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Semilocal and Local Convergence of a Three Step Fifth Order Iterative Methods under General Continuity Condition in Banach Spaces

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Abstract In this paper, First of all, we study the semilocal convergence of the fifth order iterative method using recurrence relation under the assumption that first order Fréchet derivative satisfies the more general ω-continuity condition. We calculate also the R-order of convergence and provide some a priori error bounds. Based on this, we give existence and uniqueness region of the solution for a nonlinear Hammerstein integral equation of the second kind. Next, we discuss the local convergence of iterative method under the assumptions that the first order Fréchet derivative satisfies the same $\omega-$continuity condition. Also, Numerical Example is worked out to demonstrate the efficacy of our approach.

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1. Introduction

We consider the problem of solving

$$F(x) = 0 \tag{1.1}$$

where $F : \Omega \subseteq X \to Y$ is a nonlinear Fréchet differentiable operator in an open convex domain Ω of a Banach space X with values in a Banach space Y. Newton's method and its variants are used to solve nonlinear equation (1.1). Many topics related to Newton's method still attract attentions from the researchers. We have local and semilocal convergence analysis of iterative methods. The local convergence is based on the information

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around the solution. The semilocal convergence is based on the assumption at initial approximation and the domain. The construction of a semilocal and local convergence of an iterative methods for solving nonlinear equations in Banach spaces is an important research area in the field of the numerical analysis. One of the most important problems is to find the existence and uniqueness regions of solution for iterative methods. In general, the domain of existence region is small. So, we use the semilocal and local convergence analysis to enlarge the domain of existence region. Another important problems is to find a priori error bounds. The well known Kantorovich theorem [1] gives sufficient conditions for the semilocal convergence of Newton's method as well as the error estimates and existence-uniqueness regions of solutions. We have two approaches to establish the convergence of iterative methods. Those are a majorizing sequence, recurrence relation approach. Rall in [2] suggested a recurrence relation approach for the convergence of iterative methods. The main assumption for the semilocal and local convergence of iterative methods are Lipschitz/Hölder/ω-continuity conditions. Many researchers [3–10] discussed the semilocal convergence of several iterative methods of different orders using recurrence relation approach. In recent year, the semilocal convergence of fifth order method discussed by [11] using recurrence relations approach, they used the assumption that the first order Fréchet derivative satisfies the Lipschitz continuity condition. The Semilocal and local convergence of fifth order method discussed by [12] under the assumption that the first order fréchet derivative satisfies the Hölder continuity. The main motivation of this paper is to sometimes the Hölder continuity condition fails for many examples, in this case, we use more generalized continuity condition that is $\omega-$continuity condition to discuss the semilocal and local convergence of iterative method.

In this paper, First, we analyze the semilocal convergence of a fifth-order method considered in [13] under the assumption that the first order Fréchet derivative satisfies the ω-continuity condition. We use the recurrence relation approach, where the problem in Banach space into real sequences and its properties, providing a suitable convergence domain. Finally, we apply our semilocal convergence result to a nonlinear Hammerstein integral equation of the second kind and obtain an existence and uniqueness of the solution for this type of equations. Next, we discuss the local convergence of the iterative method under the assumption that the first order Fréchet derivative satisfies the ω-continuity condition. Similarly, from the local convergence theorem we obtain the existence and uniqueness of the solution.

This paper is organized into four main sections. Section 1 is the introduction. In Section 2, the semilocal convergence of the fifth order iterative method and theorem for the existence and uniqueness region of the solution are establishing. Numerical example is worked out to demonstrate the efficacy of our convergence theorem. In Section 3, the local convergence analysis of the iterative method and the theorem for the existence and uniqueness for the solution is given. One numerical example is worked out. Finally, conclusions form the section 4.

2. Semilocal Convergence

In this section, the semilocal convergence of iterative method is established. The convergence analysis discussed under the general continuity conditions. First of all, the properties of real sequences are discussed. Also, some recurrence relations are established. A convergence theorem with the existence and uniqueness theorem for the solutions is derived. Finally, numerical example is worked out to validate our approach.

2.1. Preliminary Results

Let $x_0 \in \Omega$ and the nonlinear operator $F : \Omega \subset X \to Y$ be continuously first order Fréchet differentiable where Ω is an open set in X and Y are Banach spaces. The fifth order iterative method for solving nonlinear equation in Banach spaces write it as

$$\left.\begin{array}{l} y_n = x_n - \Gamma_n F(x_n) \\ z_n = y_n - 5\Gamma_n F(y_n) \\ x_{n+1} = z_n - \frac{1}{5}\Gamma_n(-16F(y_n) + F(z_n)). \end{array}\right\} \tag{2.1}$$

Let $F'(x_0)^{-1} = \Gamma_0 \in L(Y, X)$ exists at some $x_0 \in \Omega$, where $L(Y, X)$ is the set of bounded linear operators from Y into X. For $y_0, z_0 \in \Omega$, we assume that Kantorovich's conditions [1].

C_1. $\|\Gamma_0\| \leq \beta$

C_2. $\|\Gamma_0 F(x_0)\| \leq \eta$

C_3. $\|F'(x) - F'(y)\| \leq \omega(\|x - y\|)$, $\forall x, y \in \Omega$, where $\omega(t)$ is a non-decreasing continuous real function for $t > 0$ and satisfy $\omega(0) \geq 0$.

C_6. There exist a non-negative real function $\phi \in C[0, 1]$ with $\phi(s) \geq 1$, such that $\omega(st) \leq \phi(s)\omega(t)$, for $s \in [0, 1]$, $t \in (0, \infty)$.

Let $a_0 = \beta\omega(\eta)$ and define the sequence $a_{n+1} = a_n f(a_n)\phi(c_n)$, $c_n = f(a_n)g(a_n)$

$$f(x) = \frac{1}{1 - x\phi(1 + h(x))}, \tag{2.2}$$

$$g(x) = xM + h(x)(x + 1) + xh(x)M\phi(h(x)), \tag{2.3}$$

and

$$h(x) = \frac{4xM}{5} + \frac{xM\phi(1 + a_0M)}{5}(1 + a_0M), \tag{2.4}$$

where, $M = \int_0^1 \phi(t)$. We now describe the properties of the sequence $\{a_n\}$ and the real functions (2.2), (2.3) and (2.4) through the following Lemmas.

Lemma 2.1. *Let f, g and h be the functions defined in (2.2), (2.3) and (2.4) respectively. Then*

(i) f is a increasing function and $f(x) > 1$ for $x \in (0, r_0)$.

(ii) g and h are increasing for $x \in (0, t_p)$, $p \in (0, 1]$.

Proof. The proof is trivial and hence omitted here. ∎

Lemma 2.2. *Let $f(x)$, $g(x)$ defined above and $a_0 \in (0, r_0)$, where r_0 be the smallest positive zero of the polynomial $f(a_0)\phi(a_0) - 1 = 0$. Then,*

(i) $f(a_0)\phi(a_0) < 1$.

(ii) the sequence $\{a_n\}$ is decreasing and $a_n < r_0$ for $n \geq 0$.

Let r_0 be the smallest positive zero of the polynomial $f(a_0)\phi(a_0) - 1 = 0$. Using Taylor's expansion of $F(y_0)$ around x_0,

$$z_0 - x_0 = y_0 - x_0 - 5\Gamma_0 F(y_0) = y_0 - x_0 - 5\Gamma_0 \int_0^1 [F'(x_0 + t(y_0 - x_0)) - F'(x_0)](y_0 - x_0)dt.$$

Apply norm on both sides, we get

$$\begin{aligned}\|z_0 - x_0\| &\leq \|y_0 - x_0\| + 5\beta\eta \int_0^1 \omega(t\|y_0 - x_0\|)dt \\ &\leq \|y_0 - x_0\| + 5\beta\eta\omega(\eta) \int_0^1 \phi(t)dt \\ &= \|y_0 - x_0\| + 5a_0 \int_0^1 \phi(t)dt\|y_0 - x_0\|.\end{aligned}$$

Also,

$$\|z_0 - y_0\| \leq 5\beta\omega(\eta) \int_0^1 \phi(t)dt\|y_0 - x_0\| = 5a_0 \int_0^1 \phi(t)dt\|y_0 - x_0\|.$$

Again, use the Taylor's expansion of $F(z_0)$ and (2.1), we have

$$\begin{aligned}\|x_1 - x_0\| &\leq \|y_0 - x_0 - \frac{9}{5}\Gamma_0 \int_0^1 [F'(x_0 + t(y_0 - x_0)) - F'(x_0)]dt(y_0 - x_0) \\ &\quad +\Gamma_0 \int_0^1 [F'(x_0 + t(y_0 - x_0)) - F'(x_0)]dt(y_0 - x_0) \\ &\quad -\frac{1}{5}\Gamma_0 \int_0^1 [F'(x_0 + t(z_0 - x_0)) - F'(x_0)]dt(z_0 - x_0)\| \\ &= \|y_0 - x_0\| + \frac{4\beta\omega(\eta)M}{5}\|y_0 - x_0\| + \frac{1}{5}\beta M\omega(\|z_0 - x_0\|)\|z_0 - x_0\| \\ &= \Big(1 + \frac{4a_0M}{5} + \frac{1}{5}\beta M\omega(\|z_0 - x_0\|)\|z_0 - x_0\|\Big)\|y_0 - x_0\|. \end{aligned} \tag{2.5}$$

As,

$$\omega(\|z_0 - x_0\|)\|z_0 - x_0\| \leq \omega(\eta)\phi(1 + a_0M)(1 + a_0M)\eta. \tag{2.6}$$

Using (2.6) in (2.5), we get

$$\|x_1 - x_0\| \leq (1 + h(a_0))\eta. \tag{2.7}$$

Now, for $a_0 < r_0$ and applying assumptions (i)-(iv), we have

$$\begin{aligned}\|I - \Gamma_0F'(x_1)\| &\leq \|\Gamma_0\|\|F'(x_1) - F'(x_0)\| \\ &\leq \beta\omega(\|x_1 - x_0\|) \\ &\leq \beta\omega((1 + h(a_0))\|y_0 - x_0\|) \\ &\leq \beta\omega(\eta)\phi(1 + h(a_0)) \\ &= a_0\phi(1 + h(a_0)) < 1. \end{aligned} \tag{2.8}$$

By the Banach Lemma, Γ_1 exists and

$$\|\Gamma_1\| \leq \frac{1}{1 - a_0\phi(1 + h(a_0))}\|\Gamma_0\| = f(a_0)\|\Gamma_0\|. \tag{2.9}$$

For $a_0\phi(1+h(a_0)) < 1$, we need $a_0 < r_0$, Now we prove the following inequalities using induction,

$$\left.\begin{array}{l}(I)\ \|\Gamma_n\| \leq f(a_{n-1})\|\Gamma_{n-1}\|,\\ (II)\ \|\Gamma_n F(x_n)\| \leq c_{n-1}\|\Gamma_{n-1}F(x_{n-1})\|,\\ (III)\ \|z_n - y_n\| \leq 5Ma_nc_{n-1}\|y_{n-1}-x_{n-1}\|,\\ (IV)\ \|\Gamma_n\|\|\omega(\|y_n - x_n\|) \leq a_{n-1},\\ (V)\ \|x_n - x_{n-1}\| \leq (1+h(a_{n-1}))\|\Gamma_{n-1}F(x_{n-1}\|.\end{array}\right\} \tag{2.10}$$

Using mathematical induction, we prove that the above inequalities. For $n = 1$, (I) hold true from (2.9). To prove (II), using Taylor's formula,

$$\begin{aligned} F(x_1) &= F(y_0) + F'(y_0)(x_1 - y_0) + \int_{y_0}^{x_1}(F'(x) - F'(y_0))dx \\ &= \int_0^1 [F'(x_0 + t(y_0 - x_0)) - F'(x_0)](y_0 - x_0)dt \\ &\quad -(F'(y_0) - F'(x_0) + F'(x_0))\Gamma_0\Big(\frac{9}{5}F(y_0) + \frac{1}{5}F(z_0)\Big) \\ &\quad -\Gamma_0\Big(\frac{9}{5}F(y_0) + \frac{1}{5}F(z_0)\Big)\int_0^1 [F'(y_0 + t(x_1 - y_0)) - F'(y_0)]dt. \end{aligned}$$

Since,

$$\|\frac{9}{5}F(y_0) + \frac{1}{5}F(z_0)\| \leq \frac{\eta}{\beta}h(a_0).$$

Then, we get

$$\|F(x_1)\| \leq \omega(\eta)\eta M + \omega(\eta)\eta h(a_0) + \omega(\eta)\eta h(a_0)M\phi(h(a_0)) + \frac{\eta}{\beta}h(a_0). \tag{2.11}$$

From (2.9), (2.11), we get

$$\begin{aligned} \|\Gamma_1 F(x_1)\| &\leq \|\Gamma_1\|\|F(x_1)\| \\ &\leq f(a_0)\|\Gamma_0\|\|F(x_1)\| \\ &= f(a_0)\Big[a_0M + (a_0+1)h(a_0) + a_0h(a_0)M\phi(h(a_0))\Big]\eta \\ &= f(a_0)g(a_0)\|y_0 - x_0\| = c_0\|y_0 - x_0\|. \end{aligned} \tag{2.12}$$

Also, From (2.9), we get

$$\begin{aligned} \|z_1 - y_1\| &\leq 5\|\Gamma_1\|\|F(y_1)\| \\ &\leq 5a_0f(a_0)^2g(a_0)\phi(c_0)M\|y_0 - x_0\| \\ &= 5a_1c_0M\|y_0 - x_0\|. \end{aligned} \tag{2.13}$$

By using (I) and (II), we get (IV) hold true, that is

$$\begin{aligned} \|\Gamma_1\omega(\|\|y_1 - x_1\|) &= f(a_0)\|\Gamma_0\omega(c_0\|y_0 - x_0\|) \\ &= f(a_0)\beta\omega(\eta)\phi(c_0) \\ &= a_0f(a_0)\phi(c_0) = a_1. \end{aligned} \tag{2.14}$$

From, (2.7), we get (V) hold true for $n = 1$. Hence, by induction process, it can be proved that (I)-(V) hold true for $n + 1$.

2.2. Convergence Analysis

Theorem 2.3. *Let X and Y be Banach spaces and $F(x)$ be a nonlinear Fréchet differentiable operator in an open convex domain Ω. Let the assumptions (i)-(iii) are satisfied. Let us denote $a_0 = K\beta\eta^p$ and $a_0 < r_p$. Then, the sequence $\{x_n\}$ defined in (2.1) and starting at x_0 converge to a solution x^* of the equation (1.1). In that case the solution x^* and the iterates x_n, y_n and z_n lies in $\overline{\mathcal{B}}(x_0, R\eta)$, where, $R = \frac{\phi(h(a_0)+1)}{1-(f(a_0)g(a_0))}$.*

Proof. In order to establish the convergence of $\{x_n\}$, It is sufficient to show that $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ lie in $\overline{\mathcal{B}}(x_0, R\eta)$ and a Cauchy sequence. From (2.10), we get

$$\begin{aligned} \|y_n - x_n\| &\leq f(a_{n-1})g(a_{n-1})\|y_{n-1} - x_{n-1}\| \\ &\leq \prod_{j=0}^{n-1} f(a_j)g(a_j)\|y_0 - x_0\| \\ &\leq \prod_{j=0}^{n-1} f(a_j)g(a_j)\eta, \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} \|x_{m+n} - x_m\| &\leq \|x_{m+n} - x_{m+n-1}\| + \ldots + \|x_{m+1} - x_m\| \\ &\leq (1 + h(a_{m+n-1}))\|y_{m+n-1} - x_{m+n-1}\| + \ldots + (1 + h(a_m))\|y_m - x_m\| \\ &\leq (1 + h(a_m))\Big[\prod_{j=0}^{m+n-2} f(a_j)g(a_j) + \ldots + \prod_{j=0}^{m-1} f(a_j)g(a_j)\Big]\eta. \end{aligned} \tag{2.16}$$

Now, for $a_0 = r_0$, we obtain $f(a_0)\phi(c_0) = 1$, $a_n = a_{n-1} = ... = a_0$. This gives

$$\|y_n - x_n\| \leq (1 + h(a_0))(f(a_0)\phi(c_0))^n\|y_0 - x_0\|,$$

and

$$\|x_{m+n} - x_m\| \leq (1 + h(a_0))\|y_0 - x_0\| \sum_{i=0}^{m+n-1} (f(a_0)g(a_0))^i. \tag{2.17}$$

Hence, if we take $m = 0$, we get

$$\|x_n - x_0\| \leq (1 + h(a_0))\|y_0 - x_0\| \sum_{i=0}^{n-1}(f(a_0)g(a_0))^i. \tag{2.18}$$

Also,

$$\begin{aligned} \|y_n - x_0\| &\leq \|y_n - x_n\| + \|x_n - x_0\| \\ &\leq (1+h(a_0))(f(a_0)g(a_0))^n\|y_0-x_0\| + (1+h(a_0))\|y_0-x_0\| \sum_{i=0}^{n-1}(f(a_0)g(a_0))^i \\ &= \phi(1 + h(a_0))\Big[(f(a_0)g(a_0))^n + \sum_{i=0}^{n-1}(f(a_0)g(a_0))^i\Big]\eta \\ &= \phi(1 + h(a_0))\frac{1 - (f(a_0)g(a_0))^{n+1}}{1 - (f(a_0)g(a_0))}\eta < R\eta, \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} \|z_n - y_n\| &\leq 5\|\Gamma_n\|\|F(y_n)\| \\ &\leq 5M\beta c_0^n\|y_0 - x_0\| \\ &= 5a_0M\|y_0 - x_0\|. \end{aligned} \tag{2.20}$$

Hence,

$$\begin{aligned} \|z_n - x_0\| &\leq \|z_n - y_n\| + \|y_n - x_0\| \\ &= 5a_0M\|y_0 - x_0\| + \phi(1 + h(a_0))\frac{1 - (f(a_0)g(a_0))^{n+1}}{1 - (f(a_0)g(a_0))}\|y_0 - x_0\| \\ &< \Big(5Ma_0 + \phi(1 + h(a_0))\frac{1 - (f(a_0)g(a_0))^{n+1}}{1 - (f(a_0)g(a_0))}\Big)\eta < R\eta. \end{aligned} \tag{2.21}$$

Thus, $x_n, y_n, z_n \in \overline{\mathcal{B}}(x_0, R\eta)$. Also, we can conclude that $\{x_n\}$ is a Cauchy sequence. On taking the limit as $n \to \infty$ in (2.18), we get $x^* \in \overline{\mathcal{B}}(x_{\alpha,0}, R\eta)$. To show that x^* is a solution of $F(x) = 0$. We have that $\|F(x_n)\| \leq \|F'(x_n)\|\|\Gamma_n F(x_n)\|$ and the sequence $\{\|F'(x_n)\|\}$ is bounded as

$$\|F'(x_n)\| \leq \|F'(x_0)\| + K\|x_n - x_0\|^p < \|F'(x_0)\| + KR\eta^p.$$

Since F is continuous, by taking limit as $n \to \infty$, we get $F(x^*) = 0$. ∎

Theorem 2.4. *Let F satisfy the assumptions and assume that the equation $2\beta\omega(R + r)\int_{1/2}^1 \phi(t)dt = 1$ in r has a positive root. Then, the solution x^* is unique in $B(x_0, r) \cap \Omega$.*

Proof. To prove the uniqueness of the solution, if y^* be the another solution of (1) in $B(x_0, r) \cap \Omega$ then we have

$$0 = F(y^*) - F(x^*) = \int_0^1 F'(x^* + t(y^* - x^*))dt(y^* - x^*).$$

Clearly, $y^* = x^*$, if $\int_0^1 F'(x^* + t(y^* - x^*))dt$ is invertible. This follows from

$$\begin{aligned} \|\Gamma_0\|\|\int_0^1 [F'(x^* + t(y^* - x^*)) - F'(x_0)]dt\| &\leq \beta\int_0^1 \omega(\|x^* + t(y^* - x^*) - x_0\|)dt \\ &\leq \beta\int_0^1 \omega((1-t)\|x^* - x_0\| + t\|y^* - x_0\|)dt \\ &\leq \beta\int_0^{1/2} \omega((1-t)\|x^* - x_0\| + t\|y^* - x_0\|)dt \\ &\quad + \int_{1/2}^1 \omega((1-t)\|x^* - x_0\| + t\|y^* - x_0\|)dt \\ &\leq \beta\Big[\int_0^{1/2} \phi(1-t)\omega(R + r)dt \\ &\quad + \int_{1/2}^1 \phi(t)\omega(R + r)dt\Big] \\ &= 2\beta\omega(R + r)\int_{1/2}^1 h(t)dt = 1, \end{aligned} \tag{2.22}$$

and by Banach Lemma. Thus, $y^* = x^*$. ∎

2.3. Numerical Examples

An interesting possibility arising from the study of the convergence of the iterative methods for solving equations is to obtain results of existence and uniqueness of solutions for different types of equations. In this section, we provide some results of this type for a nonlinear Hammerstein integral equation of the second kind

Example 2.5.

$$x(s) = 1 + \int_0^1 G(s,t)\Big[x(t)^{8/5} + \frac{x(t)^2}{10}\Big]dt \;\; s \in [0,1] \tag{2.23}$$

for $x \in X = C[a,b]$ is the space of continuous functions on $[0,1]$ with max norm $\|x\| = \max_{s\in[0,1]} |x(s)|$, where $G(s,t)$ is the kernel,

$$G(s,t) = \begin{cases} (1-s)t, & t \le s, \\ s(1-t), & s \le t, \end{cases} \tag{2.24}$$

Solving (2.23) is same as solve $F(x) = 0$, where $F : \Omega \subseteq C[a,b] \to C[a,b]$ and

$$[F(x)](s) = x(s) - 1 - \int_0^1 G(s,t)\Big[x(t)^{8/5} + \frac{x(t)^2}{10}\Big]dt \;\; s \in [0,1]. \tag{2.25}$$

Now, we find the First order Fréchet derivative of (2.23),

$$F'(x)u(s) = u(s) - \int_0^1 G(s,t)\Big[\frac{8}{5}x(t)^{3/5} + \frac{x(t)}{5}\Big]u(t)dt.$$

From this,

$$\|F'(x) - F'(y)\| \le \frac{1}{5}(\|x-y\|^{3/5} + \|x-y\|).$$

Here, we observe that F' does not satisfy the Lipschitz and Hölder continuity condition for all $x, y \in \Omega$ but it satisfies the $\omega-$continuity condition. Then, it follows that $\omega(z) = \frac{1}{5}(z^{3/5} + z)$. For a fixed $x_0(s) = 1$, we have $\|\Gamma_0\| = \|F'(x_0)^{-1}\| \le \frac{40}{31} = \beta$, $\|\Gamma_0 F(x_0)\| \le \frac{11}{62} = \eta$. Using these all, we get $a_0 = \beta\omega(\eta) = 0.137224 \le 0.401291$. Hence, we observed that the the convergence theorem satisfies all the conditions. Hence, the solution of (2.23) exists in $\overline{\mathcal{B}}(x_0, 0.238277) \subseteq \Omega$ and the solution is unique in the ball $\mathcal{B}(x_0, 2.54114) \cap \Omega$.

3. Local Convergence

In this section, we shall discuss the local convergence of the iterative method (2.1). Let $\omega_0, w : [0,\infty) \to [0,\infty)$ with $\omega_0(0) = w(0) = 0$ be a nondecreasing continuous functions and r_0 be defined as follows. Let us suppose that there exist $x^* \in \Omega$ such that for each $x \in \Omega$ the following assumptions hold,

$$F(x^*) = 0, \;\; F'(x^*)^{-1} \in L(Y,X), \tag{3.1}$$

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \le \omega_0(\|x - x^*\|), \tag{3.2}$$

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \le \omega(\|x-y\|), \tag{3.3}$$

$$\|F'(x^*)^{-1}F'(x)\| \leq 1 + \omega_0(\|x - x^*\|), \tag{3.4}$$

and define,

$$r_0 = \sup\{t \geq 0 : \omega_0(t) < 1\}. \tag{3.5}$$

Under these assumptions we can show the main local convergence result for the method (2.1) in the form of following theorem.

Theorem 3.1. *Let X and Y be Banach spaces and $F(x)$ be a nonlinear Fréchet differentiable operator in an open convex domain Ω. Let us suppose that there exist $x^* \in \Omega$ such that the assumptions (3.1)-(3.4) are satisfied and $\overline{\mathcal{B}}(x^*, r) \subseteq \Omega$, , where r is the radius. Then, the sequence $\{x_n\}$ generated for initial approximation x_0 by the method (2.1) is well defined, remains in $\mathcal{B}(x^*, r)$ for $n = 0, 1, \ldots$ and converge to x^*. Moreover the following estimates hold,*

$$\|y_n - x^*\| \leq g_1(\|x_n - x^*\|)\|x_n - x^*\|, \tag{3.6}$$

$$\|z_n - x^*\| \leq g_2(\|x_n - x^*\|)\|x_n - x^*\|, \tag{3.7}$$

and

$$\|x_{n+1} - x^*\| \leq g_3(\|x_n - x^*\|)\|x_n - x^*\|, \tag{3.8}$$

where the functions g_i for $i = 1, 2, 3$ are defined. Furthermore, if $\int_0^1 \omega_0(\theta R)d\theta < 1$ for $R \geq r$, then the the point $x^*$is the only solution of $F(x) = 0$ in $\overline{\mathcal{B}}(x^*, R)$.

Proof. We show that the sequence $\{x_n\}$ is well defined and converge to the solution x^* so that the estimates (3.6)-(3.8) hold true with the help of mathematical induction. By the hypothesis $x_0 \in \mathcal{B}(x^*, r) - x^*$ Using (3.2), we have that

$$\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| \leq \omega_0(\|x_0 - x^*\|) < \omega_0(r). \tag{3.9}$$

From (3.5) we have $\omega_0(r) < 1$. Hence by Banach lemma, we get

$$\|F'(x_0)^{-1}F'(x^*)\| \leq \frac{1}{1 - \omega_0(\|x_0 - x^*\|)}. \tag{3.10}$$

From first step of the iteration method (2.1), we get

$$\begin{aligned} y_0 - x^* &= x_0 - x^* - F'(x_0)^{-1}F(x_0) \\ &= -F'(x_0)^{-1}F'(x^*) \int_0^1 F'(x^*)^{-1}[F'(x^* + \theta(x_0 - x^*)) - F'(x_0)] \\ &\quad (x_0 - x^*)d\theta \end{aligned} \tag{3.11}$$

Using, (3.3), (3.10) and (3.11), we get

$$\begin{aligned} \|y_0 - x^*\| &\leq \|F'(x_0)^{-1}F'(x^*)\| \int_0^1 \|F'(x^*)^{-1}[F'(x^* + \theta(x_0 - x^*)) - F'(x_0)]\| \|(x_0 - x^*)\| d\theta \\ &= \frac{\int_0^1 \omega((1-\theta)\|x_0 - x^*\|)d\theta \|x_0 - x^*\|}{1 - \omega_0(\|x_0 - x^*\|)} \\ &= g_1(\|x_0 - x^*\|)\|x_0 - x^*\|. \end{aligned} \tag{3.12}$$

Which shows (3.6) hold for $n = 0$ and $y_0 \in \mathcal{B}(x^*, r)$, where,

$$g_1(t) = \frac{\int_0^1 \omega((1-\theta)t)d\theta}{1-\omega_0(t)} \tag{3.13}$$

then, $h_1(t) = g_1(t) - 1$, with $h_1(0) = -1$, $h_1(t) \to +\infty$ as $t \to r_0^-$. Then by the intermediate value theorem we say that the function h_1 have smallest zero r_1 in the interval $(0, r_0)$. Then we get, $0 < r < r_1 < r_0$ and $0 \le g_1(t) < 1 \forall t \in (0, r_1)$.

We can write by (3.1) that

$$F(y_0) = F(y_0) - F(x^*) = \int_0^1 F'(x^* + \theta(x_0 - x^*))(x_0 - x^*)d\theta. \tag{3.14}$$

Note that, $\|x^* + \theta(x_0 - x^*) - x^*\| = \theta\|x_0 - x^*\| < r$, so $x^* + \theta(x_0 - x^*) \in \mathcal{B}(x^*, r)$. Hence, using (3.4), (3.12) and (3.14) we get

$$\begin{aligned}
\|F'(x^*)^{-1}F(y_0)\| &\le \int_0^1 \|F'(x^*)^{-1}F'(x^* + \theta(x_0 - x^*))(x_0 - x^*)\|d\theta \\
&\le \int_0^1 (1 + \omega_0(\theta\|y_0 - x^*\|))\|y_0 - x^*\|d\theta \\
&\le \int_0^1 (1 + \omega_0(\theta g_1(\|x_0 - x^*\|)\|x_0 - x^*\|))g_1(\|x_0 - x^*\|) \\
&\quad \times\|x_0 - x^*\|d\theta.
\end{aligned} \tag{3.15}$$

Now, from the second step of the method, we get

$$\begin{aligned}
\|z_0 - x^*\| &\le \|y_0 - x^*\| + 5\|F'(x_0)^{-1}F(x^*)\|\|F'(x^*)^{-1}F(y_0)\| \\
&= \|y_0 - x^*\| + 5\|F'(x_0)^{-1}F'(x^*)\| \int_0^1 \|F'(x^*)^{-1}F'(x^* + \theta(y_0 - x^*))(y_0 - x^*)d\theta\| \\
&= \Big(1 + \frac{5\int_0^1 (1 + \omega_0(\theta\|y_0 - x^*\|)d\theta}{1 - \omega_0(\|x_0 - x^*\|)}\Big)\|y_0 - x^*\| \\
&= \Big(1 + \frac{5\int_0^1 (1 + \omega_0(\theta g_1(\|x_0 - x^*\|)\|x_0 - x^*\|))d\theta}{(1 - \omega_0(\|x_0 - x^*\|))}\Big)g_1(\|x_0 - x^*\|)\|x_0 - x^*\| \\
&= g_2(\|x_0 - x^*\|)\|x_0 - x^*\|.
\end{aligned} \tag{3.16}$$

Which shows (3.7) hold for $n = 0$ and $z_0 \in \mathcal{B}(x^*, r)$, where,

$$g_2(t) = \Big(1 + \frac{5\int_0^1 (1 + \omega_0(\theta g_1(t)t))d\theta}{1 - \omega_0(t)}\Big)g_1(t) \tag{3.17}$$

then, $h_2(t) = g_2(t) - 1$, with $h_2(0) = -1 < 0$, $h_2(r_1) > 0$. Then by the intermediate value theorem we say that the function h_2 have smallest zero r_2 in the interval $(0, r_1)$. Then we get, $0 < r < r_2 < r_1$ and $0 \le g_2(t) < 1 \forall t \in (0, r_2)$. Also, we have as from (3.18) for

$z_0 = y_0$ that

$$\begin{aligned}\|F'(x^*)^{-1}F(z_0)\| &\leq \int_0^1 \|F'(x^*)^{-1}F'(x^*+\theta(z_0-x^*))(x_0-x^*)\|d\theta \\ &\leq \int_0^1 (1+\omega_0(\theta\|z_0-x^*\|))\|z_0-x^*\|d\theta \\ &\leq \int_0^1 (1+\omega_0(\theta g_2(\|x_0-x^*\|)\|x_0-x^*\|))g_2(\|x_0-x^*\|) \\ &\quad \times\|x_0-x^*\|d\theta. \end{aligned} \tag{3.18}$$

From the third step of the method, we get

$$\begin{aligned}\|x_1-x^*\| &\leq \|z_0-x^*\| + \frac{1}{5}\|F'(x_0)^{-1}F'(x^*)\|(\|F'(x^*)F(y_0)\| + \|F'(x^*)F(z_0)\|) \\ &= g_2(\|x_0-x^*\|)\|x_0-x^*\| \\ &+\frac{1}{1-\omega_0(\|x_0-x^*\|)}\Big[\int_0^1 (1+\omega_0(\theta g_1(\|x_0-x^*\|)\|x_0-x^*\|))g_1(\|x_0-x^*\|)\|x_0-x^*\|d\theta \\ &+\int_0^1 (1+\omega_0(\theta g_2(\|x_0-x^*\|)\|x_0-x^*\|))g_2(\|x_0-x^*\|)\|x_0-x^*\|d\theta\Big] \\ &= g_3(\|x_0-x^*\|)\|x_0-x^*\|. \end{aligned} \tag{3.19}$$

Which shows (3.8) hold for $n = 0$ and $x_1 \in \mathcal{B}(x^*, r)$, where

$$g_3(t) = g_2(t) + \frac{1}{5(1-\omega_0(t))}\Big[\int_0^1 (1+\omega_0(\theta g_1(t)t))g_1(t)d\theta + \int_0^1 (1+\omega_0(\theta g_2(t)t)g_2(t)d\theta\Big] \tag{3.20}$$

then, $h_3(t) = g_3(t) - 1$, with $h_3(0) = -1 < 0$, $h_3(r_2) > 0$. Then by the intermediate value theorem we say that the function h_3 have smallest zero r_3 in the interval $(0, r_2)$. Then we get, $0 < r < r_3 < r_2 < r_1 < r_0$ and $0 \leq g_3(t) < 1 \; \forall \; t \in (0, r)$.

$$\|x_1 - x^*\| \leq g_6(\|x_0-x^*\|)\|x_0-x^*\| < \|x_0-x^*\| < r. \tag{3.21}$$

Therefore the theorem hold true for $n = 0$. By using the mathematical induction we can prove (3.6)-(3.8) hold true for $n \geq 1$. Using the estimate $\|x_{n+1}-x^*\| \leq g_3(\|x_0-x^*\|)\|x_0-x^*\|$, where, $g_3(\|x_0-x^*\|) < 1$, we deduce that $x_n \to x^*$ as $n \to \infty$, and $x_{n+1} \in \mathcal{B}(x^*, r)$. Now, we prove the uniqueness part of the theorem. Let $y^* \in \Omega$ be another solution with $F(y^*) = 0$. Define $P = \int_0^1 F'(x^*+\theta(x^*-y^*))d\theta$. Using, (3.2), we get

$$\|F'(x^*)^{-1}\int_0^1 [F'(x^*+\theta(x^*-y^*)) - F'(x^*)]d\theta\| \leq \|\int_0^1 \omega_0(\theta\|y^*-x^*\|) \leq \int_0^1 \omega_0(\theta R) < 1.$$

From this it follows that P is invertible, then in view of identity

$$0 = F(x^*) - F(y^*) = P(x^*-y^*).$$

Hence, $x^* = y^*$ ■

3.1. Numerical Examples

In this subsection, we demonstrate the theoretical results which we have proposed in the previous section. Therefore, we consider the one numerical example in this section, which are defined as follows

Example 3.2. Let $X = Y = C[0, 1]$ and consider the nonlinear integral equations of the mixed Hammerstein type, defined by

$$x(s) = 1 + \int_0^1 G(s,t)\Big(x(t)^{3/2} + \frac{x(t)^2}{2}\Big)dt \ s \in [0,1] \tag{3.22}$$

where, the kernel G is the green's function defined on the interval $[0, 1] \times [0, 1]$ by

$$G(s,t) = \begin{cases} (1-s)t, & t \le s, \\ s(1-t), & s \le t. \end{cases} \tag{3.23}$$

The solution $x^*(s) = 0$ is the same as the solution of equation (1.1), where F is defined as

$$F(x)(s) = x(s) - 1 - \int_0^1 G(s,t)\Big(x(t)^{3/2} + \frac{x(t)^2}{2}\Big)dt \ s \in [0,1]. \tag{3.24}$$

Since, we have

$$\| \int_0^1 G(s,t)dt\| \le \frac{1}{8}. \tag{3.25}$$

Then, we get

$$F'(x)u(s) = u(s) - \int_0^1 G(s,t)\Big(\frac{3}{2}x(t)^{1/2} + x(t)\Big)dt.$$

So,

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \le \frac{1}{8}\Big(\frac{3}{2}\|x-y\|^{1/2} + \|x-y\|\Big). \tag{3.26}$$

Therefore, we can get $\omega_0(t) = \omega(t) = \frac{1}{8}\Big(\frac{3}{2}t^{1/2} + t\Big)$ and, $v(t) = 1 + \omega_0(t)$. This problem fails to satisfies the Lipschitz continuity condition. However, our results can apply. Hence, using the the above choice of the function v, ω_0, ω, we get that

$$r_0 = 3.2000, r_1 = 2.6303, r_2 = 0.4486, r_3 = 0.224209.$$

So, $r = \min\{r_0, r_1, r_2, r_3\} = 0.224209$.

4. Conclusions

The semilocal and local convergence of fifth order iterative method for solving nonlinear equations in Banach spaces is established under the assumption that the first order Fréchet derivative satisfies the $\omega-$ continuity condition. The existence and uniqueness region of solution for the method is obtained. A number of A Numerical examples are worked out to demonstrate the efficiency of our convergence analysis.

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