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Hardy-Littlevood Maximal Functions and Fractional Integrals on Hypergroups

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Abstract We define Hardy-Littlevood maximal function and fractional integrals on commutative hypergroups and investigate the L^p boundedness of the Hardy-Littlevood maximal function and the (L^p, L^q) boundedness of fractional integrals on commutative hypergroups.

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1. Introduction

The convolution of a function with a fixed density is a smoothing operation that produces a certain average of the function. Averaging is an important operation in analysis and naturally arises in many situations. The study of averages of functions is better understood and simplified by the introduction of the maximal function. This is defined as the largest average of a function over all balls containing a fixed point. Maximal functions play a key role in differentiation theory, where they are used in obtaining almost everywhere convergence for certain integral averages. Although maximal functions do not preserve qualitative information about the given functions, they maintain crucial quantitative information, a fact of great importance in the subject of Fourier analysis.

A very significant role in the estimation of different operators in analysis is played by the Hardy-Littlewood maximal function

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{|\mathcal{B}(x,r)|} \int\limits_{\mathcal{B}(x,r)} |f(y)|dy.$$

There are a lot of papers dedicated to the study of properties of the Hardy-Littlewood maximal function, its variants, and their applications.

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In 1930, Hardy and Littlewood (see [1]) proved a remarkable result, known as the Hardy-Littlewood maximal theorem, which can be formulated in the following way:

$$\|\mathcal{M}f\|_{L^p(R^n)} \leq C\|f\|_{L^p(R^n)}, forp > 0.$$

In 1939, N. Wiener (see [2]) proved a weak type $(1,1)$ inequality for the Hardy-Littlewood maximal function. Later these facts extended to various Lie groups, symmetric spaces, some measure spaces (see [3], [4] [5], [6]).

For $0 < \alpha < n$, the operator

$$R_\alpha f(x) = \int_{R^n} |x-y|^\alpha f(y)dy$$

is called a classical Riesz potential.

By the classical Hardy-Littlewood-Sobolev theorem, if $1 < p < \infty$ and $\alpha p < n$, then $R_\alpha f$ is an operator of strong type (p,q), where $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. If $p = 1$, then $R_\alpha f$ is an operator of weak type $(1,q)$, where $\frac{1}{q} = 1 - \frac{\alpha}{n}$ (see [7], [8]).

The Hardy-Littlewood-Sobolev theorem is an important result in fractional integral theory and potential theory. There are a lot of generalizations of this theorem. The Hardy-Littlewood-Sobolev theorem was proved for Riesz potentials associated to doubling measures in [9] and nondoubling measures in [10], [11]. In [12] and [13], generalized potential-type integral operators were considered and (p,q) properties of these operators were proved. In [14], [15], [16], [17] the Hardy-Littlewood-Sobolev theorem was extended to Orlicz and Musielak-Orlicz spaces for generalized Riesz potentials.

In this paper, we define the Hardy-Littlewood maximal function and the Riesz potential on the commutative hypergroup. The sufficient condition is found for a weak type $(1,1)$ and a strong type (p,p), $1 < p \leq \infty$, boundedness of the Hardy-Littlewood maximal functions. Also we prove the analogue of the Hardy-Littlewood-Sobolev theorem for the fractional integrals (Riesz potentials) on the commutative hypergroups.

2. Preliminaries

Let K be a set. A function $\rho : K \times K \to [0,\infty)$ is called quasi-metric if:

(1) $\rho(x,y) = 0 \Leftrightarrow x = y$;
(2) $\rho(x,y) = \rho(y,x)$;
(3) there exists a constant $c \geq 1$ such that for every $x,y,z \in K$

$$\rho(x,y) \leq c(\rho(x,z) + \rho(z,y)).$$

Let all balls $B(x,r) = \{y \in K : \rho(x,y) < r\}$ be λ-measurable and assume that the measure λ fulfils the doubling condition

$$0 < \lambda B(x,2r) \leq D\lambda B(x,r) < \infty. \tag{2.1}$$

A space (K,ρ,λ) which satisfies all conditions mentioned above is called a space of homogeneous type (see [4]).

In the theory of locally compact groups there arise certain spaces which, though not groups, have some of the structure of groups. Often, the structure can be expressed in terms of an abstract convolution of measures on the space.

A hypergroup $(K, *)$ consists of a locally compact Hausdorff space K together with a bilinear, associative, weakly continuous convolution on the Banach space of all bounded regular Borel measures on K with the following properties:

1. For all $x, y \in K$, the convolution of the point measures $\delta_x * \delta_y$ is a probability measure with compact support.
2. The mapping: $K \times K \to \mathcal{C}(K)$, $(x, y) \mapsto supp(\delta_x * \delta_y)$ is continuous with respect to the Michael topology on the space $\mathcal{C}(K)$ of all nonvoid compact subsets of K, where this topology is generated by the sets
$$U_{V,W} = \{L \in \mathcal{C}(K) : L \cap V \neq \varnothing, L \subset W\}$$
with V, W open in K.
3. There is an identity $e \in K$ with $\delta_e * \delta_x = \delta_x * \delta_e = \delta_x$ for all $x \in K$.
4. There is a continuous involution $\sim$ on K such that
$$(\delta_x * \delta_y)^{\sim} = \delta_{y^\sim} * \delta_{x^\sim}$$
and $e \in supp(\delta_x * \delta_y) \Leftrightarrow x = y^\sim$ for $x, y \in K$ (see [18], [19], [20], [21], [22]).

A hypergroup K is called commutative if $\delta_x * \delta_y = \delta_y * \delta_x$ for all $x, y \in K$. It is well known that every commutative hypergroup K possesses a Haar measure which will be denoted by λ (see [19]). That is, for every Borel measurable function f on K,
$$\int\limits_K f(\delta_x * \delta_y) d\lambda(y) = \int\limits_K f(y) d\lambda(y) \ \ (x \in K).$$
Define the generalized translation operators T^x, $x \in K$, by
$$T^x f(y) = \int\limits_K f d(\delta_x * \delta_y)$$
for all $y \in K$. If K is a commutative hypergroup, then $T^x f(y) = T^y f(x)$ and the convolution of two functions is defined by
$$(f * g)(x) = \int\limits_K T^x f(y) g(y^\sim) d\lambda(y).$$
Let $p > 0$. By $L^p(K, \lambda)$ denote a class of all λ-measurable functions $f : K \to (-\infty, +\infty)$ with $\|f\|_{L^p(K,\lambda)} = \left(\int\limits_K |f(x)|^p \, d\lambda(x) \right)^{\frac{1}{p}} < \infty$.
The notation $\chi_A(x)$ denotes the characteristic function of set A.
Define a function $\Lambda_x(y) = T^x \chi_{B(e,r)}(y^\sim)$.

3. Main Results

In this section we formulate the main results of this paper.
Define Hardy-Littlewood maximal function
$$Mf(x) = \sup_{r>0} \frac{1}{\lambda B(e, r)} \left(|f| * \chi_{B(e,r)} \right)(x)$$
and fractional integral (or Riesz potential)
$$I_\alpha f(x) = \left(\rho(e, \cdot)^{\alpha - N} * f \right)(x), \ 0 < \alpha < N$$
on commutative hypergroup $(K, *)$ equipped with the quasi-metric ρ.

Theorem 3.1. *Let $(K, *)$ be a commutative hypergroup, with quasi-metric ρ and doubling Haar measure λ. Assume that there exist constants $c_1 > 0$ and $c_2 > 0$ such that for every $x, y \in K$ and $r > 0$*

$$supp\Lambda_x(\cdot) \subset B(x, c_1 r)$$

and

$$\lambda B(x,r) T^x \chi_{B(e,r)}(y^{\sim}) \leq c_2 \lambda B(e,r).$$

Then

1) The maximal operator M satisfies a weak type $(1,1)$ inequality, that is, there exists a constant $C > 0$ such that for every $f \in L^1(K, \lambda)$ and $\alpha > 0$

$$\lambda\{x : Mf(x) > \alpha\} \leq \frac{C}{\alpha} \int\limits_K |f(x)| d\lambda(x).$$

2) The maximal operator M is of strong type (p,p), for $1 < p \leq \infty$, that is,

$$\|Mf\|_{L^p(K,\lambda)} \leq C_p \|f\|_{L^p(K,\lambda)},$$

for some constant C and every $f \in L^p(K, \lambda)$.

Proof. It is clear that there exists nonnegative integer m such that $c_1 \leq 2^m$ and $\lambda B(x, c_1 r) \leq D^m \lambda B(x,r)$, where D is a constant on doubling condition (2.1). Then we have

$$\begin{aligned}
Mf(x) &= \sup_{r>0} \frac{1}{\lambda B(e,r)} \int\limits_K T^x |f(y)| \chi_{B(e,r)}(y^{\sim}) d\lambda(y) \\
&= \sup_{r>0} \frac{1}{\lambda B(e,r)} \int\limits_K |f(y)| T^x \chi_{B(e,r)}(y^{\sim}) d\lambda(y) \\
&\leq \sup_{r>0} \frac{1}{\lambda B(e,r)} \int\limits_{B(x,c_1 r)} |f(y)| T^x \chi_{B(e,r)}(y^{\sim}) d\lambda(y) \\
&= \sup_{r>0} \frac{1}{\lambda B(x,r)} \int\limits_{B(x,c_1 r)} |f(y)| \frac{T^x \chi_{B(e,r)}(y^{\sim}) \lambda B(x,r)}{\lambda B(e,r)} d\lambda(y) \\
&\leq c_2 \sup_{r>0} \frac{1}{\lambda B(x,r)} \int\limits_{B(x,c_1 r)} |f(y)| d\lambda(y) \leq c_2 D^m M_\rho f(x),
\end{aligned}$$

where

$$M_\rho f(x) = \sup_{r>0} \frac{1}{\lambda B(x,r)} \int\limits_{B(x,r)} |f(y)| d\lambda(y)$$

is a maximal operator on (K, ρ, λ). It is well known that the maximal operator M_ρ is of weak type $(1,1)$ and is bounded on $L^p(K, \lambda)$ (see [4], [6]). This fact and the inequality $Mf(x) \leq c_2 D^m M_\rho f(x)$ completes the proof. ∎

Corollary 3.2. *Let $(K, *)$ be a commutative hypergroup, with quasi-metric ρ and doubling Haar measure λ. Assume that there exist constants $c_1 > 0$ and $c_2 > 0$ such that for every $x, y \in K$ and $r > 0$*

$$supp\Lambda_x(\cdot) \subset B(x, c_1 r)$$

and

$$\lambda B(x,r) T^x \chi_{B(e,r)}(y^{\sim}) \leq c_2 \lambda B(e,r).$$

If f is a locally integrable function with respect Haar measure λ on $(K,*)$, then

$$\lim_{r\to 0+}\frac{1}{\lambda B(e,r)}\int\limits_K |T^x f(y)-f(x)|\chi_{B(e,r)}(y^\sim)d\lambda(y)=0$$

for a.e. $x\in K$.

Proof. From the proof of Theorem 3.1 we have

$$\frac{1}{\lambda B(e,r)}\int\limits_K |T^x f(y)-f(x)|\chi_{B(e,r)}(y^\sim)d\lambda(y)\le\frac{1}{\lambda B(e,r)}\int\limits_K T^x|f(y)-f(x)|\chi_{B(e,r)}(y^\sim)d\lambda(y)$$

$$\le\frac{c_2D^m}{\lambda B(x,c_1r)}\int\limits_{B(x,c_1r)}|f(y)-f(x)|d\lambda(y).$$

Since

$$\lim_{r\to 0}\frac{1}{\lambda B(x,r)}\int\limits_{B(x,r)}|f(y)-f(x)|d\lambda(y)=0$$

(see [4], [6]) we have the required result. ■

Theorem 3.3. *Let $(K,*)$ be a commutative hypergroup, with quasi-metric ρ and doubling Haar measure λ and let $0<\alpha<N$, $1\le p<\frac{N}{\alpha}$ and $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{N}$. Assume that there exist positive constants c_1, c_2 and c_3 such that for every $x,y\in K$ and $r>0$*

$$supp\Lambda_x(\cdot)\subset B(x,c_1r)$$

and

$$\lambda B(x,r)T^x\chi_{B(e,r)}(y^\sim)\le c_2\lambda B(e,r)\le c_3r^N.$$

If $f\in L^p(K)$, then the integral

$$I_\alpha f(x)=\int\limits_X T^x\rho(e,y)^{\alpha-N}f(y^\sim)d\lambda(y)$$

is absolutely convergent for almost every $x\in K$.

If $1<p<\frac{N}{\alpha}$ and $f\in L^p(K,\lambda)$ then $I_\alpha f\in L^p(K,\lambda)$ and

$$\|I_\alpha f\|_{L^p(K,\lambda)}\le C_p\|f\|_{L^p(K,\lambda)}, \tag{3.1}$$

where $C_p>0$ is independent of f.

If $\frac{1}{q}=1-\frac{\alpha}{N}$ and $f\in L^1(K,\lambda)$ then

$$\lambda\{x: I_\alpha f(x)>\beta\}\le\left(\frac{C}{\beta}\|f\|_{L^1(K,\lambda)}\right)^q,\beta>0, \tag{3.2}$$

where $C>0$ is independent of f.

Proof. 1) Let $f\in L^p(K,\lambda)$ and $1\le p<\frac{N}{\alpha}$. Write $I_\alpha f(x)$ in the form

$$I_\alpha f(x)=\int\limits_{B(e,1)}\rho(e,y)^{\alpha-N}T^x f(y^\sim)d\lambda(y)$$

$$+\int\limits_{K\setminus B(e,1)}\rho(e,y)^{\alpha-N}T^x f(y^\sim)d\lambda(y)=J_1(x)+J_2(x).$$

Let us estimate $J_1(x)$. It is clear that

$$|J_1(x)| \leq \int\limits_K \rho(e,y)^{\alpha-N}\chi_{B(e,1)}(y)T^x|f(y^{\sim})|d\lambda(y)$$

By Young's inequality

$$\|J_1(\cdot)\|_{L^p(K,\lambda)} \leq \|\rho(e,\cdot)^{\alpha-N}\chi_{B(e,1)}(\cdot)\|_{L^1(K,\lambda)}\|T^x f\|_{L^p(K,\lambda)}$$
$$\leq C\|\rho(e,\cdot)^{\alpha-N}\chi_{B(e,1)}(\cdot)\|_{L^1(K,\lambda)}\|f\|_{L^p(K,\lambda)}$$

and

$$\|\rho(e,\cdot)^{\alpha-N}\chi_{B(e,1)}(\cdot)\|_{L^1(K,\lambda)} = \int\limits_{B(e,1)} \rho(e,y)^{\alpha-N}d\lambda(y)$$
$$\leq \sum_{k=1}^{\infty} \int\limits_{2^{-k}\leq\rho(e,y)<2^{-k+1}} \rho(e,y)^{\alpha-N}d\lambda(y)$$
$$\leq \sum_{k=1}^{\infty} \left(2^{-k}\right)^{\alpha-N} \int\limits_{\rho(e,y)<2^{-k+1}} d\lambda(y)$$
$$\leq C\sum_{k=1}^{\infty} 2^{(N-\alpha)k}2^{N(-k+1)} < C$$

Then

$$\|J_1(\cdot)\|_{L^p(K,\lambda)} \leq C\|f\|_{L^p(K,\lambda)},$$

e.g. $J_1(x)$ is absolutely convergent almost every $x \in K$.

By Hölder's inequality we have

$$|J_2(x)| \leq \int\limits_{K\setminus B(e,1)} \rho(e,y)^{\alpha-N}T^x|f(y^{\sim})|d\lambda(y)$$
$$\leq \|T^x f(\cdot)\|_{L^p(K,\lambda)}\left(\int\limits_{K\setminus B(e,1)} \rho(e,y)^{(\alpha-N)p'}d\lambda(y)\right)^{\frac{1}{p'}}$$
$$\leq C\|f\|_{L^p(K,\lambda)}\left(\int\limits_{K\setminus B(e,1)} \rho(e,y)^{(\alpha-N)p'}d\lambda(y)\right)^{\frac{1}{p'}}$$

and

$$\int\limits_{K\setminus B(e,1)} \rho(e,y)^{(\alpha-N)p'}d\lambda(y)$$
$$\leq \sum_{k=0}^{\infty} \int\limits_{2^k<\rho(e,y)\leq 2^{k+1}} \rho(e,y)^{(\alpha-N)p'}d\lambda(y)$$
$$\leq \sum_{k=0}^{\infty} 2^{(N-\alpha)p'k} \int\limits_{\rho(e,y)\leq 2^{k+1}} d\lambda(y)$$

$$\leq \sum_{k=0}^{\infty} 2^{(N-\alpha)p'k} 2^{(k+1)N} < C$$

Hence for $1 \leq p < \frac{N}{\alpha}$

$$|J_2(x)| \leq C\|f\|_{L^p(K,\lambda)}$$

Thus for all functions $f \in L^p(K, \lambda)$, $1 \leq p < \frac{N}{\alpha}$ the fractional integrals $I_\alpha f(x)$ are absolutely convergent for almost every $x \in K$.

2) Split $I_\alpha f(x)$ in the standard way

$$I_\alpha f(x) = \int\limits_{B(e,r)} \rho(e,y)^{\alpha-N} T^x f(y^{\sim}) d\lambda(y) + \int\limits_{K\setminus B(e,r)} \rho(e,y)^{\alpha-N} T^x f(y^{\sim}) d\lambda(y)$$

$$= U_1(x,r) + U_2(x,r).$$

Then for $U_1(x,r)$ we have the estimate

$$|U_1(x,r)| \leq \int\limits_{B(e,r)} \rho(e,y)^{\alpha-N} T^x |f(y^{\sim})| d\lambda(y).$$

$$\leq \sum_{k=1}^{\infty} \int\limits_{2^{-k}r \leq \rho(e,y) < 2^{-k+1}r} \rho(e,y)^{\alpha-N} T^x |f(y^{\sim})| d\lambda(y)$$

$$\leq \sum_{k=1}^{\infty} \left(2^{-k}r\right)^{\alpha-N} \int\limits_{\rho(e,y) < 2^{-k+1}r} T^x |f(y^{\sim})| d\lambda(y)$$

$$= \sum_{k=1}^{\infty} \left(2^{-k}r\right)^{\alpha-N} \lambda B(e, 2^{-k+1}r) \frac{1}{\lambda B(e, 2^{-k+1}r)} \int\limits_{B(e,2^{-k+1}r)} T^x |f(y^{\sim})| d\lambda(y)$$

$$\leq C r^\alpha M f(x).$$

Therefore it follows that

$$|U_1(x,r)| \leq C r^\alpha M f(x), \tag{3.3}$$

where $C > 0$ does not depend f, x and r.

Estimate $U_2(x,r)$. By Hölder's inequality we have

$$|U_2(x,r)| \leq \left(\int\limits_{K\setminus B(e,r)} |T^x f(y^{\sim})|^p d\lambda(y) \right)^{\frac{1}{p}} \left(\int\limits_{K\setminus B(e,r)} \rho(e,y)^{(\alpha-N)p'} d\lambda(y) \right)^{\frac{1}{p'}}.$$

Here

$$\left(\int\limits_{K\setminus B(e,r)} \rho(e,y)^{(\alpha-N)p'} d\lambda(y) \right)^{\frac{1}{p'}}$$

$$= \left(\sum_{k=0}^{\infty} \int\limits_{2^k r \leq \rho(e,y) < 2^{k+1}r} \rho(e,y)^{(\alpha-N)p'} d\lambda(y) \right)^{\frac{1}{p'}}$$

$$\leq \left(\sum_{k=0}^{\infty} \left(2^k r\right)^{(\alpha-N)p'} \int\limits_{\rho(e,y)<2^{k+1}r} d\lambda(y) \right)^{\frac{1}{p'}}$$

$$\leq C \left(\sum_{k=0}^{\infty} \left(2^k r\right)^{(\alpha-N)p'} \left(2^{k+1} r\right)^{N} \right)^{\frac{1}{p'}}$$

$$\leq C r^{\alpha - N + \frac{N}{p'}}$$

$$= C r^{-\frac{N}{q}}.$$

Therefore

$$|U_2(x,r)| \leq C r^{-\frac{N}{q}} \|f\|_{L^p(K,\lambda)} \tag{3.4}$$

From (3.3) and (3.4), we have

$$|I_\alpha f(x)| \leq C \left(r^\alpha M f(x) + r^{-\frac{N}{q}} \|f\|_{L^p(K,\lambda)} \right)$$

Minimum of the right-hand side is attained at $r = \left[\frac{\|f\|_{L^p(K,\lambda)}}{Mf(x)} \right]^{\frac{p}{N}}$. So

$$|I_\alpha f(x)| \leq C \left(M f(x)\right)^{\frac{p}{q}} \|f\|_{L^p(K,\lambda)}^{1-\frac{p}{q}}$$

Hence, by the Theorem 3.1 we have

$$\int\limits_K |I_\alpha f(x)|^q d\lambda(y) \leq C \|f\|_{L^p(K,\lambda)}^{q-p} \int\limits_K \left(M f(y)\right)^p d\lambda(y) \leq C \|f\|_{L^q(K,\lambda)}^q$$

3) Let $f \in L^1(K,\lambda)$. It is clear that

$$\lambda\{x \in K : |I_\alpha f(x)| > 2\beta\} \leq \lambda\{x \in K : |U_1(x,r)| > \beta\} + \lambda\{x \in K : |U_2(x,r)| > \beta\}$$

Further, from inequality (3.3) and from Theorem 3.1 we derive that

$$\beta \lambda\{x \in K : |U_1(x,r)| > \beta\} = \beta \int\limits_{\{x \in K : |U_1(x,r)| > \beta\}} d\lambda(y)$$

$$\leq \beta \int\limits_{\{x \in K : C r^\alpha M f(x) > \beta\}} d\lambda(y)$$

$$= \beta \lambda \left\{ x \in K : M f(x) > \frac{\beta}{C r^\alpha} \right\}$$

$$\leq \beta \frac{C r^\alpha}{\beta} \int\limits_K |f(y)| d\lambda(y) = C r^\alpha \|f\|_{L_1(K,\lambda)}$$

and

$$|U_2(x,r)| \leq \int\limits_{K \setminus B(e,r)} \rho(e,r)^{\alpha-N} |T^x f(y^\sim)| d\lambda(y)$$

$$\leq r^{\alpha-N} \int\limits_{K \setminus B(e,r)} |T^x f(y^\sim)| d\lambda(y)$$

$$\leq Cr^{-\frac{N}{q}} \int\limits_K |f(y)|d\lambda(y) = Cr^{-\frac{N}{q}} \|f\|_{L_1(K,\lambda)}.$$

Thus, if $\beta = r^{-\frac{N}{q}} \|f\|_{L_1(K,\lambda)}$, then $|U_2(x,r)| \leq \beta$, and, consequently, $\lambda\{x \in K : |U_2(x,r)| > \beta\} = 0$. Thus

$$\lambda\{x \in K : |I_\alpha f(x)| > 2\beta\} \leq \frac{C}{\beta} r^\alpha \|f\|_{L_1(K,\lambda)}$$

$$= Cr^{\alpha+\frac{N}{q}} = Cr^N = C\beta^{-q}\|f\|^q_{L_1(K,\lambda)} \leq \left(\frac{C}{\beta}\|f\|_{L_1(K,\lambda)}\right)^q.$$

The theorem is proved. ■

Theorem 3.4. *Let $1 < p < \frac{N}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{N}$. Then for a measure λ, finite over balls and not having any atoms, the condition*

$$\lambda B(e,r) \leq C_2 r^N \tag{3.5}$$

is necessary for the inequality (3.1) to hold

Proof. If $\lambda B(e,r) = 0$, then (3.5) is trivially true. Let $\lambda B(e,r) > 0$. Take $f(x) = \chi_{B(e,r)}(x^\sim)$. We have

$$I_\alpha f(x) = \int\limits_X T^x \rho(e,y)^{\alpha-N} \chi_{B(e,r)}(y) d\lambda(y) = \int\limits_{B(e,r)} T^x \rho(e,y)^{\alpha-N} d\lambda(y)$$

$$\geq \int\limits_{B(e,r)} T^x r^{\alpha-N} d\lambda(y) = r^{\alpha-N} \lambda B(e,r).$$

By applying (3.1) we get

$$r^{\alpha-N}\lambda B(e,r)^{1+\frac{1}{q}} = \left(\int\limits_{B(e,r)} \left(r^{\alpha-N}\lambda B(e,r)\right)^q d\lambda(x) \right)^{\frac{1}{q}}$$

$$\leq \left(\int\limits_K \left(I_\alpha \chi_{B(e,r)}(x^\sim)\right)^q d\lambda(x) \right)^{\frac{1}{q}}$$

$$\leq C \left(\int\limits_K \left(\chi_{B(e,r)}(x)\right)^p d\lambda(x) \right)^{\frac{1}{p}}$$

$$\leq C\lambda B(e,r)^{\frac{1}{p}}$$

which is equivalent to $\lambda B(e,r)^{1+\frac{1}{q}+\frac{1}{p}} \leq C(r^N)^{1-\frac{\alpha}{N}}$. Since $1 + \frac{1}{q} + \frac{1}{p} = 1 - \frac{\alpha}{N}$, the last inequality is, precisely, condition (3.5). ■

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