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A Distance Between Two Points and Nearest Points in a Metric Space of Curvature Bounded Below

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Abstract In this research, geometric properties of a subset of a metric space of curvature bounded below are investigated. A necessary and sufficient condition for being a metric space of curvature bounded below and a bound for distance between two points in the metric space are also studied. Moreover, a nearest point in a metric space of curvature bounded below is introduced.

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1. Introduction and preliminaries

The concept of the lower and upper curvature bounds on some metric spaces without the Riemannian structure is introduced by Alexandrov [1, 2]; see also [3–9]. This idea has been very fruitful because it extended many concepts to arbitrary metric spaces. Various theorems of Riemannian Geometry, in which the newly defined bounded curvature corresponds to bounded sectional curvature, could be transfered to these spaces with less stucture. Hilbert spaces, a Riemannian manifold for which sectional curvature is bounded below and its convex subsets are examples of spaces of curvature bounded below.

In this work, we study a necessary and sufficient condition for a complete metric space to be a space of curvature bounded below and then give a lower bound for a distance between any two points. We also give some remarks about nearest points in a metric space of curvature bounded below.

Let (X, d) be a metric space and $\gamma : [a, b] \to X$ a curve. The length $\ell(\gamma)$ of γ is defined by

$$\ell(\gamma) = \sup \sum_{i=1}^{k} d(\gamma(t_{i-1}), \gamma(t_i)),$$

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where the supremum is taken over all partitions $a = t_0 < t_1 < \cdots < t_k = b$ of $[a, b]$. Then

$$d^*(x, y) := \inf\{\ell(\gamma)|\ \gamma \text{ is a curve from } x \text{ to } y\},$$

for all $x, y \in X$, defines a metric on X with distance values in $[0, \infty]$. If $d = d^*$, then (X, d) is called a length space.

A geodesic in X is an isometry from $R = (-\infty, \infty)$ into X. We may also refer to the image of this isometry as a geodesic. A geodesic path joining two points x and y is a map $c : [0, l] \subset R \to X$ such that $c(0) = x$ and $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. Usually, the image $c([0, l])$ is called a geodesic segment joining x and y. If there is a unique geodesic segment joining two points x and y, then $[x, y]$ is denoted the geodesic segment joining x and y. The metric space (X, d) is called a geodesic space if each pair of two points of X is joined by a geodesic segment.

Definition 1.1. [3] Let K be a real number. The R_K is one of the following spaces, depending on the sign of K: R^2, if $K = 0$, the Euclidean sphere of radius $1/\sqrt{K}$, if $K > 0$, and the hyperbolic plane of curvature K, if $K < 0$.

Denote the diameter of R_K by D_K, i.e.,

$$D_K = \begin{cases} \frac{\pi}{\sqrt{K}} & \text{for} \quad K > 0; \\ \infty & \text{for} \quad K \leq 0. \end{cases}$$

We can get more about the spaces R_K in [3, 10]. For convenience, throughout this work, we let $\lambda = \sqrt{K}$ if $K \neq 0$. If a, b and c are arc lengths of the sides of a geodesic triangle in R_K with opposite angles α, β and γ, respectively, then the following properties are the laws of cosines [11],

$$\begin{aligned} \cosh(\lambda a) &= \cosh(\lambda b)\cosh(\lambda c) - \sinh(\lambda b)\sinh(\lambda c)\cos\alpha \quad && \text{if} \quad K < 0; \\ a^2 &= b^2 + c^2 - 2bc\cos\alpha && \text{if} \quad K = 0; \\ \cos(\lambda a) &= \cos(\lambda b)\cos(\lambda c) + \sin(\lambda b)\sin(\lambda c)\cos\alpha && \text{if} \quad K > 0. \end{aligned}$$

A geodesic triangle $\triangle(p, q, r)$ in X is a triangle with points p, q, r as its vertices and three chosen geodesics $[p, q]$, $[q, r]$, $[p, r]$ as its sides. A comparison triangle in R_K for the geodesic triangle $\triangle(p, q, r)$ in X is a triangle $\triangle(\tilde{p}, \tilde{q}, \tilde{r})$ in R_K such that $d(p, q) = d(\tilde{p}, \tilde{q})$, $d(q, r) = d(\tilde{q}, \tilde{r})$, and $d(p, r) = d(\tilde{p}, \tilde{r})$. Such a triangle $\triangle(\tilde{p}, \tilde{q}, \tilde{r})$ always exists if $d(p, q) + d(q, r) + d(p, r) < 2D_K$ and it is unique up to isometries.

Given a pair of a triangle $\triangle(p, q, r)$ in X and its comparison triangle $\triangle(\tilde{p}, \tilde{q}, \tilde{r})$ in R_K, the comparison point for a point $x \in [q, r]$ is the point denoted by $\tilde{x}$ in $[\tilde{q}, \tilde{r}]$ such that $d(q, x) = d(\tilde{q}, \tilde{x})$, and the comparison angle at q of the triangle $\triangle(p, q, r)$ is the angle at $\tilde{q}$ of the triangle $\triangle(\tilde{p}, \tilde{q}, \tilde{r})$. We denote $\widetilde{\angle}(q, p, r)$ the angle at $\tilde{p}$ of a triangle $\triangle(\tilde{p}, \tilde{q}, \tilde{r})$ in R_K. Sometimes, for convenience we let a triangle $\widetilde{\triangle}(p, q, r)$ in R_K be a comparison triangle of $\triangle(p, q, r)$ in X.

Definition 1.2. [3] Let X be a length space. A locally complete space X is a space of curvature bounded below by a real number K if every point $x \in X$ has a neighborhood $U(x)$ such that the following condition is satisfied:
(A) for any four distinct points $p, q, r, s \in U(x)$, $\widetilde{\angle}(q, s, p) + \widetilde{\angle}(q, s, r) + \widetilde{\angle}(p, s, r) \leq 2\pi$.

For spaces in which, locally, any two points are joined by a geodesic, in particular for locally compact spaces, the condition (A) in Definition 1.2 can be replaced by the condition:

(B) for any triangle $\triangle(p,q,r)$ in $U(x)$ and any point s on the side $[q,r]$, the inequality $d(p,s) \geq d(\tilde{p},\tilde{s})$ is satisfied, where $\tilde{s}$ is the corresponding point of s on the side $[\tilde{q},\tilde{r}]$ of a comparison triangle $\widetilde{\triangle}(p,q,r)$.

Let X be a space of curvature bounded below by K and α and β be two geodesics starting at a point p in X. The angle between α and β is defined by

$$\lim_{s\to 0} \cos^{-1}\left(\frac{d^2(p,\alpha(s)) + d^2(p,\beta(s)) - d^2((\alpha(s),\beta(s))}{2d(p,\alpha(s))d(p,\beta(s))}\right),$$

if the limit exists. The angle at p of a triangle $\triangle(p,q,r)$ is the angle between $[p,q]$ and $[p,r]$ and denoted by $\angle(q,p,r)$.

The condition (B) is equivalent to the following condition:
$(\widetilde{B})$ for any triangle $\triangle(p,q,r)$ in $U(x)$, $\angle(p,q,r) \geq \widetilde{\angle}(p,q,r)$, $\angle(q,r,p) \geq \widetilde{\angle}(q,r,p)$, and $\angle(r,p,q) \geq \widetilde{\angle}(r,p,q)$, where $\widetilde{\triangle}(p,q,r)$ is a comparison triangle in R_K of the triangle $\triangle(p,q,r)$.

By Definition 1.2, we have that if X is a space of curvature bounded below by K_1, then it is a space of curvature bounded below by K_2 for every $K_2 \leq K_1$. A Riemannian manifold of sectional curvature bounded below by K and its convex subsets are spaces of curvature bounded below by K, a Hilbert space is a space of curvature bounded below by 0, and R_K is a space of curvature bounded below by K, for examples.

It is worth to remark that in any metric space X of curvature bounded below by K, there is no a branch point in X. Suppose there were a branch point in X, then there exists a triangle $\triangle$ which is thinner than its comparison triangle, which is impossible.

Spaces with curvature bounded below were defined above using local conditions. However, for complete spaces, the global conditions may be deduced from the corresponding local ones. In this work, we set X being a metric space of curvature bounded above using global conditions and then we call X a metric space of curvature bounded above in the large. We can see more about the spaces of curvature bounded below in [1, 3].

Theorem 1.3. [3] *If X is a metric space of curvature bounded below by K in the large, where $K > 0$, then $\dim(X) \leq D_K$ and any triangle in X has perimeter no greater than $2D_K$.*

Theorem 1.4. *Let X be a metric space of curvature bounded below by K in the large, $\triangle(p,q,r)$ a triangle in X and $\triangle(\tilde{p},\tilde{q},\tilde{r})$ a comparison triangle in R_K. If $d(p,q) = d(\tilde{p},\tilde{q})$, $d(p,r) = d(\tilde{p},\tilde{r})$, and $\angle(q,p,r) = \angle(\tilde{q},\tilde{p},\tilde{r})$, then $d(q,r) \leq d(\tilde{q},\tilde{r})$.*

Proof. Let $d(p,q) = d(\tilde{p},\tilde{q})$, $d(p,r) = d(\tilde{p},\tilde{r})$, and $\angle(q,p,r) = \angle(\tilde{q},\tilde{p},\tilde{r})$. We shall show that $d(q,r) \leq d(\tilde{q},\tilde{r})$. Suppose that $d(q,r) > d(\tilde{q},\tilde{r})$. Let $\triangle(\tilde{p},\tilde{q},\tilde{s})$ be a comparison triangle of $\triangle(p,q,r)$. Then we have $\angle(p,q,r) \geq \angle(\tilde{p},\tilde{q},\tilde{s})$. As $d(q,r) = d(\tilde{q},\tilde{s})$, we have $d(\tilde{q},\tilde{s}) > d(\tilde{q},\tilde{r})$, and hence $\angle(\tilde{p},\tilde{q},\tilde{s}) > \angle(\tilde{p},\tilde{q},\tilde{r})$. So we have $\angle(p,q,r) > \angle(\tilde{p},\tilde{q},\tilde{r})$, which is a contradiction. Therefore $d(q,r) \leq d(\tilde{q},\tilde{r})$, as desired. ∎

2. Distance between two points

In this section we give a necessary and sufficient condition for being a space of curvature bounded below. We also give a lower bound for distance between two points in this space. Let $\triangle(\tilde{x},\tilde{y},\tilde{z})$ be a triangle in R_K. We firstly give the distance between the point $\tilde{z}$ and the midpoint $\tilde{m}$ of the geodesic segment $[\tilde{x},\tilde{y}]$, opposite the angle at $\tilde{z}$ by the following lemma.

Lemma 2.1. *Let* $\triangle(\tilde{x}, \tilde{y}, \tilde{z})$ *be a triangle in* R_K *and* $\tilde{m}$ *the midpoint between two points* $\tilde{x}$ *and* $\tilde{y}$. *Then*

$$d(\tilde{m}, \tilde{z}) = \begin{cases} \dfrac{1}{\lambda}\cosh^{-1}\left(\dfrac{\cosh[\lambda d(\tilde{x}, \tilde{z})] + \cosh[\lambda d(\tilde{y}, \tilde{z})]}{2\cosh\left[\lambda \frac{d(\tilde{x}, \tilde{y})}{2}\right]}\right) & \text{if} \quad K < 0; \\ \sqrt{\dfrac{1}{2}\left[d^2(\tilde{x}, \tilde{z}) + d^2(\tilde{y}, \tilde{z})\right] - \dfrac{1}{4}d^2(\tilde{x}, \tilde{y})} & \text{if} \quad K = 0; \\ \dfrac{1}{\lambda}\cos^{-1}\left(\dfrac{\cos[\lambda d(\tilde{x}, \tilde{z})] + \cos[\lambda d(\tilde{y}, \tilde{z})]}{2\cos\left[\lambda \frac{d(\tilde{x}, \tilde{y})}{2}\right]}\right) & \text{if} \quad K > 0. \end{cases}$$

Proof. Let $\alpha = \angle(\tilde{x}, \tilde{m}, \tilde{z})$. Now we consider two triangles $\triangle(\tilde{x}, \tilde{m}, \tilde{z})$ and $\triangle(\tilde{y}, \tilde{m}, \tilde{z})$ in three possibilities.

Case $K < 0$. By the law of cosine in R_K, we have

$$\cosh[\lambda d(\tilde{x}, \tilde{z})] = \cosh[\lambda d(\tilde{x}, \tilde{m})]\cosh[\lambda d(\tilde{m}, \tilde{z})] - \sinh[\lambda d(\tilde{x}, \tilde{m})]\sinh[\lambda d(\tilde{m}, \tilde{z})]\cos\alpha, \quad (1)$$

and

$$\cosh[\lambda d(\tilde{y}, \tilde{z})] = \cosh[\lambda d(\tilde{y}, \tilde{m})]\cosh[\lambda d(\tilde{m}, \tilde{z})] - \sinh[\lambda d(\tilde{y}, \tilde{m})]\sinh[\lambda d(\tilde{m}, \tilde{z})]\cos(\pi - \alpha). \quad (2)$$

Since $d(\tilde{x}, \tilde{m}) = d(\tilde{y}, \tilde{m})$ and $\cos(\pi - \alpha) = -\cos(\alpha)$, by (2) we have that

$$\cosh[\lambda d(\tilde{y}, \tilde{z})] = \cosh[\lambda d(\tilde{x}, \tilde{m})]\cosh[\lambda d(\tilde{m}, \tilde{z})] + \sinh[\lambda d(\tilde{x}, \tilde{m})]\sinh[\lambda d(\tilde{m}, \tilde{z})]\cos(\alpha). \quad (3)$$

Using (1) and (3), we get

$$\cosh[\lambda d(\tilde{x}, \tilde{z})] + \cosh[\lambda d(\tilde{y}, \tilde{z})] = 2\cosh[\lambda d(\tilde{x}, \tilde{m})]\cosh[\lambda d(\tilde{m}, \tilde{z})],$$

and so

$$\cosh[\lambda d(\tilde{m}, \tilde{z})] = \frac{\cosh[\lambda d(\tilde{x}, \tilde{z})] + \cosh[\lambda d(\tilde{y}, \tilde{z})]}{2\cosh[\lambda d(\tilde{x}, \tilde{m})]}.$$

Hence

$$d(\tilde{m}, \tilde{z}) = \frac{1}{\lambda}\cosh^{-1}\left(\frac{\cosh[\lambda d(\tilde{x}, \tilde{z})] + \cosh[\lambda d(\tilde{y}, \tilde{z})]}{2\cosh[\lambda d(\tilde{x}, \tilde{m})]}\right).$$

As we know that $d(\tilde{x}, \tilde{y}) = 2d(\tilde{m}, \tilde{x})$, it follows that

$$d(\tilde{m}, \tilde{z}) = \frac{1}{\lambda}\cosh^{-1}\left(\frac{\cosh[\lambda d(\tilde{x}, \tilde{z})] + \cosh[\lambda d(\tilde{y}, \tilde{z})]}{2\cos\left[\lambda \frac{d(\tilde{x}, \tilde{y})}{2}\right]}\right).$$

Case $K = 0$. By the law of cosine in R_0, we have

$$d^2(\tilde{x}, \tilde{z}) = d^2(\tilde{x}, \tilde{m}) + d^2(\tilde{m}, \tilde{z}) - 2d(\tilde{x}, \tilde{m})d(\tilde{m}, \tilde{z})\cos\alpha, \quad (4)$$

and

$$d^2(\tilde{y}, \tilde{z}) = d^2(\tilde{y}, \tilde{m}) + d^2(\tilde{m}, \tilde{z}) - 2d(\tilde{y}, \tilde{m})d(\tilde{m}, \tilde{z})\cos(\pi - \alpha). \quad (5)$$

By (4) and (5) we have

$$\begin{aligned} d^2(\tilde{x},\tilde{z}) + d^2(\tilde{y},\tilde{z}) &= d^2(\tilde{x},\tilde{m}) + d^2(\tilde{m},\tilde{z}) - 2d(\tilde{x},\tilde{m})d(\tilde{m},\tilde{z})\cos\alpha \\ &\quad + d^2(\tilde{y},\tilde{m}) + d^2(\tilde{m},\tilde{z}) - 2d(\tilde{y},\tilde{m})d(\tilde{m},\tilde{z})\cos(\pi-\alpha) \\ &= d^2(\tilde{x},\tilde{m}) + d^2(\tilde{m},\tilde{z}) + d^2(\tilde{y},\tilde{m}) + d^2(\tilde{m},\tilde{z}) \\ &= 2d^2(\tilde{x},\tilde{m}) + 2d^2(\tilde{m},\tilde{z}) \\ &= \frac{d^2(\tilde{x},\tilde{y})}{2} + 2d^2(\tilde{m},\tilde{z}). \end{aligned}$$

Hence

$$d^2(\tilde{m},\tilde{z}) = \frac{1}{2}\left(d^2(\tilde{x},\tilde{z}) + d^2(\tilde{y},\tilde{z})\right) - \frac{d^2(\tilde{x},\tilde{y})}{4}.$$

Case $K > 0$. By the law of cosine in R_K, we have

$$\cos\left[\lambda d(\tilde{x},\tilde{z})\right] = \cos\left[\lambda d(\tilde{x},\tilde{m})\right]\cos\left[\lambda d(\tilde{m},\tilde{z})\right] + \sin\left[\lambda d(\tilde{x},\tilde{m})\right]\sin\left[\lambda d(\tilde{m},\tilde{z})\right]\cos\alpha, \tag{6}$$

and

$$\cos\left[\lambda d(\tilde{y},\tilde{z})\right] = \cos\left[\lambda d(\tilde{y},\tilde{m})\right]\cos\left[\lambda d(\tilde{m},\tilde{z})\right] + \sin\left[\lambda d(\tilde{y},\tilde{m})\right]\sin\left[\lambda d(\tilde{m},\tilde{z})\right]\cos(\pi-\alpha). \tag{7}$$

Since $d(\tilde{x},\tilde{m}) = d(\tilde{y},\tilde{m})$ and $\cos(\pi-\alpha) = -\cos(\alpha)$, by (7) we have that

$$\cos\left[\lambda d(\tilde{y},\tilde{z})\right] = \cos\left[\lambda d(\tilde{x},\tilde{m})\right]\cos\left[\lambda d(\tilde{m},\tilde{z})\right] - \sin\left[\lambda d(\tilde{x},\tilde{m})\right]\sin\left[\lambda d(\tilde{m},\tilde{z})\right]\cos(\alpha). \tag{8}$$

Adding (6) and (8), we get

$$\cos\left[\lambda d(\tilde{x},\tilde{z})\right] + \cos\left[\lambda d(\tilde{y},\tilde{z})\right] = 2\cos\left[\lambda d(\tilde{x},\tilde{m})\right]\cos\left[\lambda d(\tilde{m},\tilde{z})\right],$$

and thus

$$\cos\left[\lambda d(\tilde{m},\tilde{z})\right] = \frac{\cos\left[\lambda d(\tilde{x},\tilde{z})\right] + \cos\left[\lambda d(\tilde{y},\tilde{z})\right]}{2\cos\left[\lambda d(\tilde{x},\tilde{m})\right]}.$$

Hence

$$d(\tilde{m},\tilde{z}) = \frac{1}{\lambda}\cos^{-1}\left(\frac{\cos\left[\lambda d(\tilde{x},\tilde{z})\right] + \cos\left[\lambda d(\tilde{y},\tilde{z})\right]}{2\cos\left[\lambda d(\tilde{x},\tilde{m})\right]}\right).$$

As $d(\tilde{x},\tilde{y}) = 2d(\tilde{m},\tilde{x})$, it becomes

$$d(\tilde{m},\tilde{z}) = \frac{1}{\lambda}\cos^{-1}\left(\frac{\cos\left[\lambda d(\tilde{x},\tilde{z})\right] + \cos\left[\lambda d(\tilde{y},\tilde{z})\right]}{2\cos\left[\lambda \frac{d(\tilde{x},\tilde{y})}{2}\right]}\right).$$

The proof is completed. ■

Theorem 2.2. *Let X be a complete metric space. The space X is a space of curvature bounded below by K in the large if and only if for any pair of different points $x, y \in X$, there exists the midpoint m between them such that,*

$$d(m,z) \geq \begin{cases} \dfrac{1}{\lambda}\cosh^{-1}\left(\dfrac{\cosh\left[\lambda d(x,z)\right] + \cosh\left[\lambda d(y,z)\right]}{2\cosh\left[\lambda\frac{d(x,y)}{2}\right]}\right) & \text{if} \quad K < 0; \\ \sqrt{\dfrac{1}{2}\left[d^2(x,z) + d^2(y,z)\right] - \dfrac{1}{4}d^2(x,y)} & \text{if} \quad K = 0; \\ \dfrac{1}{\lambda}\cos^{-1}\left(\dfrac{\cos\left[\lambda d(x,z)\right] + \cos\left[\lambda d(y,z)\right]}{2\cos\left[\lambda\frac{d(x,y)}{2}\right]}\right) & \text{if} \quad K > 0. \end{cases}$$

for all $z \in X$.

Proof. Suppose that X is a space of curvature bounded below by K in the large. Let $x, y, z \in X$. We choose m the midpoint of geodesic segment $[x, y]$. We consider a comparison triangle $\triangle(\tilde{x}, \tilde{y}, \tilde{z})$ in R_K of the triangle $\triangle(x, y, z)$. Let $\tilde{m}$ be the midpoint of $[\tilde{x}, \tilde{y}]$. By Lemma 2.1, we have

$$d(\tilde{m}, \tilde{z}) = \begin{cases} \frac{1}{\lambda}\cosh^{-1}\left(\frac{\cosh[\lambda d(\tilde{x},\tilde{z})] + \cosh[\lambda d(\tilde{y},\tilde{z})]}{2\cosh\left[\lambda \frac{d(\tilde{x},\tilde{y})}{2}\right]}\right) & \text{if} \quad K < 0; \\ \sqrt{\frac{1}{2}\left[d^2(\tilde{x},\tilde{z}) + d^2(\tilde{y},\tilde{z})\right] - \frac{1}{4}d^2(\tilde{x},\tilde{y})} & \text{if} \quad K = 0; \\ \frac{1}{\lambda}\cos^{-1}\left(\frac{\cos[\lambda d(\tilde{x},\tilde{z})] + \cos[\lambda d(\tilde{y},\tilde{z})]}{2\cos\left[\lambda \frac{d(\tilde{x},\tilde{y})}{2}\right]}\right) & \text{if} \quad K > 0, \end{cases}$$

and hence the result follows by $d(y, z) = d(\tilde{y}, \tilde{z})$, $d(x, z) = d(\tilde{x}, \tilde{z})$, $d(x, y) = d(\tilde{x}, \tilde{y})$ and $d(m, z) \geq d(\tilde{m}, \tilde{z})$.

Next, we shall prove the necessity. Suppose that the sufficiency holds and suppose that for a pair of different points $x, y \in X$, there exists the midpoint m between them satisfying the following:

$$d(m, z) \geq \begin{cases} \frac{1}{\lambda}\cosh^{-1}\left(\frac{\cosh[\lambda d(x,z)] + \cosh[\lambda d(y,z)]}{2\cosh\left[\lambda \frac{d(x,y)}{2}\right]}\right) & \text{if} \quad K < 0; \\ \sqrt{\frac{1}{2}\left[d^2(x,z) + d^2(y,z)\right] - \frac{1}{4}d^2(x,y)} & \text{if} \quad K = 0; \\ \frac{1}{\lambda}\cos^{-1}\left(\frac{\cos[\lambda d(x,z)] + \cos[\lambda d(y,z)]}{2\cos\left[\lambda \frac{d(x,y)}{2}\right]}\right) & \text{if} \quad K > 0. \end{cases}$$

for all $z \in X$. Let $\tilde{m}$ be the midpoint of geodesic segment $[\tilde{x}, \tilde{y}]$. We can conclude that X is a space of curvature bounded below by K in the large if $d(m, z) \geq d(\tilde{m}, \tilde{z})$. By Lemma 2.1 and $d(y, z) = d(\tilde{y}, \tilde{z})$, $d(x, z) = d(\tilde{x}, \tilde{z})$ and $d(x, y) = d(\tilde{x}, \tilde{y})$, we have

$$d(\tilde{m}, \tilde{z}) = \begin{cases} \frac{1}{\lambda}\cosh^{-1}\left(\frac{\cosh[\lambda d(x,z)] + \cosh[\lambda d(y,z)]}{2\cosh\left[\lambda \frac{d(x,y)}{2}\right]}\right) & \text{if} \quad K < 0; \\ \sqrt{\frac{1}{2}\left[d^2(x,z) + d^2(y,z)\right] - \frac{1}{4}d^2(x,y)} & \text{if} \quad K = 0; \\ \frac{1}{\lambda}\cos^{-1}\left(\frac{\cos[\lambda d(x,z)] + \cos[\lambda d(y,z)]}{2\cos\left[\lambda \frac{d(x,y)}{2}\right]}\right) & \text{if} \quad K > 0. \end{cases}$$

Then $d(m, z) \geq d(\tilde{m}, \tilde{z})$. ■

If X is a CAT(K) space (this space is introduced in [11]), then the inequality in Theorem 2.2 becomes $\leq$, see [13].

Corollary 2.3. *Let X be a metric space of curvature bounded below by 0 in the large. If a triangle $\triangle$ in X has sides of length $a, b, c > 0$ and angle α at the vertex opposite to the side of length c, then $c^2 \leq a^2 + b^2 - 2ab\cos\alpha$.*

Proof. Let $\triangle$ be a triangle in X whose sides of length are $a, b, c > 0$ and angle is α at the vertex opposite to the side of length c and $\triangle(\tilde{x}, \tilde{y}, \tilde{z})$ comparison triangle of $\triangle$ in R_0 such that $d(\tilde{x}, \tilde{y}) = a$, $d(\tilde{x}, \tilde{z}) = b$ and $d(\tilde{y}, \tilde{z}) = c$. By the law of cosine in R_0, we have

$$c^2 = a^2 + b^2 - 2ab\cos\angle(\tilde{y}, \tilde{x}, \tilde{z}). \tag{10}$$

Since X is a space of curvature bounded below by 0 in the large, we have $\alpha \geq \angle(\tilde{y}, \tilde{x}, \tilde{z})$ and hence $\cos\alpha \leq \cos\angle(\tilde{y}, \tilde{x}, \tilde{z})$. By (10), we have

$$c^2 \leq a^2 + b^2 - 2ab\cos\alpha.$$

■

If X is a CAT(0) space, then the inequality in Corollary 2.3 becomes $\geq$, see [12].

Theorem 2.4. *Let X be a metric space of curvature bounded below by K in the large and A be a bounded set of X. If any pair of different points $x, y \in X$ and the midpoint m of them satisfy the following conditions:*

(1) $d^2(x, A) + d^2(y, A) \geq 2d^2(m, A)$, *if* $K = 0$
(2) $\max\{d(x, A), d(y, A), d(m, A)\} < \frac{D_K}{2}$ *and* $\cos[\lambda d(x, A)] + \cos[\lambda d(y, A)] \leq 2\lambda d(m, A)$, *if* $K > 0$,

then

$$d(x, y) \geq \begin{cases} \dfrac{2}{\lambda}\cosh^{-1}\left(\dfrac{\cosh[\lambda d(x, A)] + \cosh[\lambda d(y, A)]}{2\cosh\left[\lambda\frac{d(x,y)}{2}\right]}\right) & ; \quad K < 0; \\ \sqrt{2[d^2(x, A) + d^2(y, A)] - 4d^2(m, A)} & ; \quad K = 0; \\ \dfrac{2}{\lambda}\cos^{-1}\left(\dfrac{\cos[\lambda d(x, A)] + \cos[\lambda d(y, A)]}{2\cos\left[\lambda\frac{d(x,y)}{2}\right]}\right) & ; \quad K > 0. \end{cases}$$

Proof. Let $x, y \in X$ and m the midpoint of them. We now consider in three possibilities.
Case $K < 0$. By Theorem 2.2, we have seen that for each $z \in A$,

$$d(m, z) \geq \frac{1}{\lambda}\cosh^{-1}\left(\frac{\cosh[\lambda d(x, z)] + \cosh[\lambda d(y, z)]}{2\cosh\left[\lambda\frac{d(x,y)}{2}\right]}\right).$$

Since the hyperbolic cosine function is increasing, it follows that for each $z \in A$,

$$\cosh[\lambda d(m, z)] \geq \frac{\cosh[\lambda d(x, z)] + \cosh[\lambda d(y, z)]}{2\cosh\left[\lambda\frac{d(x,y)}{2}\right]}.$$

Then

$$\begin{aligned}
\cosh[\lambda d(m,A)] &= \cosh[\lambda \inf_{z\in A} d(m,z)] \\
&= \inf_{z\in A} \cosh[\lambda d(m,z)] \\
&\geq \inf_{z\in A} \left(\frac{\cosh[\lambda d(x,z)] + \cosh[\lambda d(y,z)]}{2\cosh\left[\lambda \frac{d(x,y)}{2}\right]} \right) \\
&\geq \frac{\inf_{z\in A} \cosh[\lambda d(x,z)] + \inf_{z\in A} \cosh[\lambda d(y,z)]}{2\cosh\left[\lambda \frac{d(x,y)}{2}\right]} \\
&= \frac{\cosh[\lambda \inf_{z\in A} d(x,z)] + \cosh[\lambda \inf_{z\in A} d(y,z)]}{2\cosh\left[\lambda \frac{d(x,y)}{2}\right]} \\
&= \frac{\cosh[\lambda d(x,A)] + \cosh[\lambda d(y,A)]}{2\cosh\left[\lambda \frac{d(x,y)}{2}\right]},
\end{aligned}$$

which gives

$$\cosh\left(\lambda \frac{d(x,y)}{2}\right) \geq \frac{\cosh[\lambda d(x,A)] + \cosh[\lambda d(y,A)]}{2\cosh[\lambda d(m,A)]}.$$

Hence

$$d(x,y) \geq \frac{2}{\lambda} \cosh^{-1}\left(\frac{\cosh[\lambda d(x,A)] + \cosh[\lambda d(y,A)]}{2\cosh[\lambda d(m,A)]}\right).$$

Case $K = 0$. By Theorem 2.2, we have seen that for each $z \in A$,

$$d(m,z) \geq \sqrt{\frac{1}{2}\left[d^2(x,z) + d^2(y,z)\right] - \frac{1}{4} d^2(x,y)},$$

that means,

$$d^2(x,y) \geq 2d^2(x,z) + 2d^2(y,z) - 4d^2(m,z).$$

We then have

$$\begin{aligned}
d^2(x,y) &\geq \inf_{z\in A} [2d^2(x,z) + 2d^2(y,z) - 4d^2(m,z)] \\
&\geq 2\inf_{z\in A} d^2(x,z) + 2\inf_{z\in A} d^2(y,z) - 4\inf_{z\in A} d^2(m,z) \\
&= 2d^2(x,A) + 2d^2(y,A) - 4d^2(m,A).
\end{aligned}$$

Hence

$$d(x,y) \geq \sqrt{2d^2(x,A) + 2d^2(y,A) - 4d^2(m,A)}.$$

Case $K > 0$. By Theorem 2.2, we have seen that for each $z \in A$,

$$d(m,z) \geq \frac{1}{\lambda} \cos^{-1}\left(\frac{\cos[\lambda d(x,z)] + \cos[\lambda d(y,z)]}{2\cos\left[\lambda \frac{d(x,y)}{2}\right]}\right).$$

Since $\dim(X) \leq D_K$ and the cosine function decreases on $[0,\pi]$, we have that for each $z \in A$,

$$\cos[\lambda d(m,z)] \leq \frac{\cos[\lambda d(x,z)] + \cos[\lambda d(y,z)]}{2\cos\left[\lambda \frac{d(x,y)}{2}\right]}.$$

Thus

$$\begin{aligned}
\cos[\lambda d(m,A)] &= \cos[\lambda \inf_{z\in A} d(m,z)] \\
&= \sup_{z\in A} \cos[\lambda d(m,z)] \\
&\le \sup_{z\in A} \left(\frac{\cos[\lambda d(x,z)] + \cos[\lambda d(y,z)]}{2\cos\left[\lambda \frac{d(x,y)}{2}\right]} \right) \\
&\le \frac{\sup_{z\in A}\cos[\lambda d(x,z)] + \sup_{z\in A}\cos[\lambda d(y,z)]}{2\cos\left[\lambda \frac{d(x,y)}{2}\right]} \\
&= \frac{\cos[\lambda \inf_{z\in A} d(x,z)] + \cos[\lambda \inf_{z\in A} d(y,z)]}{2\cos\left[\lambda \frac{d(x,y)}{2}\right]} \\
&= \frac{\cos[\lambda d(x,A)] + \cos[\lambda d(y,A)]}{2\cos\left[\lambda \frac{d(x,y)}{2}\right]}.
\end{aligned}$$

We now have that $\max\{d(x,A), d(y,A), d(m,A)\} < \frac{D_K}{2}$, which gives $\cos[\lambda d(m,A)]$, $\cos[\lambda d(x,A)]$ and $\cos[\lambda d(y,A)]$ positive, and hence

$$\cos\left(\lambda\frac{d(x,y)}{2}\right) \le \frac{\cos[\lambda d(x,A)] + \cos[\lambda d(y,A)]}{2\cos[\lambda d(m,A)]}.$$

Therefore,

$$d(x,y) \ge \frac{2}{\lambda}\cos^{-1}\left(\frac{\cos[\lambda d(x,A)] + \cos[\lambda d(y,A)]}{2\cos[\lambda d(m,A)]}\right).$$

■

If X is a CAT(K) space, then the inequality in Theorem 2.4 becomes $\le$, see [13].

3. Nearest Points

In this section we define a nearest point in a metric space of curvature bounded below and then we give some remarks.

Definition 3.1. Let X be a metric space of curvature bounded below by K in the large and C a complete convex subset of X with induced metric. A point $w \in C$ is a nearest point of a point $x \in X$ if $d(x,w) = d(x,C) = \inf\{d(x,y) : y \in C\}$. Let $\pi(x,C)$ denote the set of all nearest points of x on C.

Let $X = \{(x,y,z) \in R^3 \mid x^2+y^2+z^2 = 1\}$ and $C = \{(x,y,z) \in R^3 \mid x^2+y^2+z^2 = 1, z \ge 0\}$. Then X is a metric space of curvature bounded below by 1 and C is a closed convex, compact, and complete subset of X. We consider a point $x = (0,0,-1) \in X$. We get $\pi(x,C) = \{(x,y,0) \mid x^2+y^2 = 1\}$. We can see that $\ell_1 := \{(x,y,0) \in R^3 \mid y = \sqrt{1-x^2}\}$ and $\ell_2 := \{(x,y,0) \in R^3 \mid y = -\sqrt{1-x^2}\}$ are geodesic segments, which all their points are nearest points, and $\ell_3 := \{(x,y,0) \in R^3 \mid y = -\sqrt{1-z^2}\}$ is a geodesic segment, whose endpoints are nearest points but the points between the endpoints are not nearest points. So we can conclude that the point between two nearest points which are the endpoints of a geodesic segment need not to be a nearest point.

Theorem 3.2. *Let X be a metric space of curvature bounded below by K in the large and C a complete convex subset and $x \in X - C$. If $y \in \pi(x, C)$ and $x' \in [x, y]$, then $y \in \pi(x', C)$.*

Proof. Let $y \in \pi(x, C)$ and $x' \in [x, y]$. Suppose that $y \notin \pi(x', C)$. Let $z \in \pi(x', C)$. Then $d(x', z) < d(x', y)$. Since $d(x, y) = d(x, x') + d(x', y)$ and $y \in \pi(x, C)$, we have $d(x, y) < d(x, z)$. As x' is not a branch point, $d(x, z) < d(x, x') + d(x', z)$ and thus

$$d(x, x') + d(x', y) = d(x, y) < d(x, z) < d(x, x') + d(x', z).$$

It follows that $d(x', y) < d(x', z)$, which is a contradiction. Therefore $y \in \pi(x', C)$, as required. ∎

Let D be a non-empty subset of X, we put $I_D(x) = \cup_{y \in X}\{y : (x, y] \cap D \neq \emptyset\} \cup \{x\}$.

Theorem 3.3. *Let X be a metric space of curvature bounded below by K in the large, C a complete convex subset of X and $z \in \pi(x, C)$. Suppose that*

(1) $\angle(x, z, w) \geq \frac{\pi}{2}$ *and* $\widetilde{\angle}(x, z, w) \geq \frac{\pi}{2}$ *for all* $w \in C$;
(2) $y \in \overline{I_C(z)} - \{z\}$;
(3) $\max\{d(x, y), d(y, z), d(x, z)\} < \frac{\pi}{2\lambda}$, *if* $K > 0$,

then $d(x, z) < d(x, y)$.

Proof. Since $y \in \overline{I_C(z)} - \{z\}$, there is a sequence $y_n \in I_C(z)$ such that $y_n \to y$. For each large positive integer n, we can find z_n in $(z, y_n] \cap C$. Because $z_n \in C - \{z\}$, by condition (1) we have that $\angle(x, z, z_n) \geq \frac{\pi}{2}$ and $\widetilde{\angle}(x, z, z_n) \geq \frac{\pi}{2}$, where $\triangle(\tilde{x}, \tilde{z}, \tilde{z_n})$ is a comparison triangle of $\triangle(x, z, z_n)$ in R_K. As $\angle(\tilde{x}, \tilde{z}, \tilde{z_n}) = \angle(\tilde{x}, \tilde{z}, \tilde{y_n})$, by the law of cosines in R_K, we have the following:

$$\begin{aligned}
\cosh[\lambda d(\tilde{x}, \tilde{y}_n)] &\geq \cosh[\lambda d(\tilde{x}, \tilde{z})]\cosh[\lambda d(\tilde{z}, \tilde{y}_n)] && \text{if} \quad K < 0;\\
d^2(\tilde{x}, \tilde{y}_n) &\geq d^2(\tilde{x}, \tilde{z}) + d^2(\tilde{z}, \tilde{y}_n) && \text{if} \quad K = 0;\\
\cos[\lambda d(\tilde{x}, \tilde{y}_n)] &\leq \cos[\lambda d(\tilde{x}, \tilde{z})]\cos[\lambda d(\tilde{z}, \tilde{y}_n)] && \text{if} \quad K > 0.
\end{aligned}$$

Since $d(x, y_n) = d(\tilde{x}, \tilde{y}_n)$, $d(x, z) = d(\tilde{x}, \tilde{z}_n)$ and $d(z, y_n) = d(\tilde{z}, \tilde{y}_n)$,

$$\begin{aligned}
\cosh[\lambda d(x, y_n)] &\geq \cosh[\lambda d(x, z)]\cosh[\lambda d(z, y_n)] && \text{if} \quad K < 0;\\
d^2(x, y_n) &\geq d^2(x, z) + d^2(z, y_n) && \text{if} \quad K = 0;\\
\cos[\lambda d(x, y_n)] &\leq \cos[\lambda d(x, z)]\cos[\lambda d(z, y_n)] && \text{if} \quad K > 0.
\end{aligned}$$

Taking $n \to \infty$, we obtain

$$\begin{aligned}
\cosh[\lambda d(x, y)] &\geq \cosh[\lambda d(x, z)]\cosh[\lambda d(z, y)] && \text{if} \quad K < 0;\\
d^2(x, y) &\geq d^2(x, z) + d^2(z, y) && \text{if} \quad K = 0;\\
\cos[\lambda d(x, y)] &\leq \cos[\lambda d(x, z)]\cos[\lambda d(z, y)] && \text{if} \quad K > 0.
\end{aligned}$$

Because $\cosh[\lambda d(z, y)] > 1$, $d^2(z, y) > 0$ and $\cos[\lambda d(z, y)] < 1$, we have

$$\begin{aligned}
\cosh[\lambda d(x, y)] &> \cosh[\lambda d(x, z)] && \text{if} \quad K < 0;\\
d^2(x, y) &> d^2(x, z) && \text{if} \quad K = 0;\\
\cos[\lambda d(x, y)] &< \cos[\lambda d(x, z)] && \text{if} \quad K > 0,
\end{aligned}$$

which give $d(x, y) > d(x, z)$ for any K, as desired.

∎

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