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On ξ-Conformally and ξ-Pseudo Projectively Flat Lorentzian Sasakian Manifolds with Tanaka-Webster Connection

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Abstract In this work, the Tanaka-Webster connection on a Lorentzian Sasakian manifold is defined and the notions ξ-quasi conformally and ξ-pseudo projectively flat structures on a Lorentzian Sasakian manifold are introduced. After that, it is proved that if any Lorentzian Sasakian manifold with Tanaka-Webster connection is an η- Einstein manifold, then the Tanaka-Webster connection $\hat{\nabla}$ is ξ-conformally flat. Furthermore, we give some structure theorems on Lorentzian Sasakian manifold with respect to the Tanaka-Webster connection.

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1. Introduction

If a differentiable manifold has a Lorentzian metric g, i.e., a symmetric nondegenerated (0,2) tensor field of index 1, then it is called a Lorentzian manifold. In generally, a differentiable manifold has a Lorentzian metric if and only if it has an 1-dimensional distribution. Hence an odd dimensional manifold is able to have a Lorentzian metric. It is very natural and interesting to define both a Sasakian structure and a Lorentzian metric on an odd dimensional manifold. In fact, odd dimensional de Sitter space and Goedell Universe, that are important examples on relativity theory, have Sasakian structure with Lorentzian metric, [1–8].

In this paper, we define the Tanaka-Webster connection on a Lorentzian Sasakian manifold and investigate some of its properties like curvature tensor, conformal curvature tensor and pseudo projective curvature tensor, see [9–12].

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2. Preliminaries

2.1. Sasakian Manifolds with Lorentzian Metric

Definition 2.1 ([6], [8])**.** Let M be a differentiable manifold of class C^∞ and let ϕ, ξ, η be a tensor field of type (1,1), a vector field and an 1-form on M, respectively, such that

$$\phi^2(X) = -X + \eta(X)\xi, \tag{2.1}$$

$$\eta(\xi) = 1, \tag{2.2}$$

$$\phi\xi = 0, \tag{2.3}$$

$$\eta(\phi X) = 0 \tag{2.4}$$

for any vector field X on M. Then M is said to have an *almost contact structure* (ϕ, ξ, η) and is called an *almost contact manifold*.

Since M has a globally defined unique vector field ξ which is also called the *Reeb vector field*, it is able to have a Lorentzian metric g.

Definition 2.2 ([6])**.** (ϕ, ξ, η, g) is called an *almost contact metric structure* on M if it is an almost contact structure on M and g is a Lorentzian metric such that

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y); X, Y \in \chi(M), \tag{2.5}$$

$$g(\xi, \xi) = -1, \tag{2.6}$$

$$g(\xi, X) = -\eta(X). \tag{2.7}$$

Definition 2.3 ([8])**.** An almost contact metric structure (ϕ, ξ, η, g) is called a *contact metric structure* if it satisfies

$$(\triangledown_X \eta)(Y) = g(\phi X, Y), X, Y \in \chi(M), \tag{2.8}$$

where $\triangledown$ is the covariant derivative with respect to g.

Definition 2.4 ([8])**.** If a contact metric structure satisfies

$$(\triangledown_X \phi)Y = -\eta(Y)X - g(X, Y)\xi, X, Y \in \chi(M), \tag{2.9}$$

it is called a *normal contact metric structure* on M. In this case we call $M(\phi, \xi, \eta, g)$ a *Sasakian manifold with the Lorentzian metric g*.

Proposition 2.5 ([8])**.** *For an almost contact metric structure* (ϕ, ξ, η, g) *on* M*, equation (2.9) implies*

$$\triangledown_X \xi = -\phi X, \qquad (\triangledown_X \eta)(Y) = g(\phi X, Y); X, Y \in \chi(M) \tag{2.10}$$

and ξ *is a Killing vector field.*

The Riemann curvature tensor R of a Sasakian manifold with Lorentzian metric satisfies

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y = g(\xi, X)Y - g(\xi, Y)X. \tag{2.11}$$

Contracting X in (2.11) it follows that

$$S(Y, \xi) = 2n\eta(Y), \tag{2.12}$$

where S is the Ricci tensor of the Lorentzian Sasakian manifold.

2.2. Tanaka-Webster Connection on a Sasakian Manifold

Now, we review the Tanaka-Webster connection on a Sasakian manifold M with Lorentzian metric g, see [6], [11] and [12]. We denote by $\triangledown$ the Lorentzian connection defined by g. Let r be arbitrary fixed real number, and let A be a tensor field of type (1,2) defined by

$$A(X)Y = g(\phi X, Y)\xi + r\eta(X)\phi(Y) + \eta(Y)\phi X \tag{2.13}$$

for all vector fields X, Y on M. Then we can define a linear connection D (D-connection) as

$$D_X Y = \triangledown_X Y + A(X)Y, \tag{2.14}$$

where $\triangledown$ is the covariant derivative with respect to g.

The tensor fields ξ, η, g and A are parallel with respect to the D- connection, see [6]. If we choose $r = 1$ in (2.13) we get the special form of D-connection which is called the Tanaka-Webster connection and denoted by $\hat{\triangledown}$, that is we will define

$$\hat{\triangledown}_X Y = \triangledown_X Y + g(\phi X, Y)\xi + \eta(X)\phi(Y) + \eta(Y)\phi X. \tag{2.15}$$

We see that the Tanaka-Webster connection $\hat{\triangledown}$ for Sasakian manifold M with Lorentzian metric g has the torsion

$$\hat{T}(X, Y) = -2g(X, \phi Y)\xi. \tag{2.16}$$

Lemma 2.6 ([6]). *The tensor field A satisfies followings*

$$\begin{aligned} A(A(Z)X)Y &= g(X, \phi Z)\phi Y - g(X, Y)\eta(Z)\xi - g(Y, Z)\eta(X)\xi \\ &\quad - \eta(Y)\eta(Z)X - \eta(X)\eta(Y)Z \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} A(Z)A(X)Y - A(X)A(Z)Y &= \eta(X)g(Z, Y)\xi - \eta(Z)g(X, Y)\xi \\ &\quad + g(\phi X, Y)\phi Z - g(\phi Z, Y)\phi X + \eta(Y)\eta(X)Z - \eta(Y)\eta(Z)X. \end{aligned} \tag{2.18}$$

3. ξ-Conformally Flat Lorentzian Sasakian Manifolds with Tanaka-Webster Connection

Since the curvature tensor $\hat{R}$ of the Tanaka-Webster connection and the curvature tensor R of the Lorentzian connection satisfies

$$\hat{R}(X, Y)Z = R(X, Y)Z + A(A(Y)X)Z - A(A(X)Y)Z + A(X)A(Y)Z - A(Y)A(X)Z,$$

from Lemma 2.1, we have the following

Proposition 3.1 ([6]). *Curvature tensors $\hat{R}$ and R satisfies following equation*

$$\begin{aligned} \hat{R}(X, Y)Z &= R(X, Y)Z + 2g(\phi X, Y)\phi Z + g(Z, Y)\eta(X)\xi - g(X, Z)\eta(Y)\xi \\ &\quad + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(\phi X, Z)\phi Y - g(\phi Y, Z)\phi X. \end{aligned} \tag{3.1}$$

Putting $Z = \xi$ in (3.1) we get $\hat{R}(X, Y)\xi = 0$. As the Reeb vector field ξ is a parallel vector field with respect to the Tanaka-Webster connection, we obtain the following

Theorem 3.2 ([6]). *Let M be a $(2n+1)$-dimensional Sasakian manifold with Lorentzian metric. Then the sectional curvature $\hat{K}(X,\xi)$ of the Tanaka-Webster connection with respect to a section spanned by ξ and X is identically zero.*

Now let $e_1, e_2, ..., e_{2n}, e_{2n+1} = \xi$ be an orthonormal frame on M. From the definition of Ricci tensor $\hat{S} = \sum_{i=1}^{2n+1} \varepsilon_i g(\hat{R}(e_i, Y)Z, e_i)$, $\varepsilon_i = 1$ for $i = 1, 2, ..., 2n$ and $\varepsilon_{2n+1} = -1$. Using $\eta(e_i) = 0$ and equations (2.1), (2.2), (2.3) and (2.4), from (3.1) we have the following equations about the Ricci tensor and the scalar curvature.

Proposition 3.3 ([6]). *The Ricci tensor $\hat{S}$ of the Tanaka-Webster connection and the Ricci tensor S of the Lorentzian connection satisfies*

$$\hat{S}(X,Y) = S(X,Y) - 2g(X,Y) - 2(n+1)\eta(X)\eta(Y). \tag{3.2}$$

The scalar curvature $\hat{\rho}$ of the Tanaka-Webster connection and the scalar curvature of the Lorentzian connection satisfies

$$\hat{\rho} = \rho - 2n. \tag{3.3}$$

The Ricci operator $\hat{Q}$ of the Lorentzian Sasakian manifold M with Tanaka-Webster connection is defined by $g(\hat{Q}X,Y) = \hat{S}(X,Y)$. Then by (3.2), we have

$$g(\hat{Q}X,Y) = S(X,Y) - 2g(X,Y) - 2(n+1)\eta(X)\eta(Y), \tag{3.4}$$

where S is the Ricci tensor of the Lorentzian connection. From (3.4), using (2.12) we get

$$g(\hat{Q}\xi,Y) = \hat{S}(\xi,Y) = 0 \tag{3.5}$$

and $\hat{Q}\xi = 0$.

Definition 3.4. A Lorentzian Sasakian manifold M is η-*Einstein* if there are functions α and β such that

$$S(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y).$$

Hence Lorentzian Sasakian manifold M with Tanaka-Webster connection is also η-Einstein for some functions $\hat{\alpha}$ and $\hat{\beta}$ such that

$$\hat{S}(X,Y) = \hat{\alpha} g(X,Y) + \hat{\beta}\eta(X)\eta(Y), \tag{3.6}$$

where $\hat{\alpha} = \alpha - 2$ and $\hat{\beta} = \beta - 2(n+1)$. Hence, we get

$$g(\hat{Q}X,Y) = \hat{\alpha} g(X,Y) - \hat{\beta}\eta(X)g(\xi,Y),$$

$$\hat{Q}X = \hat{\alpha}X - \hat{\beta}\eta(X)\xi. \tag{3.7}$$

Then using the equality (3.5) and (3.7) we obtain

$$\hat{\alpha} = \hat{\beta}. \tag{3.8}$$

Also from (3.6), it follows that

$$\hat{\rho} = Tr(\hat{Q}) = (2n+1)\hat{\alpha} - \hat{\beta}. \tag{3.9}$$

Using (3.8) in (3.9) yields

$$\hat{\rho} = 2n\hat{\alpha}. \tag{3.10}$$

Now we suppose that the Lorentzian Sasakian manifold M is η-Einstein.

In [13],[14] Weyl constructed a generalized curvature tensor on a (2n+1)-dimensional Riemannian manifold which vanishes whenever the metric is (locally) conformally equivalent to a flat metric. Conformally flat and ξ-Conformally flat manifolds are studied in [15] and [16], respectively.

Definition 3.5. The *Weyl conformal curvature tensor of Tanaka-Webster connection* is defined by

$$\begin{aligned}\hat{C}(X,Y)Z &= \hat{R}(X,Y)Z \\ &\quad - \frac{1}{2n-1}[\hat{S}(Y,Z)X - \hat{S}(X,Z)Y + g(Y,Z)\hat{Q}X - g(X,Z)\hat{Q}Y] \\ &\quad + \frac{\hat{\rho}}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y],\end{aligned} \tag{3.11}$$

where $\hat{R}$ and $\hat{\rho}$ denote the Riemannian curvature tensor and the scalar curvature of M with respect to the Tanaka-Webster connection respectively.

Definition 3.6. A Lorentzian Sasakian manifold M with Tanaka-Webster connection is called *ξ-conformally flat* if the condition $\hat{C}(X,Y)\xi = 0$ is satisfied on the manifold M.

Putting $Z = \xi$ in (3.11) and using $\hat{R}(X,Y)\xi = 0$, $\hat{S}(Y,\xi) = 0$, (3.7) and (3.10) we obtain

$$\begin{aligned}\hat{C}(X,Y)\xi &= -\frac{1}{2n-1}[-\eta(Y)(\hat{\alpha}X - \hat{\beta}\eta(X)\xi) + \eta(X)(\hat{\alpha}Y - \hat{\beta}\eta(Y)\xi)] \\ &\quad + \frac{2n\hat{\alpha}}{2n(2n-1)}[\eta(X)Y - \eta(Y)X] = 0.\end{aligned} \tag{3.12}$$

Thus, from (3.12) we have the following

Theorem 3.7. *If a Lorentzian Sasakian manifold with Tanaka-Webster connection is an η-Einstein manifold, then it is ξ-conformally flat with respect to Tanaka-Webster connection $\hat{\bigtriangledown}$.*

Definition 3.8. The *concircular curvature tensor* $\hat{\hat{C}}$ is given by

$$\hat{\hat{C}}(X,Y)Z = \hat{R}(X,Y)Z - \frac{\hat{\rho}}{2n(2n+1)}[g(Y,Z)X - g(X,Z)Y]. \tag{3.13}$$

Definition 3.9. A Lorentzian Sasakian manifold with Tanaka-Webster connection is called *ξ-concircularly flat* if $\hat{\hat{C}}(X,Y)\xi = 0$.

Then putting $Z = \xi$ in (3.13) and using $\hat{R}(X,Y)\xi = 0$ and (3.10) from (3.13) we get

$$\hat{\hat{C}}(X,Y)\xi = \frac{\alpha - 2}{2n+1}R(X,Y)\xi. \tag{3.14}$$

Thus, since $\hat{\alpha} = \hat{\beta} \neq 0$ we may express the following theorem:

Theorem 3.10. *If a Lorentzian Sasakian manifold with Tanaka-Webster connection is an η- Einstein manifold, then it is ξ-concircularly flat with respect to the Tanaka-Webster connection $\hat{\bigtriangledown}$ if $\alpha = 2$.*

Definition 3.11. The *quasi-conformal curvature tensor $\hat{W}$ on the Lorentzian Sasakian manifold with Tanaka-Webster connection M* is defined by

$$\hat{W}(X,Y)Z = -[(2n-1)b]\hat{C}(X,Y)Z + [a + (2n-1)b]\hat{\hat{C}}(X,Y)Z, \tag{3.15}$$

where a and b are arbitrary constants such that a and b are not zero simultaneously, $\hat{C}$ and $\hat{\tilde{C}}$ are conformal curvature tensor and concircular curvature tensor respectively.

Definition 3.12. A Lorentzian Sasakian manifold with Tanaka-Webster connection is called *ξ-quasi conformally flat* if $\hat{W}(X,Y)\xi = 0$.

Now putting $Z = \xi$ in (3.15), using (3.12) and (3.14) we get

$$\hat{W}(X,Y)\xi = \frac{2nb(\alpha-2)+(a-b)(\alpha-2)}{2n+1}\hat{\tilde{C}}(X,Y)\xi. \tag{3.16}$$

Hence, we may express the following theorem:

Theorem 3.13. *If a Lorentzian Sasakian manifold with Tanaka-Webster connection is an η-Einstein manifold, then it is ξ-quasi conformally flat with respect to Tanaka-Webster connection $\hat{\bigtriangledown}$ if $\alpha = 2$.*

4. ξ-Pseudo Projectively Flat Lorentzian Sasakian Manifolds with Tanaka-Webster Connection

Let M be an $(2n+1)$-dimensional Lorentzian Sasakian manifold equipped with a Tanaka-Webster connection. Since the Ricci tensor $\hat{S}$ of the Tanaka-Webster connection is symmetric, the pseudo projective curvature tensor of the Sasakian manifold with respect to the Tanaka-Webster connection can be defined by

$$\begin{aligned}\hat{\tilde{P}}(X,Y)Z &= a\hat{R}(X,Y)Z + b\{\hat{S}(Y,Z)X - \hat{S}(X,Z)Y\} \\ &\quad - \frac{\hat{\rho}}{2n+1}[\frac{a}{2n}+b](g(Y,Z)X - g(X,Z)Y),\end{aligned} \tag{4.1}$$

where a,b are constants.

Definition 4.1. A *Sasakian manifold is called ξ-pseudo projectively flat with respect to Tanaka-Webster connection* if the condition $\hat{\tilde{P}}(X,Y)Z = 0$ is satisfied on the manifold.

Putting $Z = \xi$ in (4.1) and using $\hat{R}(X,Y)\xi = 0$ and $\hat{S}(Y,\xi) = 0$, we have

$$\hat{\tilde{P}}(X,Y)\xi = \frac{\hat{\rho}}{2n+1}[\frac{a}{2n}+b]R(X,Y)\xi. \tag{4.2}$$

From equation (4.2) we have

Theorem 4.2. *A Lorentzian Sasakian manifold with Tanaka-Webster connection is ξ-pseudo projectively flat with respect to Tanaka-Webster connection $\hat{\bigtriangledown}$ if $R(X,Y)\xi = 0$.*

For $a = 1$ and $b = -\frac{1}{2n}$ (4.1) takes the form

$$\hat{\tilde{P}}(X,Y)Z = \hat{R}(X,Y)Z - \frac{1}{2n}\{\hat{S}(Y,Z)X - \hat{S}(X,Z)Y\} = \hat{P}(X,Y)Z, \tag{4.3}$$

where $\hat{P}$ is the projective curvature tensor with respect to the Tanaka-Webster connection. From (4.3) we have that the following

Theorem 4.3. *A ξ-pseudo projectively flat Lorentzian Sasakian manifold is ξ-projectively flat with respect to Tanaka-Webster connection $\hat{\bigtriangledown}$ if $a = 1$ and $b = -\frac{1}{2n}$.*

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