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Eigenvalue Problem For Perturbated p-Laplacian

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Abstract We want to study the nonlinear eigenvalue problem, for perturbated p-Laplacian operator with zero Dirichlet condition on a bounded region in $\mathbb{R}^N$. Using the Ljusternik-Schnirelman principle we show that the existence of a nondecreasing sequence of nonnegative eigenvalues and a sequence of eigenfunction that weakly convergences to zero function.

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1. Introduction

Eigenvalue problems for the p-laplacian operator subject to zero Dirichlet boundary condition on a bounded domain have been extensively studied during the past tree decades and many interesting results have been obtained. The investigations principally have related on variational methods and minimization techniques of appropriate functionals. We consider existence eigenvalue problem for following Dirichlet problem

$$D(\Omega) : \begin{cases} -\Delta_p u - g(x, u(x), \nabla u) = \lambda |u|^{p-2} u, & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega; \end{cases}$$

where Ω is a bounded region with convenient boundary in $\mathbb{R}^N$. We use the following notations in this context

$$\Delta_p u := div(|\nabla u|^{p-2} \nabla u) \text{ is the p-Laplacian operator with } p > 1,$$

$$\nabla u = Du := (D_1 u, \cdots, D_N u) \text{ is the gradient of } u \text{ in } \mathbb{R}^N.$$

Many results have been obtained on the structure of the spectrum of the Dirichlet problem

$$-\Delta_p u = \lambda |u|^{p-2} u.$$

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It is shown in [1] that there exists a nondecreasing sequence of positive eigenvalues $\{\lambda_n\}$ tending to ∞ as $n \to \infty$. Moreover, the first eigenvalue is simple and isolated, see [2–4]. In [5], a characterization of the second eigenvalue of Dirichlet problem $p-$Laplacian was also given.
For the degenerate elliptic equation

$$-\triangle_p u(x) = f(x, u(x)) \quad \text{in } \Omega$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a caratheodory function, i.e.
$x \to f(x, u(x))$ is measurable on Ω for all $u \in \mathbb{R}$,
$u \to f(x, u(x))$ is continuous for a.e. $x \in \Omega$,
Today's study perturbations of the eigenvalue problem for $p-$Laplacian there are many result for existence and smoothness solutions, e.g. [17, 18]. Laplacian problems with Dirichlet and Robin boundary condition for different perturbations, by different author's, e.g. [6–8] for

$$-\Delta_p u + f(x, u(x)) = \lambda |u|^{p-2} u.$$

Similar studies concerning positive solutions, were studied by Brezis and Oswald in [9] and by Diaz and Saa in [10] (for problems driven by the Dirichlet Laplacian). More recently, Gasinski and Papageorgiau in [11] produced analogous results for the Neuman $p-$Laplacian. The objective of this paper is to obtain for eigenvalue problem of the equation

$$-\triangle_p u + g(x, u(x), Du(x)) = \lambda |u|^{p-2} u \tag{1.1}$$

with Dirichlet boundary condition.

Purpose of this paper is to study for existence of eigenvalue perturbated $p-$Laplacian problem with the Dirichlet boundary value condition.

This paper is organized as follows: We first present the preliminary, then the sake of completeness the Ljusternik-Schnirelman principle and applications to our setting. Then we establish the existence of a sequences of eigenfunctions for the $p-$Laplacian and other suchlike of Laplacian Dirichlet problem.

2. Preliminaries

2.1. Necessaries Condition

Let X be a closed subspace of the sobolev space $W^{1,p}(\Omega)$ containing $W_0^{1,p}(\Omega)$ that $1 < p < \infty, \Omega \subset \mathbb{R}^N$ is a bounded domain with sufficiently smooth boundary, and assume the imbedding $X \hookrightarrow L^p(\Omega)$ is compact.
Instead of (1.1) we may consider a class of nonlinear elliptic operators in divergence form

$$-\sum_{j=1}^{N} D_j a_j(x, u(x), Du(x)) + a_0(x, u(x), Du(x)). \tag{2.1}$$

Assume $a_j : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, for $0 \le j \le N$ are given functions with $a_0(x, \eta, \xi) = g(x, \eta, \xi)$ in which satisfy the following conditions.
$(P1)$ Caratheodory condition:

For $0 \le j \le N$, the function a_j has the following two properties:

(i) $x \longrightarrow a_j(x,\eta,\xi)$ is measurable on Ω for all $\eta \in \mathbb{R}, \xi \in \mathbb{R}^N$.
(ii) $(\eta,\xi) \longrightarrow a_j(x,\eta,\xi)$ is continuous on $\mathbb{R} \times \mathbb{R}^N$ for almost all $x \in \Omega$.

For example, this condition is satisfied if a_j is continuous.
$(P2)$:

$$|a_j(x,\eta,\xi)| \leq c(K(x) + |\eta|^{p-1} + \|\xi\|^{p-1}); \qquad \text{a.e. } x \in \Omega, \eta \in \mathbb{R}, \xi \in \mathbb{R}^N$$

where $K(x) \in L^q(\Omega)$, $\frac{1}{p} + \frac{1}{q} = 1$.
$(P3)$:

$$\sum_{j=1}^{N} (a_j(x,\eta,\xi) - a_j(x,\eta,\tilde{\xi}))(\xi_j - \tilde{\xi}_j) > 0; \quad \text{for a.e. } x \in \Omega, \eta \in \mathbb{R}, \xi \neq \tilde{\xi} \in \mathbb{R}^N.$$

$(P4)$:

$$\frac{\sum_{j=1}^{N} a_j(x,\eta,\xi)\xi_j}{\|\xi\| + \|\xi\|^{p-1}} \longrightarrow +\infty, \qquad \text{as} \qquad \|\xi\| \longrightarrow +\infty.$$

Remark 2.1. In some parts of this paper we need a weaker condition of $(P3)$.
$(\widetilde{P3})$: There exists a constant $\widetilde{c_2} > 0$, such that for a.e. $x \in \Omega$, $\eta \in \mathbb{R}$, and any $\xi, \xi^* \in \mathbb{R}^N$

$$\sum_{j=1}^{N} \left[a_j(x,\eta,\xi) - a_j(x,\eta,\xi^*)\right] \left(\xi_j - \xi_j^*\right) \geq \widetilde{c_2} \left|\xi - \xi^*\right|^p$$

Remark 2.2. If we in the above condition, consider $a_j(x,\eta,\xi) = |\xi|^{p-2}\xi_j$ for $1 \leq j \leq N$, then (2.1) is the same left side of (1.1).

Lemma 2.3. *The functions $a_j(x,\eta,\xi) = |\xi|^{p-2}\xi_j$ for $1 \leq j \leq N$, satisfy in condition $(P1)$-$(P4)$.*

Proof. Since a_j is continuous, $(P1)$ is confirmed. $(P2), (P4)$ are trivial. For $(P3)$,

$$\begin{aligned}
&\sum_{j=1}^{N} (a_j(x,\eta,\xi) - a_j(x,\eta,\tilde{\xi}))(\xi_j - \tilde{\xi}_j) \\
&= \sum_{j=1}^{N} \left[\xi_j|\xi|^{p-2} - \tilde{\xi}_j|\tilde{\xi}|^{p-2}\right](\xi_j - \tilde{\xi}_j) \\
&= \sum_{j=1}^{N} \left(\xi_j^2|\xi|^{p-2} - \xi_j\tilde{\xi}_j|\tilde{\xi}|^{p-2} - \xi_j\tilde{\xi}_j|\xi|^{p-2} + \tilde{\xi}_j^{\,2}|\tilde{\xi}|^{p-2}\right) \\
&= |\xi|^p + |\tilde{\xi}|^p - \left(|\xi|^{p-2} + |\tilde{\xi}|^{p-2}\right)\sum_{j=1}^{N} \xi_j\tilde{\xi}_j \\
&\geq |\xi|^p + |\tilde{\xi}|^p - \left(|\xi|^{p-2} + |\tilde{\xi}|^{p-2}\right)|\xi||\tilde{\xi}| \\
&= \left(|\xi|^{p-1} - |\tilde{\xi}|^{p-1}\right)\left(|\xi| - |\tilde{\xi}|\right) \\
&\geq 0.
\end{aligned}$$

∎

Remark 2.4. Let A be a second order quasilinear elliptic operator in the divergence form:

$$(Au)(x) = \sum_{j=1}^{N} D_j \left(a_j \left(x, u(x), Du(x)\right)\right).$$

Converting A to this form $Au = \sum_{i,j=1}^{N} D_i \left(a^{ij}(x) D_j u\right)$, then clearly $(P1)-(P4)$ are satisfied with $a_j\left(x, u(x), Du(x)\right) = \sum_{i=1}^{N} D_i \left(a^{ij}(x) D_i u\right)$ and $q = 2$, provided that $a^{ij} \in L^{\infty}(\Omega)$ and

$$\sum_{i,j=1}^{N} a^{ij}(x)\xi_i\xi_j \geq \alpha|\xi|^2 \qquad \forall \xi \in \mathbb{R}^N.$$

2.2. Primary Definitions and Properties

Now we give some definitions and corollaries that in sequel are requested.
Assume that X is a real Banach space equipped with norm $\|.\|$, X^* it's topological dual and $\langle\cdot,\cdot\rangle$ is the dual pair between X and X^*.

Definition 2.5. Let $A : X \to X^*$ be an operator.

(i) A is called monotone iff

$$\langle Au - Av, u - v\rangle \geq 0 \quad \forall\, u, v \in X.$$

(ii) A is called uniformly monotone iff

$$\langle Au - Av, u - v\rangle \geq a(\|u - v\|)\|u - v\| \text{ for all } \ u, v \in X,$$

where the continuous function $a : \mathbb{R} \to \mathbb{R}$ is strictly monotone increasing with $a(0) = 0$ and $a(t) \to \infty$ as $t \to \infty$.

For example, we may choose $a(t) = c|t|^{p-1}$ with $p > 1$ and $c > 0$. In this case,

$$\langle Au - Av, u - v\rangle \geq c\|u - v\|^P \quad \text{for all} \quad u, v \in X.$$

Definition 2.6. Let X be a real Banach space . The operator $A : X \to X^*$ is called:

(i) Strongly continuous:

$$u_n \rightharpoonup u \ , \text{ implies } \ Au_n \to Au.$$

(ii) Compact: if A is continuous and maps bounded sets into relatively compact sets.

(iii) Condition $(S)_+$:

$$u_n \rightharpoonup u\,, \limsup_{n\to\infty} \langle Au_n - Au, u_n - u\rangle \leq 0; \quad \text{implies} \quad u_n \to u.$$

(iv) Condition (S):

$$u_n \rightharpoonup u\,, \lim_{n\to\infty} \langle Au_n - Au, u_n - u\rangle = 0; \quad \text{implies} \quad u_n \to u.$$

(v) Condition $(S)_0$:

$$u_n \rightharpoonup u\,, Au_n \rightharpoonup b, \lim_{n\to\infty} \langle Au_n, u_n\rangle = \langle b, u\rangle\,; \quad \text{implies} \quad u_n \to u.$$

(vi) Condition $(S)_1$:

$$u_n \rightharpoonup u\,, Au_n \to b \text{ implies } \ u_n \to u.$$

Obviously, the following holds:

$$(S)_+ \Rightarrow (S) \Rightarrow (S)_0 \Rightarrow (S)_1$$

i.e., if the operator A satisfies the condition $(S)_+$ then A also satisfies in condition (S), etc.

Remark 2.7. [15] If the operator $A : X \to X^*$ on the real reflexive Banach space X; is uniformly monotone, then A satisfies $(S)_+, (S), (S)_0$ and $(S)_1$.

Lemma 2.8. *Let $A, B : X \to X^*$ be operators on the real reflexive Banach space X.*

(i) *If the operator A satisfies $(S)_+$ and B is strongly continuous or, more generally, B is compact; then $A + B$ satisfies $(S)_+$.*

(ii) *If the operator A satisfies (S) and B is strongly continuous; then $A + B$ satisfies (S).*

Proof. (i): Let $u_n \rightharpoonup u, \limsup \langle Au_n + Bu_n - Au - Bu, u_n - u\rangle) \leq 0$ as $n \to \infty$. Since (u_n) is bounded in the reflexive Banach space X and the operator B is compact, there exists a subsequence $(u_{n'})$ such that $Bu_{n'} \to b$ as $n \to \infty$ in X^*, and hence

$$\limsup_{n\to\infty} \langle Au_{n'} - Au, u_{n'} - u\rangle) \leq 0.$$

The operator A satisfies $(S)_+$; therefore $u_{n'} \to u$ as $n \to \infty$. By the convergence principle, (u_n) converges, i.e., $u_n \to u$ as $n \to \infty$.

(ii): Let $u_n \rightharpoonup u$, $\langle Au_n + Bu_n - Au - Bu, u_n - u\rangle) \to 0$ as $n \to \infty$. The operator B is strongly continuous; therefore, $Bu_n \to Bu$ and hence

$$\langle Au_{n'} - Au, u_{n'} - u\rangle) \to 0 \quad \text{as} \quad n \to \infty.$$

The operator A satisfies (S); consequently $u_n \to u$ as $n \to \infty$. ∎

The following theorem applied very frequently. They generalize well-known convergence properties of sequences of real numbers. In the following, strong convergence means convergence in the norm.

Theorem 2.9. [14] *(Convergence Principles in Banach spaces) A sequence (x_n) in a Banach space X has the following convergence properties.*

(i) *Strong convergence. Let x be a fixed element of X. If every subsequence of (x_n) has, in turn, a subsequence which converges strongly to x, then the original sequence converges strongly to x, i.e., $x_n \to x$ as $n \to \infty$.*

(ii) *Weak convergence. Let x be a fixed element in X. If every subsequence of (x_n) has, in turn, a subsequence which converges weakly to x, then the original sequence converges weakly to x, i.e., $x_n \rightharpoonup x$ as $n \to \infty$.*

(iii) *Selection principle. If X is reflexive, then every bounded sequence (x_n) in X has a weakly convergent subsequence $(x_{n'})$, i.e., $x_{n'} \rightharpoonup x$ as $n \to \infty$.Furthermore, $x \in \overline{co}\{x_n : n \in \mathbb{N}\}$.*

(iv) *Weak convergence of bounded sequences. Let (x_n) be a bounded sequence in a reflexive Banach space X. If all the weakly convergent subsequences of(x_n) have the same limit, x, then $x_n \rightharpoonup x$ as $n \to \infty$.*

Theorem 2.10. [15] *(Majorized Convergence). We have*

$$\lim_{n\to\infty} \int_M f_n \, dx = \int_M \lim_{n\to\infty} f_n \, dx$$

where $M \subset \mathbb{R}^N$ is measurable, and all the integrals and limits exist, provided the following conditions hold.

(i) $\|f_n(x)\| \le g(x)$ *for almost all $x \in M$ and all $n \in \mathbb{N}$, and $\int_M g\,dx$ exists.*
(ii) $\lim_{n\to\infty} f_n(x)$ *exists for almost all $x \in M$ where $f_n : M \to Y$, is measurable for all $n \in \mathbb{N}$, and Banach-space Y.*

Theorem 2.11. *(Vitali's theorem) Let $M \subset \mathbb{R}^n$ be a Lebesgue measurable set. Assume that the functions $f_k : M \to \mathbb{R}$ are Lebesgue integrable, further, for a.e. $x \in M$, $\lim_{k\to\infty} f_k(x)$ exists and is finite. The functions f_k are equiintegrable in the following sense: for arbitrary $\varepsilon > 0$ there exist $\delta > 0$ and $S \subset M$ of finite measure such that for all $k \in \mathbb{N}$*

$$\int_H |f_k(x)|\,dx < \varepsilon \quad \text{if} \quad \lambda(H) < \delta \quad \text{and} \quad \int_{M-H} |f_k(x)|\,dx < \varepsilon$$

Then

$$\lim_{k\to\infty} \int_M f_k(x)\,dx = \int_M f(x)\,dx$$

Remark 2.12. It is easy to show that if $f_k \to f$ in $L^1(M)$ then (f_k) is equi integrable. Further, by Hölder's inequality one obtains: if $(|g_k|^p)$ is equi integrable and (h_k) is bounded in $L^p(M)$, $(1 < p < \infty)$ then $(g_k h_k)$ is equi integrable.

Remark 2.13. Theorem (2.11) remains true if we replace the above assumption with:
$\|f_n(x)\| \le g_n(x)$, for almost all $x \in M$ and all $n \in \mathbb{N}$.
All the functions $g_n, g : M \to \mathbb{R}$, are integrable and we have the convergence $g_n \to g$, almost everywhere on M as $n \to \infty$, along with $\int_M g_n(x)\,dx \to \int_M g\,dx$, as $n \to \infty$.

Proposition 2.14. *A sequence (u_n) in $W^{m,p}, 1 < p < \infty$, is weakly convergent if and only if $(\partial^\alpha u_n)$ is weakly convergent in L^p for all α, with $|\alpha| \le m$. In this case we have $u_n \rightharpoonup u$ in $W^{m,p}$ if and only if for each α, with $|\alpha| \le m$ we have $\partial^\alpha u_n \rightharpoonup \partial^\alpha u$ in L^p.*

Proof. We may assume all sequences are bounded. If $u_n \rightharpoonup u$ in $W^{m,p}$, then for $\varphi \in C_0^\infty(\Omega)$ we have

$$\langle \partial^\alpha u_n, \varphi\rangle = (-1)^{|\alpha|} \langle u_n, \partial^\alpha \varphi\rangle \to (-1)^{|\alpha|} \langle u, \partial^\alpha \varphi\rangle = \langle \partial^\alpha u, \varphi\rangle$$

and $(\partial^\alpha u_n)$ is bounded in L^p with density of C_0^∞ in L^p, so $\partial^\alpha u_n \rightharpoonup \partial^\alpha u$ in L^p.

Conversely, let $u_n \rightharpoonup u$ and $\partial^\alpha u_n \rightharpoonup v_\alpha$ in L^p; each ∂^α operator is weakly closed, so $\partial^\alpha u_n \rightharpoonup \partial^\alpha u$. The result follows from the representation of $(W^{\alpha,p})'$ ∎

In sequal we need the following inequality for real number.

Lemma 2.15. [15] *For $0 < r < \infty$, and all nonnegative real numbers $\xi_1, \dots, \xi_N$, we have the following inequality*

$$\left(\sum_{i=1}^N \xi_i\right)^r \le c \sum_{i=1}^N \xi_i^r,$$

where the positive constant c depend only on N, r.

2.3. Concept of Genus

Definition 2.16. Let X be a real Banach space.To each symmetric set K, (i.e. if $u \in K$ then $-u \in K$) we assign a number $genK$ (which is called the genus of K) in the following way:

(i) $gen\,\emptyset = 0$.
(ii) If $K \neq \emptyset$, then let $gen\,K$ be the smallest natural number $n \geq 1$ for which a zero-free mapping $f : K \to \mathbb{R}^n - \{0\}$ that is odd and continuous exists.
(iii) If for $K \neq \emptyset$ there does not exist such n, then we set $gen\,K = +\infty$.

For example if K is the boundary of the unit disk in $\mathbb{R}^2$, then $gen\,K = 2$ and for sphere $S = \{u \in X : \|u\| = 1|\}$ in the real Banach space X, $gen\,S = \, dim\,X$.

Corollary 2.17. [16] *For any symmetric sets K, K_1, K_2 in Banach pace X, the following four assertions hold:*

(i) *$gen\,K_1 = gen\,K_2$ provided K_1 and K_2 are homoeomorphic with respect to an odd homeomorphism.*
(ii) *$gen\,K_1 < \infty$ implies $gen\,(\overline{K_2 - K_1}) \geq gen\,K_2 - gen\,K_1$.*
(iii) *$gen\,K \leq \, dim\,X$.*
(iv) *From $gen\,K > m$, $1 \leq m < \infty$, it follows that $K \cap (I - P)(X) \neq \emptyset$ when $P : X \to X_1$ is a continuous linear projection operator on the m-dimensional subspace X_1 of X.*

2.4. Sobolev Embedding Theorem

We recall the Sobolev embedding theorem.

Theorem 2.18. [12] *Let $\Omega \subset \mathbb{R}^N$ be open, bounded and have smooth boundary. Let $p \geq 1$.*

(i) *If $p < N$, then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for every $q \in \left[1, \frac{Np}{N-p}\right]$; and this embedding is compact for every $q \in \left[1, \frac{Np}{N-p}\right)$.*
(ii) *If $p = N$, then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for every $q \in [1, +\infty]$; and this embedding is compact.*
(iii) *If $p > N$, then $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$; the embedding is compact.*

Proposition 2.19. [14] *Let Ω be a bounded domain in $\mathbb{R}^N$ with $N \geq 1$. Then the following norms are equivalent on $W^{1,p}$*

$$\|u\| = \left(\int_\Omega \left(|u|^p + \sum_{i=1}^N |D_i u|^p \right) dx \right)^{\frac{1}{p}}$$

and

$$\|u\| = \left(\int_\Omega \left(\sum_{i=1}^N |D_i u|^p \right) dx \right)^{\frac{1}{p}}.$$

3. The Ljusternik-Schnirelman Principle in Banach Space

Let X be a real reflexive Banach space with $dimX = \infty$ and F, G be two functionals on X. for fixed $\alpha > 0$ consider the eigenvalue problem

$$F'(u) = \lambda G'(u) \qquad u \in N_\alpha, \lambda \in \mathbb{R} \tag{3.1}$$

that $N_\alpha := \{u \in X; G(u) = \alpha\}$ is the level set of G and following assumption holds:

- $(H1)$ Functionals $F, G \in C^1(X, \mathbb{R})$ are even functionals such that $F(0) = G(0) = 0$ (in particular F', G' are odd potential operators).
- $(H2)$ The operator F' is strongly continuous and if $F(u) \neq 0$ for $u \in \overline{co}N_\alpha$, implies that $F'(u) \neq 0$.
- $(H3)$ The operator G' is uniformly continuous on bounded sets and satisfies condition $(S)_1$.
- $(H4)$ The level set N_α is bounded and $u \neq 0$ implies

$$\langle G'(u), u\rangle > 0, \lim_{t\to+\infty} G(tu) = +\infty, and \inf_{u\in N_\alpha} \langle G'(u), u\rangle > 0.$$

It is known that (u, λ) solve (3.1) if and only if u is a critical point of F with respect to N_α, see [16].
For any $n \in \mathbb{N}$, denote by $\mathcal{A}_n$, the class of all compact, symmetric subsets K of N_α. We define:

$$\pm c_n = \begin{cases} \sup_{K\in\mathcal{A}_n} \inf_{u\in K} \pm F(u) & \text{if} \quad \mathcal{A}_n \neq \emptyset \\ 0 & \text{if} \quad \mathcal{A}_n = \emptyset \end{cases}$$

for $n = 1, 2, ...$; and

$$\chi_\pm := \begin{cases} \sup\{n \in \mathbb{N} : \pm c_n > 0\} & \text{if} \quad c_1 > 0 \\ 0 & \text{if} \quad c_1 = 0 \end{cases}$$

Theorem 3.1. [16]*(Ljusternik-Schnirelman; L-S principle) Under the assumptions* $(H1)$ - $(H4)$ *the following assertion holds:*

- (i) *Existence of an eigenvalue: if* $\pm c_n > 0$ *, (+ or -) then (3.1) possesses a pair* (u_n, u_{-n}) *of eigenvectors with eigenvalue* $\lambda_n \neq 0$ *and* $F(u_n) = c_n$*. Moreover, if* F' *and* G' *are positive homogeneous,(i.e.* $F'(tu) = tF'(u)$ *and* $G'(tu) = tG'(u)$ *for all* $u \in X, t > 0$ *) then* $c_n = \alpha\lambda_n$*.*
- (ii) *Multiplicity: (3.1) has at least* $\chi_+ + \chi_-$ *pairs* $(u, -)$ *of eigenvectors with eigenvalues that are different from zero. If* $\pm c_m = \pm c_{m+1} = ... = \pm c_{m+p} > 0$*,* $p > 1$*, (+ 0r -) then the set of all eigenvectors of (3.1) such that* $F(u) = c$ *has genus greater than or equal to* $p + 1$*. In particular, this set is infinite.*
- (iii) *Critical levels:* $\pm\infty > \pm c_1 \geq \pm c_2 \geq ... \geq 0$ *and* $c_m \to 0$ *as* $m \to \infty$*.*
- (iv) *Infinitely many eigenvalues: If* $\chi_+ = \infty$ *or* $\chi_- = \infty$ *and* $F(u) = 0$*, for* $u \in \overline{co}N_\alpha$ *implies that* $\langle F'(u), u\rangle = 0$*; then there exists a sequence* (λ_m) *of infinitely many distinct eigenvalues for (3.1) such that* $\lambda \to 0$ *as* $m \to \infty$*.*
- (v) *Weak convergence of eigenvectors: Assume that* $F(u) = 0, c \in \overline{co}N_\alpha$*, implies* $u = 0$*. then* $\max(\chi_+, \chi_-) = +\infty$*, and there exists a sequence of eigensolutions* (u_m, λ_m) *of (3.1) such that* $u_m \rightharpoonup 0, \lambda_m \to 0$*, as* $m \to \infty$ *and* $\lambda_m \neq 0$ *for all* m*.*

we have the following corollaries:

Corollary 3.2. *By virtue of a radical projection, the level set* N_α *is homomorphic to the unit sphere in* X *and* $0 \notin N_\alpha$*.*

Corollary 3.3. *$\chi_{\pm} > dim X_1$ provided there exists a linear subspace X_1 of X such that $\pm F > 0$ on $N_\alpha \cap X_1$, (+ or -).*

Corollary 3.4. *$\chi_+ = \infty$ or $\chi_- = \infty$, when the set of zeros $N_\alpha^o := \{u \in N_\alpha; F(u) = 0\}$ is compact or more generally there exists a closed linear subspace X_1 of X such that $dim(\frac{X}{X_1}) = \infty$ and $dist(\|u\|^{-1}u, X_1) < \eta$ for all $u \in N_\alpha^o$ and fixed $\eta \in]0,1[$.*

Example 3.5. The assumptions $(H1)$-$(H4)$ are fulfilled provided the following hold:

(i) X is a real separable $H-$space with $dim X = \infty$. We identify X with X^*.

(ii) $A : X \to X$ is a linear, compact, and symmetric operator. We set $F(u) = 2^{-1}\langle Au, u\rangle$ and $G(u) = 2^{-1}\langle u, u\rangle$.

Then $F' = A, G' = I$, N_α is a sphere, and eigenvalue problem

$$F'(u) = \lambda G'(u) \quad u \in N_\alpha,\ \lambda \in \mathbb{R}$$

corresponds to

$$Au = \lambda u, \quad \lambda \in \mathbb{R}$$

with the normalizing condition $G(u) = \alpha$, i.e., $u \in N_\alpha$. It can be shown that

$$c_m^{\pm} = \alpha\lambda_m^{\pm} \quad \text{when} \quad \pm c_m^{\pm} > 0.$$

Here, $\lambda_m^{\pm}$, is the eigenvalue of A. All eigenvalues that are different from zero of A are obtained from $c_m^{\pm}$ according to their multiplicity. Therefore, A has at least $\chi_+ + \chi_-$ pairs $(u, -u)$ of eigenvectors on N_α with the corresponding eigenvalues that are different from zero. If $\lambda_m^{\pm}$ has the multiplicity $p+1$, i.e., $\lambda_m^{\pm} = \lambda_{m+1}^{\pm} = \cdots = \lambda_{m+p}^{\pm}$, then the corresponding eigenvectors on N_α form a $p-$dimensional sphere and the genus of this set is $p+1$ according to the genus of spheres.

Example 3.6. As an another application of the this theorem, we consider the classical boundary eigenvalue problem

$$-\lambda \sum_{i=1}^{N} D_i \left(|D_i u(x)|^{p-2} D_i u(x)\right) = g'(u(x)), \tag{3.2}$$

with Dirichlet condition on boundary.

$(E1)$ Let Ω be a bounded region in $\mathbb{R}^N, N \geq 1$. Furthermore, let $p \geq 2$. we set $\xi = (\xi_l, \cdots, \xi_N), D_i; = \frac{\partial}{\partial \xi_i}$.

$(E2)$ $g : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, with $g(0) = 0$ and $g'(u)u > 0$ for all real numbers $u \neq 0$. There exist constants $c, d > 0$ such that the following growth condition holds for all $u \in \mathbb{R}$:

$$|g(u)| \leq c(1 + |u|^p), \ |g'(u)| \leq d(1 + |u|^{p-1}).$$

Let $X = W_0^{1,p}(\Omega)$. The generalized problem for (3.2) reads as follows: We seek $u \in X, \lambda \in \mathbb{R}$ such that

$$\lambda b(u, v) = a(u, v) \quad \text{for all} \quad v \in X,\ G(u) = \alpha \tag{3.3}$$

for fixed $\alpha > 0$. Here,

$$G(u) = p^{-1} \int_\Omega \sum_{i=1}^{N} |D_i u|^p \, dx, \ \ F(u) = \int_\Omega g(u) \, dx,$$

$$b(u,v)=\int_\Omega \sum_{i=1}^N |D_i u|^{p-2} D_i u D_i v\,dx,\ \ a(u,v)=\int_\Omega g'(u)v\,dx,$$

it is easy to see that With the assumptions $(E1)$ and $(E2)$, the following two assertions hold:

(i) (3.3) has an eigensolution (u,λ), with $u\neq 0, \lambda>0$.

(ii) If g is even, then (3.3) has infinitely many eigensolutions (u_m,λ_m), with $u_m\neq 0, \lambda_m>0$ for all $m\in\mathbb{N}$ such that $u_m\rightharpoonup 0$ in X as well as $\lambda_m\to 0$ as $m\to\infty$.

4. Perturbated p-Laplacian Operator

In this section we consider $X=W^{1,p}(\Omega)$, the sobolev space and following nonlinear elliptic equation in divergence form.

$$-\sum_{j=1}^N D_j a_j\left(x,u(x),Du(x)\right)+a_0(x,u(x),Du(x))=f, \tag{4.1}$$

where Ω is a bounded domain in $\mathbb{R}^N, N\geq 1$.
The weak solution u for (4.1) reads as follows:

$$\sum_{j=1}^N\int_\Omega a_j\left(x,u(x),Du(x)\right)D_j\varphi\,dx+\int_\Omega a_0\left(x,u(x),Du(x)\right)\varphi\,dx=\int_\Omega f\varphi\,dx \tag{4.2}$$

for all $\varphi\in C_o^\infty(\Omega)$.

Remark 4.1. Under assumptions of sufficient smoothness of the boundary of the domain Ω, the functions a_j and the weak solution $u(x)$, in the case $X=W_0^{1,p}$ we obtain that $u(x)$is a solution of the equation (4.1) in the usual sense, and satisfying the boundary conditions $u(x)=0$, for $x\in\partial\Omega$, i.e. $u(x)$ is a solution of the Dirichlet problem.

We define the nonlinear operator $A:X\to X^*$ by the equality

$$\langle Au,\varphi\rangle=\sum_{j=1}^N\int_\Omega a_j\left(x,u(x),Du(x)\right)D_j\varphi\,dx+\int_\Omega a_0\left(x,u(x),Du(x)\right)\varphi\,dx \tag{4.3}$$

for $u,\varphi\in X$.

Proposition 4.2. *Under the condition $(P1),(P2)$ the operator $A:X\to X^*$ defined by (4.3) is continuous and bounded.*

Proof. Boundedness: condition $(P1)$, implies that the function $x\to a_j\left(x,u(x),Du(x)\right)$ is measurable for arbitrary $u\in X$. Further, by $(P2)$ and lemma (2.15)

$$\begin{aligned}&\int_\Omega |a_j\left(x,u(x),Du(x)\right)|^q\,dx\\ &\leq const\times\left[\int_\Omega K(x)^q dx+\int_\Omega |u(x)|^{(p-1)q}dx+\int_\Omega |Du(x)|^{(p-1)q}dx\right]\\ &\leq const\left[const+\|u\|_X^p\right]\leq const\left[1+\|u\|_X^p\right]\end{aligned}$$

so

$$\left(\int_{\Omega} |a_j\left(x, u(x), Du(x)\right)|^q \, dx\right)^{\frac{1}{q}} \leq const\left[1 + \|u\|_X^{\frac{p}{q}}\right].$$

Hölder's inequality implies that

$$\begin{aligned}|\langle Au, \varphi\rangle| &\leq \sum_{j=1}^{N}\left[\int_{\Omega} |a_j(x, u(x), Du(x))|^q \; dx\right]^{\frac{1}{q}} \|D_j\varphi\|_{L^p(\Omega)} \\ &\quad + \left[\int_{\Omega} |a_0(x, u(x), Du(x))|^q \; dx\right]^{\frac{1}{q}} \|\varphi\|_{L^p(\Omega)} \\ &\leq const\left[1 + \|u\|_X^{\frac{p}{q}}\right] \|\varphi\|_{L^p(X)}.\end{aligned}$$

It follows that A is a bounded operator on X and $\|Au\|_{X^*} \leq const\left[1 + \|u\|_X^{\frac{p}{q}}\right]$. Contiuity: to prove this, let $u_n \to u$ in X, as $n \to \infty$.The compact embedding of X to $L^p(\Omega)$, implies that there exists a subsequence $(u_{n'})$ and a function $v(x) \in L^p(\Omega)$ in which $u_{n'} \to u(x)$ as $n \to \infty$ for almost all $x \in \Omega$, and the majorant condition $|u_{n'}(x)| \leq v(x)$ for all n' and almost all $x \in \Omega$.
Therefore, by the proposition (2.19), lemma (2.15) and $(P2)$,

$$\begin{aligned}&\|Au_{n'} - Au\|_q^q \\ &= \sup_{\|\varphi\|=1} |\langle Au_{n'} - Au, \varphi\rangle|^q \\ &= \sup_{\|\varphi\|=1} \Bigg|\sum_{j=1}^{N}\int_{\Omega} \big(a_j\left(x, u_{n'}, Du_{n'}\right) - a_j(x, u, Du)\big) D_j\varphi dx \\ &\quad + \int_{\Omega} \left(a_0(x, u_{n'}, Du_{n'}) - a_0(x, u, Du)\right)\varphi dx\Bigg|^q \\ &\leq const \int_{\Omega} \left(|K(x)|^q + |v(x)|^q + |u(x)|^q\right) \, dx.\end{aligned}$$

Now majorant convergence gives $\|Au_n - Au\|_q \to 0$, i.e. $Au_n \to Au$ in $L^q(\Omega)$ as $n \to \infty$ and by convergence principle, the entire sequence converges, i.e. $Au \to Au$ in $L^q(\Omega)$. ■

Proposition 4.3. *Under the condition* $(P1), (P2), (\widetilde{P3})$*; the operator* $A : X \to X^*$ *defined by (4.3) satisfies condition* $(S)_+$.

Proof. Assume that

$$(u_k) \rightharpoonup u \quad \text{ in } V, \quad \limsup_{k\to\infty} \langle Au_k, u_k - u\rangle \leq 0 \tag{4.4}$$

since $W^{1,p}(\Omega)$ is compactly embedded into $L^p(\Omega)$ (for bounded Ω with sufficiently smooth boundary, (see theorem 2.18)), there is a subsequence of (u_k), again denoted by (u_k), such that

$$u_k \to u \;\text{ in }\; L^p(\Omega) \;\text{ and a.e. in }\; \Omega \tag{4.5}$$

since (D_ju_k) is bounded in $L^p(\Omega)$, we may assume (on the subsequence) that

$$D_ju_k \rightharpoonup D_ju \;\text{ in }\; L^p(\Omega) \text{ for } \; j = 1, \cdots, n$$

further

$$\begin{aligned}\langle Au_k, u_k - u\rangle = &\int_\Omega a_0(x, u_k, Du_k)(u_k - u)\\ &+ \sum_{j=1}^{N} \int_\Omega \left[a_j(x, u_k, Du_k) - a_j(x, u_k, Du)\right](D_j u_k - D_j u)\, dx \\ &+ \sum_{j=1}^{N} \int_\Omega a_j(x, u_k, Du)(D_j u_k - D_j u)\, dx.\end{aligned} \tag{4.6}$$

The first term on the right-hand side tends to zero, from (4.5) and Hölder's inequality and since the multipliers of $(u_k - u)$ are bounded in $L^p(\Omega)$ by $(P2)$. Further, the third term on the right-hand side converges to 0 too, by (2.14) and since (4.5), $(P1), (P2)$ and Vitali's theorem imply that

$$a_j(x, u_k, Du) \to a_j(x, u, Du) \ \text{ in } \ L^q(\Omega).$$

Consequently,from (4.4) and (4.6)

$$\limsup_{k\to\infty} \langle Au_k, u_k - u\rangle \leq 0$$

so

$$\limsup_{k\to\infty} \sum_{j=1}^{N} \left[a_j(x, u_k, Du_k) - aj(x, u_k, Du)\right](D_j u_k - D_j u) \leq 0$$

and from $(\widetilde{P3})$,

$$\lim_{k\to\infty} \int_\Omega |Du_k - Du|^p\ dx = 0 \quad \text{ or } \quad Du_k \to Du \quad \text{in} \quad L^p(\Omega).$$

Moreover toward subsequence

$$(Du_k) \to Du \quad \text{a.e. in} \quad \Omega.$$

So then

$$\|u_k - u\|_X = \left(\int_\Omega \sum_{j=1}^{N} |D_j u_k - D_j u|^p\, dx + \int_\Omega |u_k - u|^p\, dx\right)^{\frac{1}{p}} \to 0.$$

∎

Instead of $(\widetilde{P3})$ we may assume $(P3)$ (In the linear replace
$(P4')$: There exist a constant $c_2 > 0$ and $k_2 \in L^1(\Omega)$ such that

$$\sum_{j=1}^{N} aj(x, \eta, \zeta)\xi_j \geq c_2|\zeta|^p - k_2(x).$$

Theorem 4.4. *Assume $(P1), (P2), (P3'), (P4')$. Then the (bounded) operator A, defined by (4.3) with an arbitrary (possibly unbounded) domain $\Omega \subset \mathbb{R}^N$, satisfied in condition $(S)_+$.*

Proof. Assume that $u_k \rightharpoonup u$, in X and $\limsup_{k|\infty} \langle A(u_k), u_k - u\rangle \leq 0$. We have shown that $u_k \to u$, in X.
We will show, this is true for a suitable subsequence of (u_k). Then by Cantor's trick this will imply for (u_k), too.
Assume that (Ω_m) is a sequence of bounded domains with sufficiently smooth boundary $\partial\Omega_m$ such that $\Omega_m \subseteq \Omega_{m+1}$ and $\Omega = \bigcup_{m=1}^{\infty} \Omega_m$.
From Sobolev embedding theorem, for any fixed m, there is a subsequence of (u_k) which is convergent in $L^p(\Omega_m)$, and so a subsequence of this subsequence is a.e. convergent to u in Ω_m. Using a "diagonal process" one obtains a subsequence of (u_k), which converges to u a.e. in Ω.
For simplicity, we shall denote this subsequence also by (u_k), so

$$u_k \to u \text{ a.e. in } \Omega \tag{4.7}$$

We claim that

$$Du_k \to Du \text{ a.e. in } \Omega. \tag{4.8}$$

For this, we set

$$\begin{aligned} P_k(x) = &\sum_{j=1}^{N} [a_j(x, u_k, Du_k) - a_j(x, u, Du)] (D_j u_k - D_j u) \\ &+ [a_0(x, u_k, Du_k) - a_0(x, u, Du)] (u_k - u). \end{aligned} \tag{4.9}$$

Then

$$\langle A(u_k) - A(u), u_k - u\rangle = \int_\Omega P_k(x)\, dx$$

and by the assumption

$$\limsup_{k\to\infty} \int_\Omega P_k(x)\, dx \leq 0. \tag{4.10}$$

Due to (4.9)

$$P_k(x) = \sum_{j=1}^{N} a_j(x, u_k, Du_k) u_k - g_k(x),$$

where

$$\begin{aligned} g_k(x) = &\left[\sum_{j=1}^{N} a_j(x, u, Du)(D_j u_k - D_j u) + a_0(x, u, Du)(u_k - u)\right] \\ &+ \left[\sum_{j=1}^{N} a_j(x, u_k, Du_k) D_j u + a_0(x, u_k, Du_k) u\right] \end{aligned} \tag{4.11}$$

By $(P2)$

$$\begin{aligned} |g_k(x)| \leq &c_4 \left[|u|^{p-1} + |Du|^{p-1} + k_1(x)\right] [|u_k| + |Du_k| + |u| + |Du|] + \\ &c_5 \left[|u_k|^{p-1} + |Du_k|^{p-1} + k_1(x)\right] [|u| + |Du|]. \end{aligned} \tag{4.12}$$

Hölder's inequality implies that the sequence (g_k) is equiintegrable.
Further, by Young's inequality from (4.12), for any $\varepsilon > 0$, there exist a constant $c(\varepsilon)$ and

a function $k_4 \in L^1(\Omega)$ such that

$$|g_k(x)| \leq \varepsilon |Du_k|^p + c(\varepsilon)\left[|u_k|^p + |u|^p + |Du|^p + k_4(x)\right] \tag{4.13}$$

Choosing sufficiently small $\varepsilon > 0$, one obtains from $(P4')$, (4.13) and definition of $P_k(x)$

$$\begin{aligned} P_k(x) &\geq c_2|Du_k|^p - k_2(x) - |g_k(x)| \\ &\geq \frac{c_2}{2}|Du_k|^p - c_6\left[|u_k|^p + |u|^p + |Du|^p + k_5(x)\right] \end{aligned} \tag{4.14}$$

for some constant c_6 and $k_5 \in L^1(\Omega)$. Let

$$P_k^+(x) = \max\{P_k(x), 0\} \quad , \quad P_k^-(x) = -\min\{P_k(x), 0\}$$

then by (4.14),

$$0 \leq P_k^-(x) \leq k_2(x) + |g_k(x)|$$

where the sequence on the right hand side is equiintegrable. Hence, the sequence $(P_k^-)_{k\in\mathbb{N}}$ is equiintegrable.
We show that P_k^- converges to 0 a.e. in Ω. Indeed, P_k can be written in the form

$$P_k(x) = q_k(x) + r_k(x) + s_k(x) \tag{4.15}$$

where

$$\begin{aligned} q_k(x) &= \sum_{j=1}^{N}\left[a_j(x, u_k, Du_k) - a_j(x, u_k, Du)\right](D_j u_k - D_j u), \\ r_k(x) &= \sum_{j=1}^{N}\left[a_j(x, u_k, Du) - a_j(x, u, Du)\right](D_j u_k - D_j u), \\ s_k(x) &= \left[a_0(x, u_k, Du_k) - a_0(x, u, Du)\right](u_k - u). \end{aligned}$$

Denote by χ_k the characteristic function of the set $\{x : P_k^-(x) > 0\}$ then

$$-P_k^- = \chi_k q_k + \chi_k r_k + \chi_k s_k. \tag{4.16}$$

From (4.14)

$$\frac{c_2}{2}|Du_k|^p \leq c_6\left[|u_k|^p + |u|^p + |Du|^p + k_5(x)\right] \text{ if } \quad P_k(x) < 0.$$

Hence, (4.7) the sequence $(\chi_k Du_k)$ is bounded for a.e. x.
From (4.7) and $(P2)$

$$(\chi_k r_k) \to 0 \text{ a.e. and } \ (\chi_k s_k) \to 0 \text{ a.e..}$$

Since $\chi_k q_k \geq 0$ a.e., it follows from (4.16)

$$(P_k^-) \to 0 \quad \text{a.e..} \tag{4.17}$$

Thus, by equiintegrality of $(P_k^-)_{k\in\mathbb{N}}$ and Vitali's theorem

$$\lim_{k\to\infty}\int_\Omega P_k^- \, dx = 0 \tag{4.18}$$

Since $0 \leq P_k^+ = P_k + P_k^-$, from (4.10), (4.18) we obtain $\lim_{k\to\infty}\int_\Omega P_k^+ \, dx = 0$ and $(P_k^+) \to 0$ a.e. for a subsequence (again denoted by (P_k^+), for simplicity).
$(P_k^-) \to 0$,a.e. implies that $(P_k) \to 0$ a.e.
Hence, (4.14) implies that for a.e. $x \in \Omega$ the sequence $(Du_k(x))$ is bounded.
Consider such a fixed $x \in \Omega$. Assuming that (4.8) is not valid there is a subsequence

of $(Du_k(x))$, (again denoted by $(Du_k(x))$, for simplicity), in which converges to some $\zeta^* \neq (Du)(x)$. Since

$$u_k(x) \to u(x)\ ,\ r_k(x) \to 0\ ,\ s_k(x) \to 0;$$

we obtain that

$$0 = \lim_{k\to\infty} p_k(x) = \sum_{j=1}^{N} \left[a_j(x, u(x), \zeta^*) - a_j(x, u(x), Du(x))\right] (\xi_j^* - D_j u(x))$$

Thus $(P3')$ implies $\zeta^* = Du(x)$, which contradicts to $\zeta^* \neq Du(x)$.
So, we have shown

$$Du_k \to Du \quad \text{a.e. in} \quad \Omega$$

and

$$\|u_k - u\|_X \to 0 \quad \text{or} \quad u_k \to u \quad \text{in} \quad X.$$

■

5. Application of L-S Principle to Perturbated p-Laplacian

In sequel, we assume that for left hand of (4.1), there exists an even Frechet differentiable functional $G_1 \in C^1(X, \mathbb{R})$, such that

$$\langle G_1' u, v\rangle = -\sum_{j=1}^{N} \int_\Omega D_j a_j\left(x, u(x), Du(x)\right) v(x)\, dx.$$

Moreover we assume that $G_1(0) = 0$, and for $u \neq 0$,

$$\lim_{t\to\infty} G_1(tu) = +\infty, \quad \text{and} \quad \langle G' u, u\rangle > 0.$$

This assumption is requirement for applying the L-S Theorem for p-Laplacian.

Remark 5.1. We point that the goal is investigation of p-Laplacian equation with a perturbation, such as $a_0(x, u, Du)$, that in this case we may get $G_1(u) = \frac{1}{p}|u|^p$ and see that $\langle G'(u), v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v$ ([12]).

However, for a detailed discussion in regards with Frechet-differentiability of G, one can refer to ([13]).

Remark 5.2. In general for a differential operator A, in the form

$$Au = \sum_{|\alpha|,|\beta|} D^\alpha \left(A_\alpha\left(x, D^\beta u\right)\right)$$

that arise naturally in physics, we note that, the operators of this type are generally the Euler-Lagrange equations of some energy functionals of the form $I\left(x, u, Du, \cdots, D^m u\right)$.

But for therm $a_0(x, u, Du)$ in (4.1), we use of the following lemma.

Lemma 5.3. *If $a_0(x, \eta, \xi)$ satisfy Hölder condition of order $\alpha > 1$, on X, respect to η, then the functional $G_2 : X \to \mathbb{R}$ defined by*

$$G_2(u) = \int_\Omega a_0(x, u(x), Du(x))\, dx$$

is continuously Frechet-differentiable and

$$\langle DG_2(u), v\rangle = \int_\Omega a_0(x, u(x), Du(x)) v(x)\, dx.$$

Proof. We have

$$\begin{aligned}
&\lim_{\|v\|\to 0} \frac{G_2(u+v) - G_2(u) - \langle DG_2(u), v\rangle}{\|v\|} \\
&= \lim_{\|v\|\to 0} \frac{\int_\Omega [a_0(x, u+v, Du+Dv)(u+v) - a_0(x,u,Du)u - a_0(x,u,Du)v] \ dx}{\|v\|} \\
&= \lim_{\|v\|\to 0} \frac{1}{\|v\|} \int_\Omega [a_0(x, u+v, Du+Dv) - a_0(x,u,Du)] (u+v)\, dx \\
&\le \int_\Omega \|v\|^{\alpha-1}(u+v)\, dx = 0.
\end{aligned}$$

the continuiuty of DG_2 is obvious from condition on a_0. ■

Now we ready to apply the L-S theorem for (1.1).
We consider $\lambda h'(u)$, instead of $\lambda|u|^{p-2}u$ in right-hande of (1.1), and assume that

(H) $h : R \to R$ is even and continuously differentiable function with $h(0) = 0$, and $h'(u)u > 0$, for all real number $u \neq 0$, and there exist constants $c, d > 0$ such that the following growth condition holds for all $u \in \mathbb{R}$,

$$|h(x)| \le c(1 + |x|^p)$$

$$|h'(x)| \le d(1 + |x|^{p-1})$$

So the weak solution of

$$D(\Omega) : \begin{cases} Au = \lambda h'(u), & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega; \end{cases} \tag{5.1}$$

reeds as follows:

$$\langle Au, v\rangle = \lambda \int_\Omega h'(x) v\, dx \tag{5.2}$$

for all $v \in C_0^\infty(\Omega)$, that A defined by (4.3).
Now we define functional $H, G : X \to \mathbb{R}$, by

$$H(u) = \int_\Omega h(u(x))\, dx;$$

and

$$G(u) = G_1(u) + G_2(u).$$

Lemma 5.4. *H and G are continuously Frechet-differentiable on X with*

$$\langle H'(u), v\rangle = \int h'(u) v\, dx \tag{5.3}$$

$$\langle G'(u), v\rangle = \langle Au, v\rangle \tag{5.4}$$

for all $u, v \in X$.

Proof. To check this, first we prove that H is Gateaux differentiable, and H'_G is continuous.
From the growth assumption on h, $H(u)$ is well defined via the sobolev inequality and sobolev embedding theorem. Now for fixed $u, v \in X$, it is obvious that, for almost every

$x \in \mathbb{R}$

,

$$\lim_{t\to 0} \frac{h(u(x)+tv(x))-h(u(x))}{t} = h'(u(x))v(x)$$

so there exists a real number θ such that $|\theta| \le |t|$ and

$$\begin{aligned}\left|\frac{h(u(x)+tv(x))-h(u(x))}{t}\right| &= |h'(u(x)+\theta v(x))v(x)| \\ &\le d(1+|u+\theta v|^{p-1})|v| \\ &\le (constant)(|v|+|u|^{p-1}|v|+|v|^p)\end{aligned}$$

As the function $|v|+|u|^{p-1}|v|+|v|^p$ is in $L^1(\Omega)$ by dominated convergence, we have

$$\lim_{t\to 0}\int_\Omega \frac{h(u+tv)-h(u)}{t} = \int_\Omega h'(u)v\,dx$$

The right hand side is a function of v and is a continuous linear functional on $W_{1,p}(\Omega)$. It is the Gateaux differential of H and therefore (Frechet) differentiable.
H is continuous, so take a sequence $\{u_k\}$ in $W_{1,p}$ such that $u_k \to u$. up to a subsequences, by the Hölder inequality;

$$\begin{aligned}|(H(u_k)-H(u))v| &\le \int_\Omega |h'(u_k)-h'(u)|\,|v|\;dx \\ &\le \left(\int_\Omega |h'(u_k)-h'(u)|^p\;dx\right)^{\frac{1}{p}} \left(\int_\Omega |v|^q\,dx\right)^{\frac{1}{q}}\end{aligned}$$

since $\lim_{k\to\infty} |h'(u_k(x)-h'(u(x)))| = 0$ a.e. in Ω and

$$\begin{aligned}|h'(u_k)-h'(u)|^p &\le c\left(1+|u_k|^{p-1}+|u|^{p-1}\right)^p \\ &\le c\left(1+|w|^{p-1}+|u|^{p-1}\right)^p \in L^1(\Omega)\end{aligned}$$

then by dominated convergence, $\int |h'(u_k)-h'(u)|^p\,dx \to 0$ so that

$$\begin{aligned}\|H(u_k)-H(u)\| &= \sup\{(H(u_k)-H(u))(v);\ \|v\|=1, v\in W_{1,p}\} \\ &\le \left(\int |h'(u_k)-h'(u)|^p\,dx\right)^{\frac{1}{p}} \to 0\end{aligned}$$

as $k \to 0$.
For $G(u)$, this is result of above argument in begining of this section. ■

Lemma 5.5. *H' is strongly continuous.*

Proof. by embedding theorem of Sobolev spaces, we have embedding $X \hookrightarrow L_p(\Omega)$ is compact. so if $u_n \rightharpoonup u$ in X then $u_n \to u$ in $L_p(\Omega)$ and since H' is continuous $H'(u_n) \to H'(u)$. ■

6. Main Results

Theorem 6.1. *With assumption $(H),(P1),(P2),(P3),(P4')$ on the coefficient function of the equation (5.2), the following two assertion hold:*

(i) (5.2) has an eigensolution with $\lambda > 0$, and eigenfunction $u \neq 0$.
(ii) If h is even, then (5.2) has infinitely many eigensolutions (u_n, λ_n), with $u_n \neq 0$, $\lambda_n > 0$, for all $n \in \mathbb{N}$, such that $\lambda_n \to 0$.

Proof. According to lemma (5.5),(5.4) and propositions (4.2) and (4.3), H and G are continuously Frechet-differentiable on X. H' is strongly continuous, G' is continuous and bounded.
Condition (H) yields that $\langle H'(u), u\rangle > 0$ for all $u \neq 0$ in X.
Now we verify the assumption $(H1)$-$(H2)$ of theorem 3.1.

$(H1)$ This is obviously fulfilled.
$(H2)$ We have $H(u) = 0 \iff \langle H'(u), u\rangle > 0 \iff u = 0$, since for $u \neq 0$. have $\langle H'(u), u\rangle > 0$ and $H(u) = \int_0^1 \langle H'(tu), u\rangle\, dt$.
$(H3)$ this is obvious from lemma (4.3), proposition (4.3) and comment after definition (2.6).
$(H4)$ In begining of this section, we assumed this condition on G.

∎

Remark 6.2. The above arguments can easily applied to other laplacian equation with Dirichlet boundary condition, and get similar results on existence of eigenvalues and eigenfunctions.

We note that if $f \in C^1$ and $Tu = f(x, u, \ldots, D^m u)$, we have

$$\langle T'u, v\rangle = \sum_{|\alpha|\leq m} f_\alpha\,(x, u, \ldots, D^m u)\, D^\alpha v \quad \text{where} \quad f_\alpha \equiv \frac{\partial f}{\partial \xi^\alpha}.$$

Proposition 6.3. *Suppose Ω is a bounded domain in $\mathbb{R}^N$, whose boundary $\partial\Omega$ is regular, and suppose $f(x, y_1, \ldots, y_k)$ satisfies the Caratheodory condition and the growth conditions,*

$$|f(x, y_1, \ldots, y_m)| \leq c\left\{1 + \sum_{\alpha=1}^{m} |y-\alpha|^{\sigma_\alpha}\right\} \tag{6.1}$$

where c is an absolute constant and y_m is a vector variable. the $f(u) = f(x, u, Du, \ldots, D^m u)$ defines a bounded continuous mapping from $W^{m,p}(\Omega)$ to $L^s(\Omega)$, provided the numbers $\{\sigma_\alpha\}$ satisfy the inequalities

$$\sigma_\alpha < \frac{1}{s}\left\{\frac{1}{p} - \frac{m-|\alpha|}{N}\right\}^{-1}$$

Proof. By Sobolev's inequality, we note that for $u \in W^{m,p}(\Omega), D^\alpha u \in L^{p(\alpha)}(\Omega)$, that $\frac{1}{p(\alpha)} \geq \frac{1}{p} - \frac{m-|\alpha|}{N}$. Consequently, $|D^\alpha u|^{\sigma_\alpha} \in L^s(\Omega)$ provided $\sigma_\alpha s \leq p(\alpha)$, i.e.$\sigma_\alpha \leq \frac{p(\alpha)}{s}$. Thus the result follows from proposition 4.2 and theorem 2.18. Since these results ∎

Now we have the following theorem.

Theorem 6.4. *suppose A is a bounded operator from $W_0^{m,p}(\Omega)$ to $W_0^{-m,q}(\Omega)$, defined implicitly by*

$$\langle Au, v\rangle = \sum_{|\alpha|\leq m-1} \int_{\Omega} A_{\alpha}\left(x, u, Du, \ldots, D^{m-1}u\right) D^{\alpha}v$$

that the continuous functions $A_{\alpha}\left(x, u, Du, \ldots, D^{m-1}u\right)$ satisfy the growth condition 6.1, then A is a compact operator. In fact A maps weakly convergent sequences in $W_0^{m,p}(\Omega)$ into strongly convergent sequences.

Proof. Let $u_n \rightharpoonup u$ in $W_0^{m,p}(\Omega)$, then

$$\|Au_n - Au\| = \sup_{\|v\|=1} \langle Au_n - Au, v\rangle$$

and

$$\begin{aligned}
&\langle Au_n - Au, v\rangle \\
&= \sum_{|\alpha|\leq m-1} \int_{\Omega} \left[A_{\alpha}\left(x, u_n, Du_n, \ldots, D^{m-1}u_n\right) - A_{\alpha}\left(x, u, Du, \ldots, D^{m-1}u\right)\right] D^{\alpha}v \\
&\leq \sum_{|\alpha|\leq m-1} K_{\alpha} \left\|A_{\alpha}\left(x, u_n, Du_n, \ldots, D^{m-1}u_n\right) - A_{\alpha}\left(x, u, Du, \ldots, D^{m-1}u\right)\right\|_{L^{q_\alpha}(\Omega)}
\end{aligned}$$

with Sobolev embedding theorem q_α so chosen that $A_{\alpha}\left(x, u, Du, \ldots, D^{m-1}u\right) : W_0^{m-1,p^*}(\Omega) \to L^{q_\alpha}$ is compact and $p^* < \frac{Np}{N-p}$. Now if $u_n \rightharpoonup u$ in $W_0^{m,p}(\Omega)$, so $u_n \to u$ (strongly) in $W_0^{m-1,p^*}(\Omega)$. On the other hand, by virtue of the hypothesis on the growth condition and proposition 6.3, $A_{\alpha}\left(x, u, Du, \ldots, D^{m-1}u\right)$ is a continuous function from $W_0^{m-1,p}(\Omega)$ to L^{q_α}. consequently, $\|Au_n - Au\| \to 0$ as $n \to \infty$, so that A is a compact mapping. ∎

We can apply this theorem in many different cases, for example in perturbed p-Laplacian with addition condition on perturbed term, and used of results in spectrum theory for compact operators, that characterize the spectrum.

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