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Recursion Formulas for Bernoulli Numbers

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Abstract In this paper we establish simple recursion formulas for Bernoulli numbers, for instance,

$$\sum_{k=1}^{n} \binom{4n+2}{4k} (-1)^k 2^{2k-1} B_{4k} = n$$

and

$$\sum_{k=0}^{n} \binom{4n+4}{4k+2} (-1)^k 2^{2k} B_{4k+2} = n+1$$

in Theorem 1.1. Furthermore applying a Lucas sequence V_n, we obtain

$$\sum_{k=1}^{n} \binom{8n+4}{8k} (-1)^k 2^{2k-1} B_{8k} V_{4n-4k+2} = nV_{4n+2}$$

and

$$\sum_{k=0}^{n} \binom{8n+8}{8k+4} (-1)^k 2^{2k} B_{8k+4} V_{4n-4k+2} = -(n+1)V_{4n+3}$$

in Theorem 1.2.

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1. Introduction

Let $\mathbb{N}$ be the sets of positive integers. The Bernoulli numbers $\{B_n\}$ defined by $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, $B_4 = -\frac{1}{30}$, ..., etc., and

$$\sum_{k=0}^{n-1} \binom{n}{k} B_k = 0 \qquad (n \geq 2).$$

It is well known that

$$\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = \frac{x}{e^x - 1} \qquad (|x| < 2\pi).$$

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In 1911 Ramanujan [1, 2] discovered some recursion formulas with gaps for Bernoulli numbers. In particular, he proved that if n is odd, then

$$\sum_{k\equiv 3 \pmod 6} \binom{n}{k} B_{n-k} = \begin{cases} -\dfrac{n}{6} & \text{if } n \equiv 1 \pmod 6, \\ \dfrac{n}{3} & \text{if } n \equiv 3,5 \pmod 6 \end{cases}$$

and

$$\sum_{k\equiv 5 \pmod{10}} \binom{n}{k} (L_k+1) B_{n-k} = \begin{cases} \dfrac{n}{5}(L_n+1) & \text{if } n \equiv 5,7 \pmod{10}, \\ \dfrac{n}{10}(L_{n-1}-3) & \text{if } n \equiv 1 \pmod{10}, \\ \dfrac{n}{5}(L_{n-2}-2) & \text{if } n \equiv 3,9 \pmod{10}, \end{cases}$$

where $\{L_n\}$ is the Lucas sequence given by $L_0 = 2$, $L_1 = 1$ and $L_{n+1} = L_n + L_{n-1}$. From the above Ramanujan's identities we see [3] that

$$\sum_{k=0}^{n-1} \binom{6n+3}{6k+3} B_{6n-6k} = 2n$$

and

$$\sum_{k=0}^{n-1} \binom{10n+5}{10k+5} (L_{10k+5}+1) B_{10n-10k} = 2n (L_{10n+5}+1).$$

Based on this inspiration we decide the main theme of this article, that is, our aim is to obtain some recursion formulas for Bernoulli numbers for example, similar to Ramanujan's result [2]

$$\sum_{k=0}^{[\frac{n}{2}]} \binom{2n+2}{4k+2} (-1)^k 2^{n-2k} B_{2n-4k} = (-1)^{[\frac{n}{2}]}(n+1) \qquad \text{for } n \geq 0,$$

where $[\cdot]$ denotes the greatest integer function, moreover resembling

$$\sum_{k=0}^{n} \binom{4n+4}{4k+2} (-1)^k 2^{2k+1}(2^{4k+2}-1) B_{4k+2} = 2n+1$$

in [3]. Here we perform the analogous method of Z.H. Sun's mathematical skill in [3] to get as follows :

Theorem 1.1. *For $n \in \mathbb{N}$ we have*

(a)

$$\sum_{k=1}^{n} \binom{4n+2}{4k} (-1)^k 2^{2k-1} B_{4k} = n,$$

(b)

$$\sum_{k=0}^{n} \binom{4n+4}{4k+2} (-1)^k 2^{2k} B_{4k+2} = n+1.$$

Theorem 1.2. *Let* $V_0 = V_1 = 2$ *and* $V_{n+1} = 2V_n + V_{n-1}$ $(n \geq 1)$. *Then for* $n \in \mathbb{N}$ *we have*

(a)

$$\sum_{k=1}^{n} \binom{8n+4}{8k} (-1)^k 2^{2k-1} B_{8k} V_{4n-4k+2} = nV_{4n+2},$$

(b)

$$\sum_{k=0}^{n} \binom{8n+8}{8k+4} (-1)^k 2^{2k} B_{8k+4} V_{4n-4k+2} = -(n+1)V_{4n+3}.$$

2. Proofs of Theorem 1.1 and Theorem 1.2

In order to obtain some recursion formulas for Bernoulli numbers, we consider the following proposition which is the powerful fundamental identities :

Proposition 2.1. ([3, Theorem 4.1]) *For* $m \in \mathbb{N}$, $n \in \{0, 1, 2, ...\}$ *and* $t \in \{0, 1, ..., m-1\}$ *let*

$$\alpha_n^{(m)} = \sum_{k_1+\cdots+k_m=mn} e^{2\pi i \frac{k_1+2k_2+\cdots+mk_m}{m}} \frac{1}{(2k_1+1)!\cdots(2k_m+1)!}.$$

Then

$$\sum_{k=\max\{0,1-t\}}^{n} \alpha_{n-k}^{(m)} \frac{2^{2km+2t-1} B_{2km+2t}}{(2km+2t)!}$$
$$= \frac{1}{m} \sum_{k_1+\cdots+k_m=mn+t} e^{2\pi i \frac{k_1+2k_2+\cdots+mk_m}{m}} \left(\sum_{r=1}^{m} k_r e^{-2\pi i \frac{rt}{m}} \right) \frac{1}{\prod_{r=1}^{m}(2k_r+1)!}. \tag{2.1}$$

In particular, for $t = 0$ *we have*

$$\sum_{k=1}^{n} \alpha_{n-k}^{(m)} \frac{2^{2km-1} B_{2km}}{(2km)!} = n\alpha_n^{(m)} \qquad (n \geq 1). \tag{2.2}$$

In advance we set

$$T_{r(m)}^{n} := \sum_{\substack{k=0 \\ k \equiv r \pmod{m}}}^{n} \binom{n}{k} \tag{2.3}$$

then we can find Zhi-hong Sun's results in [4, 5] as

$$T_{1(4)}^{n} = T_{3(4)}^{n} = 2^{n-2}, \qquad \text{if } n \equiv 0 \pmod{4}, \tag{2.4}$$

$$T_{1(4)}^{n} = \frac{2^{n-1} + (-1)^{[\frac{n}{4}]} 2^{[\frac{n}{2}]}}{2}, \qquad \text{if } n \equiv 2 \pmod{4}, \tag{2.5}$$

$$T_{3(4)}^{n} = \frac{2^{n-1} - (-1)^{[\frac{n}{4}]} 2^{[\frac{n}{2}]}}{2}, \qquad \text{if } n \equiv 2 \pmod{4}, \tag{2.6}$$

$$T_{0(4)}^{n}=\frac{2^{n-1}-(-1)^{[\frac{n}{4}]}2^{[\frac{n}{2}]}}{2},\qquad \text{if } n\equiv 3 \pmod 4, \tag{2.7}$$

and

$$T_{2(4)}^{n}=\frac{2^{n-1}+(-1)^{[\frac{n}{4}]}2^{[\frac{n}{2}]}}{2},\qquad \text{if } n\equiv 3 \pmod 4. \tag{2.8}$$

Similarly we can see that

$$T_{2(8)}^{n}-T_{6(8)}^{n}=(-1)^{\frac{n-3}{8}}2^{\frac{n-7}{4}}V_{\frac{n+1}{2}},\qquad \text{if } n\equiv 3 \pmod 8 \tag{2.9}$$

and

$$T_{2(8)}^{n}-T_{6(8)}^{n}=(-1)^{\frac{n-7}{8}}2^{\frac{n-7}{4}}V_{\frac{n-1}{2}},\qquad \text{if } n\equiv 7 \pmod 8. \tag{2.10}$$

Proof of Theorem 1.1. (a) By the definition of $\alpha_n^{(m)}$ in Proposition 2.1 we have

$$\begin{aligned}
\alpha_n^{(2)} &= \sum_{k_1+k_2=2n} e^{2\pi i\frac{k_1+2k_2}{2}}\frac{1}{(2k_1+1)!(2k_2+1)!}\\
&= \sum_{k_1+k_2=2n} (-1)^{k_1+2k_2}\frac{1}{(2k_1+1)!(2k_2+1)!}\\
&= \sum_{k_1=0}^{2n} (-1)^{k_1}\frac{1}{(2k_1+1)!(2(2n-k_1)+1)!}\\
&= \sum_{k_1=0}^{2n} (-1)^{k_1}\frac{(4n+2)!}{(2k_1+1)!(4n-2k_1+1)!}\cdot\frac{1}{(4n+2)!}\\
&= \frac{1}{(4n+2)!}\sum_{k_1=0}^{2n}(-1)^{k_1}\binom{4n+2}{2k_1+1}\\
&= \frac{1}{(4n+2)!}\left(\sum_{\substack{k_1=0\\ k_1\equiv 0 \pmod 2}}^{2n}\binom{4n+2}{2k_1+1}-\sum_{\substack{k_1=0\\ k_1\equiv 1 \pmod 2}}^{2n}\binom{4n+2}{2k_1+1}\right)\\
&= \frac{1}{(4n+2)!}\left(\sum_{\substack{K=1\\ K\equiv 1 \pmod 4}}^{4n+1}\binom{4n+2}{K}-\sum_{\substack{K=1\\ K\equiv 3 \pmod 4}}^{4n+1}\binom{4n+2}{K}\right)\\
&= \frac{1}{(4n+2)!}\left(\sum_{\substack{K=0\\ K\equiv 1 \pmod 4}}^{4n+2}\binom{4n+2}{K}-\sum_{\substack{K=0\\ K\equiv 3 \pmod 4}}^{4n+2}\binom{4n+2}{K}\right)
\end{aligned}$$

so by (2.3), (2.5), and (2.6) the above identity can be written as

$$\begin{aligned}\alpha_n^{(2)} &= \frac{1}{(4n+2)!}\left(T_{1(4)}^{4n+2} - T_{3(4)}^{4n+2}\right) \\ &= \frac{1}{(4n+2)!}\left\{\frac{1}{2}\left(2^{4n+1} + (-1)^n 2^{2n+1}\right) - \frac{1}{2}\left(2^{4n+1} - (-1)^n 2^{2n+1}\right)\right\} \\ &= \frac{(-1)^n 2^{2n+1}}{(4n+2)!}.\end{aligned} \tag{2.11}$$

Then by (2.2) and (2.11) we obtain

$$\begin{aligned}n\frac{(-1)^n 2^{2n+1}}{(4n+2)!} = n\alpha_n^{(2)} &= \sum_{k=1}^{n} \alpha_{n-k}^{(2)} \frac{2^{4k-1}B_{4k}}{(4k)!} \\ &= \sum_{k=1}^{n} \frac{(-1)^{n-k}2^{2(n-k)+1}}{(4(n-k)+2)!} \cdot \frac{2^{4k-1}B_{4k}}{(4k)!}\end{aligned}$$

and so

$$\begin{aligned}n &= \frac{(4n+2)!}{(-1)^n 2^{2n+1}} \sum_{k=1}^{n} \frac{(-1)^{n-k}2^{2(n-k)+1}}{(4(n-k)+2)!} \cdot \frac{2^{4k-1}B_{4k}}{(4k)!} \\ &= \sum_{k=1}^{n}(-1)^k 2^{2k-1} \frac{(4n+2)!}{(4(n-k)+2)!(4k)!} B_{4k} \\ &= \sum_{k=1}^{n}(-1)^k 2^{2k-1} \binom{4n+2}{4k} B_{4k}.\end{aligned}$$

(b) If $t = 1$ and $m = 2$ in Eq. (2.1), then we observe that

$$\begin{aligned}&\sum_{k=0}^{n} \alpha_{n-k}^{(2)} \frac{2^{4k+1}B_{4k+2}}{(4k+2)!} \\ &= \frac{1}{2} \sum_{k_1+k_2=2n+1} e^{2\pi i \frac{k_1+2k_2}{2}} \left(\sum_{r=1}^{2} k_r e^{-2\pi i \frac{r}{2}}\right) \frac{1}{\prod_{r=1}^{2}(2k_r+1)!} \\ &= \frac{1}{2} \sum_{k_1+k_2=2n+1} (-1)^{k_1+2k_2}\left(k_1 e^{-\pi i} + k_2 e^{-2\pi i}\right) \frac{1}{(2k_1+1)!(2k_2+1)!} \\ &= \frac{1}{2} \sum_{k_1+k_2=2n+1} (-1)^{k_1}(-k_1+k_2) \frac{1}{(2k_1+1)!(2k_2+1)!} \\ &= \frac{1}{2} \sum_{k_1=0}^{2n+1} (-1)^{k_1}(2n+1-2k_1) \frac{1}{(2k_1+1)!(2(2n+1-k_1)+1)!} \\ &= \frac{1}{2}\left[(2n+2)\sum_{k_1=0}^{2n+1}(-1)^{k_1} \frac{1}{(2k_1+1)!(2(2n+1-k_1)+1)!}\right. \\ &\qquad \left. - \sum_{k_1=0}^{2n+1}(2k_1+1)(-1)^{k_1} \frac{1}{(2k_1+1)!(2(2n+1-k_1)+1)!}\right]\end{aligned}$$

$$
\begin{aligned}
&= \frac{n+1}{(4n+4)!}\sum_{k_1=0}^{2n+1}(-1)^{k_1}\binom{4n+4}{2k_1+1} \\
&\quad - \frac{1}{2}\sum_{k_1=0}^{2n+1}(-1)^{k_1}\frac{(4n+3)!}{(2k_1)!(2(2n+1-k_1)+1)!}\cdot\frac{1}{(4n+3)!} \\
&= \frac{n+1}{(4n+4)!}\sum_{k_1=0}^{2n+1}(-1)^{k_1}\binom{4n+4}{2k_1+1} - \frac{1}{2\cdot(4n+3)!}\sum_{k_1=0}^{2n+1}(-1)^{k_1}\binom{4n+3}{2k_1}
\end{aligned}
$$

and so by (2.4), (2.7), and (2.8) we have

$$
\begin{aligned}
&\sum_{k=0}^{n}\alpha_{n-k}^{(2)}\frac{2^{4k+1}B_{4k+2}}{(4k+2)!} \\
&= \frac{n+1}{(4n+4)!}\left(\sum_{\substack{k_1=0\\ k_1\equiv 0 \pmod 2}}^{2n+1}\binom{4n+4}{2k_1+1} - \sum_{\substack{k_1=0\\ k_1\equiv 1 \pmod 2}}^{2n+1}\binom{4n+4}{2k_1+1}\right) \\
&\quad - \frac{1}{2\cdot(4n+3)!}\left(\sum_{\substack{k_1=0\\ k_1\equiv 0 \pmod 2}}^{2n+1}\binom{4n+3}{2k_1} - \sum_{\substack{k_1=0\\ k_1\equiv 1 \pmod 2}}^{2n+1}\binom{4n+3}{2k_1}\right) \\
&= \frac{n+1}{(4n+4)!}\left(\sum_{\substack{K=0\\ K\equiv 1 \pmod 4}}^{4n+4}\binom{4n+4}{K} - \sum_{\substack{K=0\\ K\equiv 3 \pmod 4}}^{4n+4}\binom{4n+4}{K}\right) \\
&\quad - \frac{1}{2\cdot(4n+3)!}\left(\sum_{\substack{K=0\\ K\equiv 0 \pmod 4}}^{4n+3}\binom{4n+3}{K} - \sum_{\substack{K=0\\ K\equiv 2 \pmod 4}}^{4n+3}\binom{4n+3}{K}\right) \\
&= \frac{n+1}{(4n+4)!}\left(T_{1(4)}^{4n+4} - T_{3(4)}^{4n+4}\right) - \frac{1}{2\cdot(4n+3)!}\left(T_{0(4)}^{4n+3} - T_{2(4)}^{4n+3}\right) \\
&= \frac{n+1}{(4n+4)!}\left(2^{4n+2} - 2^{4n+2}\right) \\
&\quad - \frac{1}{2\cdot(4n+3)!}\left(\frac{1}{2}\left(2^{4n+2} - (-1)^n 2^{2n+1}\right) - \frac{1}{2}\left(2^{4n+2} + (-1)^n 2^{2n+1}\right)\right) \\
&= \frac{(-1)^n 2^{2n}}{(4n+3)!},
\end{aligned}
$$

which concludes that

$$
\sum_{k=0}^{n}\alpha_{n-k}^{(2)}\frac{2^{4k+1}B_{4k+2}}{(4k+2)!} = \frac{(-1)^n 2^{2n}}{(4n+3)!}. \tag{2.12}
$$

Finally combining (2.11) and (2.12) we obtain

$$\begin{aligned}\frac{(-1)^n 2^{2n}}{(4n+3)!} &= \sum_{k=0}^{n} \frac{(-1)^{n-k} 2^{2(n-k)+1}}{(4(n-k)+2)!} \cdot \frac{2^{4k+1} B_{4k+2}}{(4k+2)!} \\ &= \frac{(-1)^n 2^{2n}}{(4n+4)!} \sum_{k=0}^{n} (-1)^k 2^{2k+2} B_{4k+2} \frac{(4n+4)!}{(4(n-k)+2)!(4k+2)!} \\ &= \frac{(-1)^n 2^{2n}}{(4n+4)!} \sum_{k=0}^{n} \binom{4n+4}{4k+2} (-1)^k 2^{2k+2} B_{4k+2}\end{aligned}$$

and

$$4n+4 = \sum_{k=0}^{n} \binom{4n+4}{4k+2} (-1)^k 2^{2k+2} B_{4k+2}.$$

Thus the proof is complete. ■

Lemma 2.2. *Let $n \in \mathbb{N}$. Then we have* $\alpha_n^{(4)} = \sum_{k=0}^{2n} (-1)^k \alpha_k^{(2)} \alpha_{2n-k}^{(2)}$.

Proof. In Proposition 2.1, $\alpha_n^{(m)}$ constructs that

$$\begin{aligned}\sum_{k=0}^{2n} (-1)^k \alpha_k^{(2)} \alpha_{2n-k}^{(2)} &= \sum_{k=0}^{2n} (-1)^k \left(\sum_{k_1+k_2=2k} e^{2\pi i \frac{k_1+2k_2}{2}} \frac{1}{(2k_1+1)!(2k_2+1)!} \right) \\ &\quad \times \left(\sum_{k_3+k_4=2(2n-k)} e^{2\pi i \frac{k_3+2k_4}{2}} \frac{1}{(2k_3+1)!(2k_4+1)!} \right) \\ &= \sum_{k_1+k_2+k_3+k_4=4n} (-1)^{-\frac{k_1+k_2}{2}} e^{2\pi i \frac{k_1+2k_2+k_3+2k_4}{2}} \\ &\quad \times \frac{1}{(2k_1+1)! \cdots (2k_4+1)!} \\ &= \sum_{k_1+k_2+k_3+k_4=4n} e^{2\pi i \frac{-k_1-k_2}{4}} e^{2\pi i \frac{k_1+2k_2+k_3+2k_4}{2}} \\ &\quad \times \frac{1}{(2k_1+1)! \cdots (2k_4+1)!} \\ &= \sum_{k_1+k_2+k_3+k_4=4n} e^{2\pi i \frac{k_1+2k_3+3k_2+4k_4}{4}} \frac{1}{(2k_1+1)! \cdots (2k_4+1)!} \\ &= \alpha_n^{(4)}.\end{aligned}$$

■

Proof of Theorem 1.2. (a) Since $\alpha_n^{(4)} = \sum_{k=0}^{2n}(-1)^k \alpha_k^{(2)}\alpha_{2n-k}^{(2)}$ in Lemma 2.2, we apply (2.11) and deduce that

$$\begin{aligned}
\alpha_n^{(4)} &= \sum_{k=0}^{2n}(-1)^k \frac{(-1)^k 2^{2k+1}}{(4k+2)!}\cdot\frac{(-1)^{2n-k}2^{2(2n-k)+1}}{(4(2n-k)+2)!} \\
&= 2^{4n+2}\sum_{k=0}^{2n}(-1)^k \frac{1}{(4k+2)!(4(2n-k)+2)!} \\
&= \frac{2^{4n+2}}{(8n+4)!}\sum_{k=0}^{2n}(-1)^k\binom{8n+4}{4k+2} \\
&= \frac{2^{4n+2}}{(8n+4)!}\left(\sum_{\substack{k=0\\ k\equiv 0 \pmod 2}}^{2n}\binom{8n+4}{4k+2} - \sum_{\substack{k=0\\ k\equiv 1 \pmod 2}}^{2n}\binom{8n+4}{4k+2}\right) \\
&= \frac{2^{4n+2}}{(8n+4)!}\left(\sum_{\substack{K=2\\ K\equiv 2 \pmod 8}}^{8n+2}\binom{8n+4}{K} - \sum_{\substack{K=2\\ K\equiv 6 \pmod 8}}^{8n+2}\binom{8n+4}{K}\right) \\
&= \frac{2^{4n+2}}{(8n+4)!}\left(\sum_{\substack{K=0\\ K\equiv 2 \pmod 8}}^{8n+4}\binom{8n+4}{K} - \sum_{\substack{K=0\\ K\equiv 6 \pmod 8}}^{8n+4}\binom{8n+4}{K}\right) \\
&= \frac{2^{4n+2}}{(8n+4)!}\left(T_{2(8)}^{8n+4} - T_{6(8)}^{8n+4}\right).
\end{aligned} \tag{2.13}$$

Now using (2.9) and the following facts

$$T_{r(m)}^n = T_{n-r(m)}^n \qquad \text{and} \qquad T_{r(m)}^{n+1} = T_{r(m)}^n + T_{r-1(m)}^n, \tag{2.14}$$

we have

$$\begin{aligned}
T_{2(8)}^{8n+4} - T_{6(8)}^{8n+4} &= T_{2(8)}^{8n+3} + T_{1(8)}^{8n+3} - \left(T_{6(8)}^{8n+3} + T_{5(8)}^{8n+3}\right) \\
&= T_{2(8)}^{8n+3} + T_{8n+2(8)}^{8n+3} - \left(T_{6(8)}^{8n+3} + T_{-2(8)}^{8n+3}\right) \\
&= T_{2(8)}^{8n+3} + T_{2(8)}^{8n+3} - \left(T_{6(8)}^{8n+3} + T_{6(8)}^{8n+3}\right) \\
&= 2\left(T_{2(8)}^{8n+3} - T_{6(8)}^{8n+3}\right) \\
&= (-1)^n 2^{2n} V_{4n+2}.
\end{aligned}$$

Employing the above identity to (2.13) we obtain

$$\alpha_n^{(4)} = \frac{2^{4n+2}}{(8n+4)!}\cdot(-1)^n 2^{2n}V_{4n+2} = \frac{(-1)^n 2^{6n+2}V_{4n+2}}{(8n+4)!}. \tag{2.15}$$

From (2.2) and (2.15) with $m = 4$ we observe that

$$\begin{aligned} n\frac{(-1)^n 2^{6n+2} V_{4n+2}}{(8n+4)!} &= n\alpha_n^{(4)} \\ &= \sum_{k=1}^{n} \alpha_{n-k}^{(4)} \frac{2^{8k-1} B_{8k}}{(8k)!} \\ &= \sum_{k=1}^{n} \frac{(-1)^{n-k} 2^{6(n-k)+2} V_{4(n-k)+2}}{(8(n-k)+4)!} \cdot \frac{2^{8k-1} B_{8k}}{(8k)!} \\ &= \frac{(-1)^n 2^{6n+2}}{(8n+4)!} \sum_{k=1}^{n} (-1)^k 2^{2k-1} V_{4n-4k+2} B_{8k} \binom{8n+4}{8k} \end{aligned}$$

and so

$$nV_{4n+2} = \sum_{k=1}^{n} \binom{8n+4}{8k} (-1)^k 2^{2k-1} V_{4n-4k+2} B_{8k}.$$

(b) If $t = 2$ and $m = 4$ in Eq. (2.1), then we have

$$\begin{aligned} &\sum_{k=0}^{n} \alpha_{n-k}^{(4)} \frac{2^{8k+3} B_{8k+4}}{(8k+4)!} \\ &= \frac{1}{4} \sum_{k_1+\cdots+k_4=4n+2} e^{2\pi i \frac{k_1+2k_2+3k_3+4k_4}{4}} \left(\sum_{r=1}^{4} k_r e^{-\pi i r} \right) \frac{1}{\prod_{r=1}^{4} (2k_r+1)!} \\ &= \frac{1}{4} \sum_{k_1+\cdots+k_4=4n+2} e^{2\pi i \frac{k_1+2k_2+3k_3+4k_4}{4}} (-k_1+k_2-k_3+k_4) \\ &\quad \times \frac{1}{(2k_1+1)! \cdots (2k_4+1)!}. \end{aligned} \tag{2.16}$$

First we show that the right hand side of (2.16) is equal to the following identity :

$$\begin{aligned} &\sum_{k=0}^{2n+1} (-1)^k (2n+1-2k) \alpha_k^{(2)} \alpha_{2n+1-k}^{(2)} \\ &= \sum_{k=0}^{2n+1} (-1)^k (2n+1-2k) \sum_{k_1+k_2=2k} e^{2\pi i \frac{k_1+2k_2}{2}} \frac{1}{(2k_1+1)!(2k_2+1)!} \\ &\quad \times \sum_{k_3+k_4=2(2n+1-k)} e^{2\pi i \frac{k_3+2k_4}{2}} \frac{1}{(2k_3+1)!(2k_4+1)!} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{2n+1} \sum_{\substack{k_1+k_2=2k \\ k_3+k_4=2(2n+1-k)}} (-1)^{\frac{k_1+k_2}{2}} e^{2\pi i\left(\frac{k_1+2k_2}{2}+\frac{k_3+2k_4}{2}\right)} \\
&\quad \times \left(\frac{k_3+k_4}{2}+k-2k\right) \frac{1}{(2k_1+1)!\cdots(2k_4+1)!} \\
&= \sum_{k=0}^{2n+1} \sum_{\substack{k_1+k_2=2k \\ k_3+k_4=2(2n+1-k)}} \left(e^{2\pi i\cdot\frac{1}{2}}\right)^{-\frac{k_1+k_2}{2}} e^{2\pi i\left(\frac{k_1+2k_2}{2}+\frac{k_3+2k_4}{2}\right)} \\
&\quad \times \left(\frac{k_3+k_4}{2}-\frac{k_1+k_2}{2}\right) \frac{1}{(2k_1+1)!\cdots(2k_4+1)!} \\
&= \frac{1}{2} \sum_{k_1+k_2+k_3+k_4=4n+2} e^{2\pi i\frac{k_1+2k_3+3k_2+4k_4}{4}} \left(-k_1-k_2+k_3+k_4\right) \\
&\quad \times \frac{1}{(2k_1+1)!\cdots(2k_4+1)!}
\end{aligned}
$$

and so by exchanging the index k_2 with k_3 we obtain

$$
\begin{aligned}
&\sum_{k=0}^{2n+1} (-1)^k \left(2n+1-2k\right) \alpha_k^{(2)} \alpha_{2n+1-k}^{(2)} \\
&= \frac{1}{2} \sum_{k_1+k_2+k_3+k_4=4n+2} e^{2\pi i\frac{k_1+2k_2+3k_3+4k_4}{4}} \left(-k_1+k_2-k_3+k_4\right) \\
&\quad \times \frac{1}{(2k_1+1)!\cdots(2k_4+1)!}.
\end{aligned} \tag{2.17}
$$

Equating (2.17) with (2.16) we have

$$
2\sum_{k=0}^{n} \alpha_{n-k}^{(4)} \frac{2^{8k+3}B_{8k+4}}{(8k+4)!} = \sum_{k=0}^{2n+1} (-1)^k \left(2n+1-2k\right) \alpha_k^{(2)} \alpha_{2n+1-k}^{(2)}. \tag{2.18}
$$

Second we evaluate the right hand side of (2.18). From (2.11) we deduce that

$$
\begin{aligned}
&\sum_{k=0}^{2n+1} (-1)^k \left(2n+1-2k\right) \alpha_k^{(2)} \alpha_{2n+1-k}^{(2)} \\
&= \sum_{k=0}^{2n+1} (-1)^k \left(2n+1-2k\right) \cdot \frac{(-1)^k 2^{2k+1}}{(4k+2)!} \cdot \frac{(-1)^{2n+1-k} 2^{2(2n+1-k)+1}}{(4(2n+1-k)+2)!} \\
&= -\sum_{k=0}^{2n+1} (-1)^k \left(2n+2-2k-1\right) \cdot \frac{2^{4n+4}}{(4k+2)!(8n+6-4k)!} \\
&= -\sum_{k=0}^{2n+1} (-1)^k \left(2n+2\right) \cdot 2^{4n+4} \cdot \frac{(8n+8)!}{(4k+2)!(8n+6-4k)!} \cdot \frac{1}{(8n+8)!}
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{k=0}^{2n+1} (-1)^k \, (2k+1) \cdot \frac{2^{4n+4}}{(4k+2)!(8n+6-4k)!} \\
&= -\frac{2^{4n+2}\,(8n+8)}{(8n+8)!} \sum_{k=0}^{2n+1} (-1)^k \binom{8n+8}{4k+2} \\
&\quad + 2^{4n+3} \sum_{k=0}^{2n+1} (-1)^k \, (4k+2) \cdot \frac{(8n+7)!}{(4k+2)!(8n+6-4k)!} \cdot \frac{1}{(8n+7)!} \\
&= -\frac{2^{4n+2}}{(8n+7)!} \sum_{k=0}^{2n+1} (-1)^k \binom{8n+8}{4k+2} + \frac{2^{4n+3}}{(8n+7)!} \sum_{k=0}^{2n+1} (-1)^k \binom{8n+7}{4k+1} \\
&= -\frac{2^{4n+2}}{(8n+7)!} \left\{ \sum_{\substack{k=0 \\ k\equiv 0 \pmod 2}}^{2n+1} \binom{8n+8}{4k+2} - \sum_{\substack{k=0 \\ k\equiv 1 \pmod 2}}^{2n+1} \binom{8n+8}{4k+2} \right\} \\
&\quad + \frac{2^{4n+3}}{(8n+7)!} \left\{ \sum_{\substack{k=0 \\ k\equiv 0 \pmod 2}}^{2n+1} \binom{8n+7}{4k+1} - \sum_{\substack{k=0 \\ k\equiv 1 \pmod 2}}^{2n+1} \binom{8n+7}{4k+1} \right\} \\
&= -\frac{2^{4n+2}}{(8n+7)!} \left\{ \sum_{\substack{K=2 \\ K\equiv 2 \pmod 8}}^{8n+6} \binom{8n+8}{K} - \sum_{\substack{K=2 \\ K\equiv 6 \pmod 8}}^{8n+6} \binom{8n+8}{K} \right\} \\
&\quad + \frac{2^{4n+3}}{(8n+7)!} \left\{ \sum_{\substack{K=1 \\ K\equiv 1 \pmod 8}}^{8n+5} \binom{8n+7}{K} - \sum_{\substack{K=1 \\ K\equiv 5 \pmod 8}}^{8n+5} \binom{8n+7}{K} \right\}
\end{aligned}
$$

and so by (2.3) we can write

$$
\begin{aligned}
&\sum_{k=0}^{2n+1} (-1)^k \, (2n+1-2k) \, \alpha_k^{(2)} \alpha_{2n+1-k}^{(2)} \\
&= -\frac{2^{4n+2}}{(8n+7)!} \left\{ \sum_{\substack{K=0 \\ K\equiv 2 \pmod 8}}^{8n+8} \binom{8n+8}{K} - \sum_{\substack{K=0 \\ K\equiv 6 \pmod 8}}^{8n+8} \binom{8n+8}{K} \right\} \\
&\quad + \frac{2^{4n+3}}{(8n+7)!} \left\{ \sum_{\substack{K=0 \\ K\equiv 1 \pmod 8}}^{8n+7} \binom{8n+7}{K} - \sum_{\substack{K=0 \\ K\equiv 5 \pmod 8}}^{8n+7} \binom{8n+7}{K} \right\} \\
&= -\frac{2^{4n+2}}{(8n+7)!} \left(T_{2(8)}^{8n+8} - T_{6(8)}^{8n+8} \right) + \frac{2^{4n+3}}{(8n+7)!} \left(T_{1(8)}^{8n+7} - T_{5(8)}^{8n+7} \right).
\end{aligned}
\tag{2.19}
$$

Since Eq. (2.10) and (2.14) shows that

$$\begin{aligned} T_{2(8)}^{8n+8} - T_{6(8)}^{8n+8} &= T_{2(8)}^{8n+7} + T_{1(8)}^{8n+7} - \left(T_{6(8)}^{8n+7} + T_{5(8)}^{8n+7}\right) \\ &= T_{8n+5(8)}^{8n+7} + T_{8n+6(8)}^{8n+7} - T_{6(8)}^{8n+7} - T_{5(8)}^{8n+7} \\ &= T_{5(8)}^{8n+7} + T_{6(8)}^{8n+7} - T_{6(8)}^{8n+7} - T_{5(8)}^{8n+7} \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} T_{1(8)}^{8n+7} - T_{5(8)}^{8n+7} &= T_{8n+6(8)}^{8n+7} - T_{8n+2(8)}^{8n+7} \\ &= T_{6(8)}^{8n+7} - T_{2(8)}^{8n+7} \\ &= (-1)^{n+1} 2^{2n} V_{4n+3}, \end{aligned}$$

therefore (2.19) becomes

$$\begin{aligned} \sum_{k=0}^{2n+1} (-1)^k (2n+1-2k)\, \alpha_k^{(2)} \alpha_{2n+1-k}^{(2)} &= \frac{2^{4n+3}}{(8n+7)!} \cdot (-1)^{n+1} 2^{2n} V_{4n+3} \\ &= \frac{(-1)^{n+1} 2^{6n+3}}{(8n+7)!} V_{4n+3}. \end{aligned} \tag{2.20}$$

Equating (2.20) with (2.18) we have

$$2 \sum_{k=0}^{n} \alpha_{n-k}^{(4)} \frac{2^{8k+3} B_{8k+4}}{(8k+4)!} = \frac{(-1)^{n+1} 2^{6n+3}}{(8n+7)!} V_{4n+3}$$

and so by (2.15) we deduce that

$$\begin{aligned} &2 \sum_{k=0}^{n} \frac{(-1)^{n-k} 2^{6(n-k)+2} V_{4(n-k)+2}}{(8(n-k)+4)!} \cdot \frac{2^{8k+3} B_{8k+4}}{(8k+4)!} \\ &= \frac{(-1)^n 2^{6n+6}}{(8n+8)!} \sum_{k=0}^{n} (-1)^k \cdot 2^{2k} V_{4n-4k+2} B_{8k+4} \cdot \frac{(8n+8)!}{(8(n-k)+4)!(8k+4)!} \\ &= -\frac{8 \cdot (-1)^{n+1} 2^{6n+3}}{8(n+1) \cdot (8n+7)!} \sum_{k=0}^{n} \binom{8n+8}{8k+4} (-1)^k \cdot 2^{2k} V_{4n-4k+2} B_{8k+4} \\ &= \frac{(-1)^{n+1} 2^{6n+3}}{(8n+7)!} V_{4n+3} \end{aligned}$$

and

$$\sum_{k=0}^{n} \binom{8n+8}{8k+4} (-1)^k \cdot 2^{2k} V_{4n-4k+2} B_{8k+4} = -(n+1) V_{4n+3}.$$

■

References

[1] M. Chellali, Accélération de calcul de nombres de Bernoulli, J. Number Theory 28 (1988) 347–362.

[2] S. Ramanujan, Some properties of Bernoulli's numbers, J. Indian Math. Soc. 3 (1911) 219–234.

[3] Z.H. Sun, On the properties of Newton-Euler pairs, Journal of Number Theory 114 (1) (2005) 88–123.

[4] Z.H. Sun, Combinatorial sum $\sum_{\substack{k=0 \\ k\equiv r \pmod m}}^{n} \binom{n}{k}$ and its applications in number theory I, J. Nanjing Univ. Math. Biquarterly, 9 (1992) 227–240, MR94a:11026.

[5] Z.H. Sun, Combinatorial sum $\sum_{\substack{k=0 \\ k\equiv r \pmod m}}^{n} \binom{n}{k}$ and its applications in number theory II, J. Nanjing Univ. Math. Biquarterly, 10 (1993) 105–118, MR94j:11021.