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Weak and strong convergence theorems for a finite family of nonexpansive and asymptotically nonexpansive mappings in Banach spaces

Satit Saejung and Kittipong Sitthikul

Abstract : In this paper, weak and strong convergence of finite step iteration sequences with errors to a common fixed point for a pair of a finite family of nonexpansive mappings and a finite family of asymptotically nonexpansive mappings in nonempty closed convex subset of uniformly convex Banach spaces are presented.

Keywords : Nonexpansive mapping, asymptotically nonexpansive mapping, common fixed point, finite-step iterative sequence.
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1 Introduction

The concept of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [4] in 1972. They also proved that every asymptotically nonexpansive mapping of a nonempty closed bounded subset of a uniformly convex Banach space always has a fixed point. Since then many authors have studied iterative approximation methods of fixed points for asymptotically nonexpansive mappings. In 1991, Schu [13], [14] introduced the modified Mann iteration method and proved that such iterative sequences converge strongly to a fixed point of an asymptotically nonexpansive mapping in a Hilbert space. Rhoades [12] extended the results in [13] to uniformly convex Banach spaces and to the modified Ishikawa iteration methods.

Recently, Gu and He [6] studied a multi-step iterative sequence involving finite nonexpansive mappings in a uniformly convex Banach space. They obtained weak and strong convergence theorems for approximating common fixed points of nonexpansive mappings. Liu et.al. [8, 9] introduced new iterative methods, the modified three-step and the modified Ishikawa iteration methods with respect to a pair of mappings. They also proved some convergence theorems which improve and unify many results due to Chang [1], Liu and Kang[7], Osilike and Aniagbosor [11], Rhoades [12], and Schu [13, 14] and others.

Inspired and motivated by the works in [8, 9], we introduce a new iterative method with respect to finite mappings, and establish some strong and weak convergence theorems of our iteration method in uniformly convex Banach spaces. The results presented in this paper generalize, improve and unify many results due to Liu et.al. [8, 9] and also Gu and He [6].

2 Preliminaries

Let K a nonempty subset of a real Banach space E and $T : K \to K$ be a mapping with the fixed point set $F(T)$, i.e., $F(T) = \{x \in K : x = Tx\}$.

Definition 2.1. *A mapping $T : K \to K$ is said to be*

1. asymptotically nonexpansive *if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $\|T^n x - T^n y\| \le k_n\|x - y\|$ for all $x, y \in K$ and $n \ge 1$;*

2. nonexpansive *if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in K$;*

3. semi-compact *if K is closed and for any bounded sequence $\{x_n\}$ in K with $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, there exist a subsequence $\{x_{n_k}\} \subset \{x_n\}$ and $x \in K$ such that $\lim_{k\to\infty} x_{n_k} = x$;*

4. demi-closed at a point $p \in K$ *if whenever $\{x_n\}$ is a sequence in K which converges weakly to a point $x \in K$ and $\{Tx_n\}$ converges strongly to p, it follows that $Tx = p$.*

It is clear that every nonexpansive mapping is asymptotically nonexpansive. But the converse is not true (see [4]).

Definition 2.2 ([3])**.** *A Banach space E is* uniformly convex *if for all $\{x_n\}, \{y_n\} \subset \{z \in X : \|z\| = 1\}$ such that $\left\|\frac{x_n+y_n}{2}\right\| \to 1$, we have $\|x_n - y_n\| \to 0$.*

Definition 2.3 ([10])**.** *A Banach space E satisfies* Opial's condition *if for each sequence $\{x_n\}$ in E which converges weakly to a point $x \in E$, we have*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \quad \textit{for all } y \in E \textit{ with } y \neq x.$$

Let K be a nonempty subset of a Banach space E. Let $S_1, S_2, \ldots, S_N : K \to K$ be N nonexpansive mappings, $T_1, T_2, \ldots, T_N : K \to K$ be N asymptotically nonexpansive mappings. Then the sequence $\{x_n\}$ defined by

$$\begin{cases} x_1 \in K, \\ x_n^{(N)} = x_n, \\ x_n^{(N-1)} = a_n^{(N)} T_N^n x_n^{(N)} + b_n^{(N)} S_N x_n + c_n^{(N)} u_n^{(N)}, \\ x_n^{(N-2)} = a_n^{(N-1)} T_{N-1}^n x_n^{(N-1)} + b_n^{(N-1)} S_{N-1} x_n + c_n^{(N-1)} u_n^{(N-1)}, \\ \vdots \\ x_n^{(2)} = a_n^{(3)} T_3^n x_n^{(3)} + b_n^{(3)} S_3 x_n + c_n^{(3)} u_n^{(3)}, \\ x_n^{(1)} = a_n^{(2)} T_2^n x_n^{(2)} + b_n^{(2)} S_2 x_n + c_n^{(2)} u_n^{(2)}, \\ x_{n+1} = a_n^{(1)} T_1^n x_n^{(1)} + b_n^{(1)} S_1 x_n + c_n^{(1)} u_n^{(1)}, \quad n \geq 1, \end{cases} \tag{2.1}$$

is called *the N-step iterative sequence*, where $\{u_n^{(i)}\}$ are bounded sequences in K and $\{a_n^{(i)}\}_{n=1}^\infty$, $\{b_n^{(i)}\}_{n=1}^\infty$, $\{c_n^{(i)}\}_{n=1}^\infty \subset [0,1]$ such that $a_n^{(i)} + b_n^{(i)} + c_n^{(i)} = 1$, for all $i = 1, 2, \ldots, N$.

The purpose of this paper is to study the weak and strong convergence of finite-step iteration sequence with errors terms $\{x_n\}$ defined by (2.1) to a common fixed point for a pair of a finite family of nonexpansive mappings and a finite family of asymptotically nonexpansive mappings in a uniformly convex Banach space.

The following lemmas are our main tool for proving the results.

Lemma 2.1 ([5]). *Let E be a uniformly convex Banach space and K be a nonempty closed convex subset of E. If $T : K \to K$ is an asymptotically nonexpansive mapping, then $I - T$ is demiclosed at zero.*

Lemma 2.2 ([14]). *Let E be a uniformly convex Banach space, $\{t_n\} \subseteq [b, c] \subset (0, 1)$, $\{x_n\}$ and $\{y_n\}$ be sequences in E. If $\limsup_{n\to\infty} \|x_n\| \leq a$, $\limsup_{n\to\infty} \|y_n\| \leq a$ and $\lim_{n\to\infty} \|t_n x_n + (1-t_n) y_n\| = a$ for some $a \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.3 ([11]). *Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be sequences of nonnegative numbers satisfying the inequality*

$$a_{n+1} \leq (1 + c_n) a_n + b_n, \quad \textit{for all } n \geq 1.$$

If $\sum_{n=1}^\infty c_n < \infty$ and $\sum_{n=1}^\infty b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. In particular, if $\{a_n\}$ has a subsequence which converges to zero, then $\lim_{n\to\infty} a_n = 0$.

Proposition 2.4 ([15]). *Let K be a nonempty subset of a Banach space E and $T_1, T_2 \ldots, T_N : K \to K$ be N asymptotically nonexpansive mappings. Then there exists a sequence $\{k_n\} \subset [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$ and*

$$\|T_i^n x - T_i^n y\| \leq k_n \|x - y\| \tag{2.2}$$

for all $x, y \in K$, $n \geq 1$ and $i = 1, 2, \ldots, N$.

Proof. Since each $T_i : K \to K$ is an asymptotically nonexpansive mapping, there exists a sequence $\{k_n^{(i)}\} \subset [1, \infty)$ such that $\lim_{n\to\infty} k_n^{(i)} = 1$ and

$$\|T_i^n x - T_i^n y\| \leq k_n^{(i)} \|x - y\| \quad \text{for all } n \geq 1,$$

for all $i = 1, 2, \ldots, N$ Letting

$$k_n = \max\{k_n^{(1)}, k_n^{(2)}, \ldots, k_n^{(N)}\},$$

so we have $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ and (2.2) is satisfied. □

3 Main Results

Since the proof for the N-step iterative scheme is almost the same as the case $N = 3$, we may consider the following scheme instead:

$$\begin{cases} x_1 \in K, \\ z_n = a_n^{(3)} T_3^n x_n + b_n^{(3)} S_3 x_n + c_n^{(3)} u_n^{(3)}, \\ y_n = a_n^{(2)} T_2^n z_n + b_n^{(2)} S_2 x_n + c_n^{(2)} u_n^{(2)}, \\ x_{n+1} = a_n^{(1)} T_1^n y_n + b_n^{(1)} S_1 x_n + c_n^{(1)} u_n^{(1)}, \quad n \geq 1, \end{cases} \tag{3.1}$$

$\{u_n^{(i)}\}$ are bounded sequences in K and $\{a_n^{(i)}\}_{n=1}^{\infty}, \{b_n^{(i)}\}_{n=1}^{\infty}, \{c_n^{(i)}\}_{n=1}^{\infty} \subset [0, 1]$ such that $a_n^{(i)} + b_n^{(i)} + c_n^{(i)} = 1$, for all $i = 1, 2, 3$.

Lemma 3.1. *Let K be a nonempty convex subset of a real Banach space E. Let $S_1, S_2, S_3 : K \to K$ be nonexpansive mappings, $T_1, T_2, T_3 : K \to K$ be asymptotically nonexpansive mappings with a sequence $\{k_n\}$ given in Proposition 2.4 and $\cap_{i=1}^3 F(S_i) \cap F(T_i) \neq \varnothing$. If*

$$\sum_{n=1}^{\infty} (k_n - 1) < \infty, \tag{3.2}$$

and

$$\sum_{n=1}^{\infty} c_n^{(i)} < \infty \quad \textit{for all } i = 1, 2, 3, \tag{3.3}$$

then $\lim_{n\to\infty} \|x_n - q\|$ exists for any $q \in \cap_{i=1}^3 F(S_i) \cap F(T_i)$, where $\{x_n\}$ is defined by the iterative scheme (3.1).

Proof. Let $q \in \cap_{i=1}^3 F(S_i) \cap F(T_i)$. Since S_1, S_2 and S_3 are nonexpansive and T_1, T_2 and T_3 are asymptotically nonexpansive, it follows from (3.1) that

$$\begin{aligned} \|z_n - q\| &\leq a_n^{(3)} \|T_3^n x_n - q\| + b_n^{(3)} \|S_3 x_n - q\| + c_n^{(3)} \|u_n^{(3)} - q\| \\ &\leq a_n^{(3)} k_n \|x_n - q\| + b_n^{(3)} \|x_n - q\| + c_n^{(3)} \|u_n^{(3)} - q\| \\ &\leq a_n^{(3)} k_n \|x_n - q\| + b_n^{(3)} k_n \|x_n - q\| + c_n^{(3)} \|u_n^{(3)} - q\| \\ &\leq (a_n^{(3)} + b_n^{(3)}) k_n \|x_n - q\| + c_n^{(3)} \|u_n^{(3)} - q\| \\ &\leq k_n \|x_n - q\| + t_n^{(3)}, \quad \text{where } t_n^{(3)} = c_n^{(3)} \|u_n^{(3)} - q\|. \end{aligned} \tag{3.4}$$

Since $\{u_n^{(3)}\}$ is bounded and $\sum_{n=1}^{\infty} c_n^{(3)} < \infty$, $\sum_{n=1}^{\infty} t_n^{(3)} < \infty$, and from (3.4), we have

$$\begin{aligned}
\|y_n - q\| &\le a_n^{(2)}\|T_2^n z_n - q\| + b_n^{(2)}\|S_2 x_n - q\| + c_n^{(2)}\|u_n^{(2)} - q\| \\
&\le a_n^{(2)} k_n \|z_n - q\| + b_n^{(2)}\|x_n - q\| + c_n^{(2)}\|u_n^{(2)} - q\| \\
&\le a_n^{(2)} k_n^2 \|x_n - q\| + a_n^{(2)} t_n^{(3)} + b_n^{(2)} k_n^2 \|x_n - q\| + c_n^{(2)}\|u_n^{(2)} - q\| \\
&\le (a_n^{(2)} + b_n^{(2)}) k_n^2 \|x_n - q\| + a_n^{(2)} t_n^{(3)} + c_n^{(2)}\|u_n^{(2)} - q\| \\
&\le k_n^2 \|x_n - q\| + t_n^{(2)}, \quad \text{where } t_n^{(2)} = a_n^{(2)} t_n^{(3)} + c_n^{(2)}\|u_n^{(2)} - q\|.
\end{aligned} \tag{3.5}$$

From $\{u_n^{(2)}\}$ is bounded, $\sum_{n=1}^{\infty} c_n^{(2)} < \infty$ and $\sum_{n=1}^{\infty} t_n^{(3)} < \infty$, $\sum_{n=1}^{\infty} t_n^{(2)} < \infty$. Then, by (3.1) and (3.5),

$$\begin{aligned}
\|x_{n+1} - q\| &\le a_n^{(1)}\|T_1^n y_n - q\| + b_n^{(1)}\|S_1 x_n - q\| + c_n^{(1)}\|u_n^{(1)} - q\| \\
&\le a_n^{(1)} k_n \|y_n - q\| + b_n^{(1)}\|x_n - q\| + c_n^{(1)}\|u_n^{(1)} - q\| \\
&\le a_n^{(1)} k_n^3 \|x_n - q\| + a_n^{(1)} t_n^{(2)} + b_n^{(1)} k_n^3 \|x_n - q\| + c_n^{(1)}\|u_n^{(1)} - q\| \\
&\le (a_n^{(1)} + b_n^{(1)}) k_n^3 \|x_n - q\| + a_n^{(1)} t_n^{(2)} + c_n^{(1)}\|u_n^{(1)} - q\| \\
&\le (1 + (k_n^3 - 1))\|x_n - q\| + t_n^{(1)}, \quad \text{for } n \ge 1,
\end{aligned} \tag{3.6}$$

where $t_n^{(1)} = a_n^{(1)} t_n^{(2)} + c_n^{(1)}\|u_n^{(1)} - q\|$. Since $\{u_n^{(1)}\}$ is bounded, $\sum_{n=1}^{\infty} c_n^{(1)} < \infty$ and $\sum_{n=1}^{\infty} t_n^{(2)} < \infty$, $\sum_{n=1}^{\infty} t_n^{(1)} < \infty$. Notice that (3.2) holds if and only if $\sum_{n=1}^{\infty}(k_n^3 - 1) < \infty$. By Lemma 2.3, we have $\lim_{n\to\infty}\|x_n - q\|$ exists. This completes the proof. $\square$

Lemma 3.2. *Let K be a nonempty convex subset of a uniformly convex Banach space E. Let $S_1, S_2, S_3 : K \to K$ be nonexpansive mappings, $T_1, T_2, T_3 : K \to K$ be asymptotically nonexpansive mappings with a sequence $\{k_n\}$ given in Proposition 2.4 and $\cap_{i=1}^3 F(S_i) \cap F(T_i) \ne \varnothing$. Suppose that (3.2) and (3.3) hold and*

$$\|x - T_i y\| \le \|S_i x - T_i y\| \quad \text{for all } x, y \in K \text{ and } i = 1, 2, 3. \tag{3.7}$$

Suppose that there is $\delta > 0$ such that

$$\delta \le a_n^{(i)} \le 1 - \delta \quad \text{for all } n \ge 1 \text{ and } i = 1, 2, 3. \tag{3.8}$$

If $\{x_n\}$ is defined by the iterative scheme (3.1), then

$$\lim_{n\to\infty}\|x_n - S_i x_n\| = \lim_{n\to\infty}\|x_n - T_i x_n\| = 0,$$

for all $i = 1, 2, 3$.

Proof. Let $q \in \cap_{i=1}^3 F(S_i) \cap F(T_i)$. By Lemma 3.1, we have

$$d = \lim_{n\to\infty}\|x_n - q\| \text{ exists.} \tag{3.9}$$

It follows from (3.4), (3.5), (3.9) and $\lim_{n\to\infty} k_n = 1$ that

$$\limsup_{n\to\infty} \|z_n - q\| \le d, \tag{3.10}$$

and

$$\limsup_{n\to\infty} \|y_n - q\| \le d. \tag{3.11}$$

Moreover,

$$\begin{aligned} d &= \lim_{n\to\infty} \|x_{n+1} - q\| \\ &= \lim_{n\to\infty} \|a_n^{(1)}(T_1^n y_n - q + c_n^{(1)}(u_n^{(1)} - S_1 x_n)) + (1 - a_n^{(1)})(S_1 x_n - q + c_n^{(1)}(u_n^{(1)} - S_1 x_n))\|. \end{aligned}$$

From S_1 is nonexpansive, T_1 is asymptotically nonexpansive, (3.9), and (3.11), we have

$$\limsup_{n\to\infty} \|S_1 x_n - q + c_n^{(1)}(u_n^{(1)} - S_1 x_n)\| \le d,$$

and

$$\limsup_{n\to\infty} \|T_1^n y_n - q + c_n^{(1)}(u_n^{(1)} - S_1 x_n)\| \le d.$$

By Lemma 2.2, we get

$$\lim_{n\to\infty} \|S_1 x_n - T_1^n y_n\| = \lim_{n\to\infty} \|(S_1 x_n - q + c_n^{(1)}(u_n^{(1)} - S_1 x_n)) - (T_1^n y_n - q + c_n^{(1)}(u_n^{(1)} - S_1 x_n))\| = 0. \tag{3.12}$$

It follows from (3.7) that,

$$\lim_{n\to\infty} \|x_n - T_1^n y_n\| = 0. \tag{3.13}$$

Consequently,

$$\begin{aligned} d &= \liminf_{n\to\infty} \|x_n - q\| \\ &\le \liminf_{n\to\infty} \|x_n - T_1^n y_n\| + \|T_1^n y_n - q\| \\ &= \liminf_{n\to\infty} \|T_1^n y_n - q\| \\ &\le \liminf_{n\to\infty} k_n \|y_n - q\| \\ &= \liminf_{n\to\infty} \|y_n - q\| \\ &\le \limsup_{n\to\infty} \|y_n - q\| \le d. \end{aligned}$$

Hence,

$$d = \lim_{n\to\infty} \|y_n - q\| = \lim_{n\to\infty} \|a_n^{(2)}(T_2^n z_n - q + c_n^{(2)}(u_n^{(2)} - S_2 x_n)) + (1 - a_n^{(2)})(S_2 x_n - q + c_n^{(2)}(u_n^{(2)} - S_2 x_n))\|.$$

From S_2 is nonexpansive, T_2 is asymptotically nonexpansive, (3.9), and (3.10), we have

$$\limsup_{n\to\infty} \|S_2x_n - q + c_n^{(2)}(u_n^{(2)} - S_2x_n)\| \le d,$$

and

$$\limsup_{n\to\infty} \|T_2^n z_n - q + c_n^{(2)}(u_n^{(2)} - S_2x_n)\| \le d.$$

Applying Lemma 2.2, we have

$$\lim_{n\to\infty} \|S_2x_n - T_2^n z_n\| = \lim_{n\to\infty} \|(S_2x_n - q + c_n^{(2)}(u_n^{(2)} - S_2x_n)) - (T_2^n z_n - q + c_n^{(2)}(u_n^{(2)} - S_2x_n))\| = 0. \tag{3.14}$$

Again, it follows from (3.7) that

$$\lim_{n\to\infty} \|x_n - T_2^n z_n\| = 0. \tag{3.15}$$

Consequently,

$$\begin{aligned} d &= \liminf_{n\to\infty} \|x_n - q\| \\ &\le \liminf_{n\to\infty} \|x_n - T_2^n z_n\| + \|T_2^n z_n - q\| \\ &= \liminf_{n\to\infty} \|T_2^n z_n - q\| \\ &\le \liminf_{n\to\infty} k_n\|z_n - q\| \\ &= \liminf_{n\to\infty} \|z_n - q\| \\ &\le \limsup_{n\to\infty} \|z_n - q\| \le d. \end{aligned}$$

Hence,

$$d = \lim_{n\to\infty} \|z_n - q\| = \lim_{n\to\infty} \|a_n^{(3)}(T_3^n y_n - q + c_n^{(3)}(u_n^{(3)} - S_3x_n)) + (1 - a_n^{(3)})(S_3x_n - q + c_n^{(3)}(u_n^{(3)} - S_3x_n))\|.$$

As before, from S_3 is nonexpansive, T_3 is asymptotically nonexpansive, and (3.9), we have

$$\limsup_{n\to\infty} \|S_3x_n - q + c_n^{(3)}(u_n^{(3)} - S_3x_n)\| \le d, \tag{3.16}$$

and

$$\limsup_{n\to\infty} \|T_1^n y_n - q + c_n^{(1)}(u_n^{(1)} - S_1x_n)\| \le d. \tag{3.17}$$

Using Lemma 2.2, we have

$$\lim_{n\to\infty} \|S_3x_n - T_3^n x_n\| = \lim_{n\to\infty} \|(S_3x_n - q + c_n^{(3)}(u_n^{(3)} - S_3x_n)) - (T_3^n y_n - q + c_n^{(3)}(u_n^{(3)} - S_3x_n))\| = 0. \tag{3.18}$$

By (3.7), it follows that

$$\lim_{n\to\infty} \|x_n - T_3^n x_n\| = 0. \tag{3.19}$$

Therefore, by (3.12), (3.13), (3.14), (3.15), (3.18), and (3.19), we have

$$\lim_{n\to\infty} \|x_n - S_i x_n\| = 0 \quad \text{for } i = 1, 2, 3. \tag{3.20}$$

Using (3.15), (3.19), (3.20), and

$$\begin{aligned}
\|x_n - T_2^n x_n\| &\le \|x_n - T_2^n z_n\| + \|T_2^n z_n - T_2^n x_n\| \\
&\le \|x_n - T_2^n z_n\| + k_n \|z_n - x_n\| \\
&\le \|x_n - T_2^n z_n\| + k_n (c_n \|T_3^n x_n - x_n\| + (1 - c_n)\|S_3 x_n - x_n\|),
\end{aligned}$$

we have

$$\lim_{n\to\infty} \|x_n - T_2^n x_n\| = 0. \tag{3.21}$$

Next, using (3.13), (3.15), (3.20), and

$$\begin{aligned}
\|x_n - T_1^n x_n\| &\le \|x_n - T_1^n y_n\| + \|T_1^n y_n - T_1^n x_n\| \\
&\le \|x_n - T_1^n y_n\| + k_n \|y_n - x_n\| \\
&\le \|x_n - T_1^n y_n\| + k_n (b_n \|T_2^n z_n - x_n\| + (1 - b_n)\|S_2 x_n - x_n\|),
\end{aligned}$$

we get

$$\lim_{n\to\infty} \|x_n - T_1^n x_n\| = 0. \tag{3.22}$$

From (3.19), (3.21), and (3.22), we have

$$\lim_{n\to\infty} \|x_n - T_i^n x_n\| = 0, \quad \text{for } i = 1, 2, 3. \tag{3.23}$$

Next, we consider

$$\begin{aligned}
\|x_n - x_{n+1}\| &\le a_n \|x_n - T_1^n y_n\| + (1 - a_n)\|x_n - S_1 x_n\| \\
&\le \|x_n - T_1^n y_n\| + \|x_n - S_1 x_n\| \to 0,
\end{aligned} \tag{3.24}$$

and hence

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{3.25}$$

It follows from (3.23) and (3.25) that

$$\begin{aligned}
\|x_{n+1} - T_i x_{n+1}\| \le& \|x_{n+1} - T_i^{n+1} x_{n+1}\| + \|T_i^{n+1} x_{n+1} - T_i x_{n+1}\| \\
\le& \|x_{n+1} - T_i^{n+1} x_{n+1}\| + k_1 \|T_i^n x_{n+1} - x_{n+1}\| \\
\le& \|x_{n+1} - T_i^{n+1} x_{n+1}\| + k_1 (\|T_i^n x_{n+1} - T_i^n x_n\| \\
&+ \|T_i^n x_n - x_n\| + \|x_n - x_{n+1}\|) \\
\le& \|x_{n+1} - T_i^{n+1} x_{n+1}\| + k_1 (1 + k_n)\|x_n - x_{n+1}\| \\
&+ k_1 \|x_n - T_i^n x_n\|
\end{aligned}$$

for $i = 1, 2, 3$. This implies that

$$\lim_{n\to\infty} \|x_n - T_i x_n\| = 0, \quad \text{for } i = 1, 2, 3.$$

□

We are ready to establish weak and strong convergence theorems of our iteration.

Theorem 3.3. *Let E be a uniformly convex Banach space satisfying Opial's condition and K be a nonempty closed convex subset of E. Let $S_1, S_2, S_3 : K \to K$ be nonexpansive mappings, $T_1, T_2, T_3 : K \to K$ be asymptotically nonexpansive mappings with a sequence $\{k_n\}$ given by in Proposition 2.4 and $\cap_{i=1}^3 F(S_i) \cap F(T_i) \neq \varnothing$. If the conditions (3.2), (3.3), (3.7) and (3.8) are satisfied, then the three-step iteration sequence $\{x_n\}$ defined by (3.1) converges weakly to a common fixed point of $S_1, S_2, S_3, T_1, T_2,$ and T_3.*

Proof. It follows from Lemma 3.1 that $\{x_n\}$ is bounded. Hence $\{x_n\}$ has a subsequence $\{x_{n_j}\}$ which converges weakly to p. Since $\{x_{n_j}\} \subset K$ and K is weakly closed, $p \in K$. From Lemmas 3.2 and 2.1, we deduce that all the mappings $I - T_i$ and $I - S_i$ are demiclosed at zero. Hence $(I - T_i)p = (I - S_i)p = 0$ for all $i = 1, 2, 3$. That is, $p \in \cap_{i=1}^3 F(S_i) \cap F(T_i)$. Suppose that $\{x_n\}$ does not converge weakly to p. Then there exists another subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to some $q \neq p$. Arguing as above, we have $q \in \cap_{i=1}^3 F(S_i) \cap F(T_i)$. By Lemma 3.1, we have the limits $a := \lim_{n\to\infty} \|x_n - p\|$ and $b := \lim_{n\to\infty} \|x_n - q\|$ exist. Because E satisfies the Opial's condition, so

$$\begin{aligned} a &= \liminf_{j\to\infty} \|x_{n_j} - p\| < \liminf_{j\to\infty} \|x_{n_j} - q\| = b \\ &= \liminf_{k\to\infty} \|x_{n_k} - q\| < \liminf_{k\to\infty} \|x_{n_k} - p\| = a, \end{aligned}$$

which is a contradiction. Hence, $\{x_n\}$ converges weakly to $p \in \cap_{i=1}^3 F(S_i) \cap F(T_i)$. □

Theorem 3.4. *Let E be a uniformly convex Banach space and K be a nonempty closed convex subset of E. Let $S_1, S_2, S_3 : K \to K$ be nonexpansive mappings, $T_1, T_2, T_3 : K \to K$ be asymptotically nonexpansive mappings with a sequence $\{k_n\}$ given by in Proposition 2.4 and $\cap_{i=1}^3 F(S_i) \cap F(T_i) \neq \varnothing$. Suppose that the conditions (3.2), (3.3), (3.7) and (3.8) are satisfied. If one of mappings $T_1, T_2,$ and T_3 is semi-compact, then the three-step iteration sequence $\{x_n\}$ defined by (3.1) converges strongly to a common fixed point of $S_1, S_2, S_3, T_1, T_2,$ and T_3.*

Proof. It follows from Lemma 3.2, we have $\lim_{n\to\infty} \|T_i x_n - x_n\| = 0$ for all $i = 1, 2, 3$. Since one of mappings $T_1, T_2,$ and T_3 is semi-compact, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \to q \in K$ as $k \to \infty$. By the continuity of all the mappings S_i and T_i and Lemma 3.2, we conclude that

$$\|T_i q - q\| = \lim_{k\to\infty} \|T_i x_{n_k} - x_{n_k}\| = 0,$$

and

$$\|S_i q - q\| = \lim_{k\to\infty} \|S_i x_{n_k} - x_{n_k}\| = 0,$$

for all $i = 1, 2, 3$. That is, $q \in \cap_{i=1}^{3} F(S_i) \cap F(T_i)$. It follows from Lemma 3.1 that $\lim_{n\to\infty} \|x_n - q\| = 0$ and this completes the proof. $\square$

Using the same techniques as Theorems 3.3 and 3.4, we have the following

Theorem 3.5. *Let K be a nonempty closed convex subset of a uniformly convex Banach space E. Let $S_1, S_2, \dots, S_N : K \to K$ be nonexpansive mappings, $T_1, T_2, \dots, T_N : K \to K$ be asymptotically nonexpansive mappings with a sequence $\{k_n\}$ given by in Proposition 2.4 and $\cap_{i=1}^{N} F(S_i) \cap F(T_i) \neq \varnothing$. Suppose that*

$$\sum_{n=1}^{\infty} (k_n - 1) < \infty,$$

$$\sum_{n=1}^{\infty} c_n^{(i)} < \infty,$$

and

$$\|x - T_i y\| \le \|S_i x - T_i y\| \quad \text{for all } x, y \in K \text{ and } i = 1, 2, \dots, N.$$

Let $\{x_n\}$ be the N-step iteration sequence defined by (2.1) such that there is $\delta > 0$ such that

$$\delta \le a_n^{(i)} \le 1 - \delta, \quad \text{for all } n \ge 1 \text{ and } i = 1, 2, \dots, N.$$

(i) If X has the Opial's condition, then $\{x_n\}$ converges weakly to a common fixed point of $S_1, S_2, \dots, S_N, T_1, T_2, \dots, T_N$.

(ii) If one of the mappings $T_1, T_2, \dots, T_N$ is semi-compact, then $\{x_n\}$ converges strongly to a common fixed point of $S_1, S_2, \dots, S_N, T_1, T_2, \dots, T_N$.

Since there is no further generality obtained in using the scheme with error terms rather than the one considered in this paper, it follows from letting

$$S_1 = S_2 = \cdots = S_N = \text{ the identity mapping}$$

that Theorem 3.5 extends the corresponding results in [6, 8, 9].

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Satit Saejung
Department of Mathematics,
Khon Kaen University, THAILAND.
e-mail : saejung@kku.ac.th

Kittipong Sitthikul
Department of Mathematics,
Khon Kaen University, THAILAND.
e-mail : sittikul_n@hotmail.com