that it is the sum of the intensities of the light source at distances $\frac{x+y}{2}$ and $x+y$. The right-hand side of (1.5) implies that it is the ratio of 5 times multiplied with the product of the intensities of light source at distances x and y to the square of sum of square of the distances x and y.

Thus, we have given rise the interpretation of our equations (1.4) and (1.5) through inverse square law.

6. Discussions and Conclusions

This is our first effort to deal with inverse reciprocal functional equations with arguments in rational form. In this work, we have proved that the solution of the equations (1.4) and (1.5) is inverse-quadratic mapping. We also conclude that the stability results hold good for these equations in the frame work of non-Archimedean fields except at some singular cases. To justify the instability of these equations, we have illustrated proper examples. Using inverse square law; we have encountered with two new relations through equations (1.4) and (1.5).

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