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Admitted Lie Group of the Pion Meson Motion Equation

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Abstract: In this article it is found the admitted Lie group of the dynamics equation of pion meson motion. This is the first and necessary step in application of group analysis method to partial differential equation.

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1 Introduction

The pion meson equation has been playing an important role in nuclear and particle physics over decades: it is believed that the self-interaction of pion meson particles may be modelled by the pion meson dynamics equation. This paper is concerned with the methodology for finding exact solutions of partial differential equations by using group analysis method. The group analysis method, described in Ovsiannikov (1978), is used to derive the admitted Lie group of the pion meson equation. Many applications of group analysis one can find in the Handbook of Lie group analysis edited by Ibragimov (1994), (1995), (1996).

2 Pion Meson Equation

The equation describing a motion of a pion meson particle in atom is the following equation[1]

$$\Box u + m^2 u + \lambda u^3 = 0, \tag{1}$$

where u is a function of x, y, z, t, m is the mass of pion meson,

$$\Box \equiv \frac{\partial^2}{\partial t^2} - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right).$$

[1] See, for example, [1, 2, 3]

The cubic term in (1) describes the pion self-interaction with the effective coupling constant λ. For the convenience of the calculations we rewrite the dynamics equation (1) as the following

$$F \equiv u_{tt} - (u_{xx} + u_{yy} + u_{zz} + au + bu^3) = 0, \tag{2}$$

where $a = -m^2, \quad b = -\lambda$. Equation (2) is studied further.

3 Determining Equations

The first step in finding the admitted Lie group of equations (2) is a construction of determining equations [4]. The determining equations are linear partial differential equations for the coefficients of the infinitesimal generator

$$X = \xi^x \partial_x + \xi^y \partial_y + \xi^z \partial_z + \xi^t \partial_t + \zeta \partial_u. \tag{3}$$

Here $\xi^x, \xi^y, \xi^z, \xi^t, \zeta$ are dependent functions of x, y, z, t, u. The prolongation (4) of the generator (3)

$$\underset{2}{X} = X + \zeta_i^\alpha \partial_{u_i^\alpha} \ + \zeta_{i_1 i_2}^\alpha \partial_{u_{i_1 i_2}^\alpha} \ , (\alpha = 1, ..., m) \tag{4}$$

is given by the formulas:

$$\begin{aligned} \zeta_i^\alpha &= D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j), (i = 1, ..., n), \\ \zeta_{i_1 i_2}^\alpha &= D_{i_2}(\zeta_{i_1}^\alpha) - u_{j i_1}^\alpha D_{i_2}(\xi^j), (i_1, i_2 = 1, ..., n), \\ D_i &= \partial_{x_i} + \sum_\alpha u_i^\alpha \partial_{u^\alpha} + \sum_{\alpha,\beta} u_{i\beta}^\alpha \partial_{u_\beta^\alpha} + ..., \end{aligned} \tag{5}$$

Here we used the notations $x_1 = x, x_2 = y, x_3 = z, x_4 = t$, and for the derivatives

$$u_i^\alpha = \frac{\partial u^\alpha}{\partial x_i}, u_{ij}^\alpha = \frac{\partial^2 u^\alpha}{\partial x_i \partial x_j}, \dots .$$

A collection of all derivatives of the k-th order is denoted by $u_{(k)} = \left\{u_{j_1 \dots j_k}^\alpha\right\}$.

A one parameter Lie group is called admitted by partial differential equation (2) if

$$\underset{(2)}{X} F(x, u, u_{(1)}, u_{(2)})_{|(F)} = 0, \tag{6}$$

where the sign $_{|(F)}$ means that equation (6) is considered on the manifold defined by equations (2). The process of obtaining equations (6) consists of the following steps. The first step is to get the second prolongation of the generator X. The second step is acting of the second prolongation $\underset{2}{X}$ on the equation $F = 0$. The next step is a transition onto the manifold, defined by (2). The obtained equation is called a determining equation. The generators, which coefficients satisfy this equation compose a Lie algebra of admitted generators. The Lie group corresponding to this Lie algebra is called an admitted Lie group. For all complicate calculations we use the symbolic manipulation program REDUCE [8].

4 Solving the Determining Equation

The dermining equation (6) can be split with respect to parametric derivatives: all derivatives of first and second order, except the derivative u_{tt}. After splitting it, one gets the overdetermined system of equations for the coefficients of the generators

$$\begin{gathered}\zeta^u_{uu}=0,\ \xi^t_u=0,\ \xi^x_u=0,\ \xi^y_u=0,\ \xi^z_u=0,\\ \xi^x_x=\xi^y_y=\xi^z_z=\xi^t_t,\\ \xi^t_x=\xi^x_t,\quad \xi^t_y=\xi^y_t,\quad \xi^t_z=\xi^z_t,\\ \xi^y_x=-\xi^x_y,\quad \xi^z_x=-\xi^x_z,\quad \xi^z_y=-\xi^y_z,\\ \xi^y_{zz}=-\xi^t_{ty},\ \xi^t_{xx}+\xi^t_{yy}+\xi^t_{zz}-3\xi^t_{tt}=0,\\ \xi^t_{ttt}-a\xi^t_t=0,\ \xi^x_{zz}+\xi^x_{yy}+2\xi^t_{tx}=0.\end{gathered} \tag{7}$$

Solving these equations one obtains the coefficients of the infinitesimal generators admitted by equation (2).

Note that equations (7) mean that the coefficients ξ^t,ξ^x,ξ^y,ξ^z do not depend on the variable u and the coefficient ζ^u is linear with respect to u: $\zeta^u = uh_1(x,y,z,t)+h_2(x,y,z,t)$.

Let us start analysis of the remained equations from the equations

$$\xi^t_x=\xi^x_t,\ \xi^x_x=\xi^t_t, \tag{8}$$

$$\xi^t_{xx}+\xi^t_{yy}+\xi^t_{zz}-3\xi^t_{tt}=0, \tag{9}$$

$$\xi^x_{zz}+\xi^x_{yy}+2\xi^t_{tx}=0. \tag{10}$$

The general solution of the first equation in (8) is $\xi^x=\varphi_x,\ \xi^t=\varphi_t$ with some function $\varphi=\varphi(x,y,z,t)$. Note that the equations $\xi^y_x+\xi^x_y=0$ and $\xi^z_x+\xi^x_z=0$ give

$$\xi^y=-\varphi_y+\widetilde{\xi^y}(t,y,z),\ \xi^z=-\varphi_z+\widetilde{\xi^z}(t,y,z).$$

After substituting ξ^x and ξ^t into equations (8), (9) and (10), one has

$$\varphi_{xx}=\varphi_{tt}, \tag{11}$$

$$\varphi_{tyy}+\varphi_{tzz}-2\varphi_{txx}=0, \tag{12}$$

$$\varphi_{xzz}+\varphi_{xyy}+2\varphi_{txx}=0. \tag{13}$$

The general solution of the wave equation (11) (D'Alembert solution) is

$$\varphi=H(t-x,y,z)+\psi(t+x,y,z)=0, \tag{14}$$

where y and z are considered as parameters. Substituting (14) into (12) and (13) gives

$$H_{122}+\psi_{122}+H_{133}+\psi_{133}-2H_{111}-2\psi_{111}=0, \tag{15}$$

$$-H_{133}+\psi_{133}-H_{122}+\psi_{122}-2H_{111}+2\psi_{111}=0. \tag{16}$$

Here the numbers 1, 2 or 3 mean the partial derivative of the functions H and ψ with respect to the first, second and third independent variables, respectively.

Taking linear combinations of equations (15) and (16) one can rewrite them as follows

$$\psi_{122} + \psi_{133} - 2H_{111} = 0, \tag{17}$$

$$H_{122} + H_{133} - 2\psi_{111} = 0. \tag{18}$$

Since the function H depends on $t - x$ and the function ψ depends on $t + x$, from the last equations one obtains

$$H_{111} = g(y, z),\ \psi_{122} + \psi_{133} = 2g(y, z), \tag{19}$$

$$\psi_{111} = h(y, z),\ H_{122} + H_{133} = 2h(y, z). \tag{20}$$

Here g and h only depend on y and z.

After integrating the first equations in (19) and (20) with respect to the first argument, one obtains

$$\begin{array}{l} H = \frac{(t-x)^3}{6} g(y, z) + \frac{(t-x)^2}{2} \alpha_1(y, z) + (t - x)\beta_1(y, z) + \gamma(y, z), \\ \psi = \frac{(t+x)^3}{6} h(y, z) + \frac{(t+x)^2}{2} \alpha_2(y, z) + (t + x)\beta_2(y, z), \end{array} \tag{21}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1$ and γ_2 are some functions obtained after integrating. Substituting the expressions of the functions H, ψ (21) into the remained equations of (19), (20), one has

$$\frac{(t + x)^2}{2}(h_{yy} + h_{zz}) + (t + x)(\alpha_{2yy} + \alpha_{2zz}) + \beta_{2yy} + \beta_{2zz} - 2g = 0, \tag{22}$$

$$\frac{(t - x)^2}{2}(g_{yy} + g_{zz}) + (t - x)(\alpha_{1yy} + \alpha_{1zz}) + \beta_{1yy} + \beta_{1zz} - 2h = 0. \tag{23}$$

The left hand sides of (22), (23) are polynomials with respect to t. Splitting these equations with respect to t, one obtains the following equations

$$\begin{array}{c} g_{yy} + g_{zz} = 0,\ h_{yy} + h_{zz} = 0, \\ \alpha_{1yy} + \alpha_{1zz} = 0,\ \alpha_{2yy} + \alpha_{2zz} = 0, \\ \beta_{2yy} + \beta_{2zz} = 2g,\ \beta_{1yy} + \beta_{1zz} = 2h. \end{array} \tag{24}$$

Analysis of the remained equations in (7) is done on computer by using REDUCE [8]. The methology for this is: substituting the representations of $\xi^x, \xi^y, \xi^z, \xi^t$ and splitting them with respect to x (and later with respect to t). For example, from the equations $\xi^y_t - \xi^t_y = 0$ and $\xi^z_t - \xi^t_z = 0$ one obtains

$$\begin{array}{c} g_y = g_z = h_y = h_z = 0,\ \alpha_2 = \alpha_1 + k, \\ \widetilde{\xi^y} = 4\alpha_{1_y} t^2 + 2t(\beta_{1_y} + \beta_{2_y}) + \psi^y, \\ \widetilde{\xi^z} = 4\alpha_{1_z} t^2 + 2t(\beta_{1_z} + \beta_{2_z}) + \psi^z. \end{array}$$

where $\psi^y = \psi^y(y, z), \psi^z = \psi^z(y, z)$ are some functions and k is constant. Further substitutions into the equations $\xi^y_y = \xi^z_z = \xi^t_t$ and $\xi^z_y + \xi^y_z = 0$ leads us to

$$\begin{array}{c} \alpha_{1_{zz}} = \alpha_{1_{yy}} = \alpha_{1_{yz}} = 0, \\ \beta_{1_{yy}} = h,\ \beta_{2_{yy}} = g,\ \beta_{1_{yz}} = \beta_{2_{yz}} = 0. \end{array}$$

The last equations together with (24) can easily be integrated for the functions $\alpha_1, \alpha_2, \beta_1$ and β_2. Then these solutions have to be substituted in the remained equations (7). The result of all calculations is the following.

The kernel of admitted Lie groups corresponds to the generators

$$\begin{gathered} X_1 = \partial_x, \quad X_2 = \partial_y, \quad X_3 = \partial_z, \quad X_4 = t\partial_x + x\partial_t, \\ X_5 = t\partial_y + y\partial_t, \quad X_6 = t\partial_z + z\partial_t, \quad X_7 = y\partial_z - z\partial_y, \\ X_8 = z\partial_x - x\partial_z, \quad X_9 = y\partial_x - x\partial_y, X_{10} = \partial_t. \end{gathered}$$

An extension of the kernel can be for $a = 0$:

$$\begin{gathered} X_{11} = t\partial_t + x\partial_x + y\partial_y + z\partial_z - u\partial_u, \\ X_{12} = 2tx\partial_t + (t^2 + x^2 - y^2 - z^2)\partial_x + 2xy\partial_y + 2xz\partial_z - 2xu\partial_u, \\ X_{13} = 2ty\partial_t + 2xy\partial_x + (t^2 - x^2 + y^2 - z^2)\partial_y + 2yz\partial_z - 2yu\partial_u, \\ X_{14} = 2tz\partial_t + 2xz\partial_x + 2yz\partial_y + (t^2 - x^2 - y^2 + z^2)\partial_z - 2zu\partial_u, \\ X_{15} = (t^2 + x^2 + y^2 + z^2)\partial_t + 2tx\partial_x + 2ty\partial_y + 2tz\partial_z - 2tu\partial_u \end{gathered}$$

The table of commutators $[X_i, X_j]$ is

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0	0	0	10	0	0	0	-3	-2	0	1	$2*11$	$2*9$	$2*8$	$2*4$
2		0	0	0	10	0	3	0	1	0	2	$-2*9$	$2*11$	$-2*7$	$2*5$
3			0	0	0	10	-2	1	0	0	3	$-2*8$	$2*7$	$2*11$	$2*6$
4				0	-9	-8	0	-6	-5	-1	0	15	0	0	12
5					0	7	6	0	4	-2	0	0	15	0	13
6						0	-5	4	0	-3	0	0	0	15	14
7							0	9	-8	0	0	0	-14	13	0
8								0	7	0	0	14	0	-12	0
9									0	0	0	13	-12	0	0
10										0	10	$2*4$	$2*5$	$2*6$	$2*11$
11											0	12	13	14	15
12												0	0	0	0
13													0	0	0
14														0	0

Here for convenience instead of the generator X_i it is written its integer number i, and also it is so for right hand side number if two numbers are separated by the sign $*$.

5 One-Dimensional Case

For the one-dimensional case (the case of the two independent variables (x, t)) the kernel of the admitted groups corresponds to the three generators

$$Y_1 = \partial_x, \; Y_2 = \partial_t, \; Y_3 = t\partial_x + x\partial_t.$$

For this algebra it is easily to construct an optimal system [**?**] of admitted subalgebras:

$$\{Y_1, Y_2, Y_3\}, \{Y_1, Y_2\}, \{Y_1, Y_3\}, \{Y_2 + DY_1\}, \{Y_1\}, \{Y_3\},$$

where D is an arbitrary constant. Invariant solutions can only be constructed for the one-dimensional subalgebras.

The representation of invariant solution for the subalgebra $\{Y_2 + DY_1\}$ is a travelling wave

$$u = u(x - Dt).$$

In this case the dynamics equation (1) is reduced to the following ordinary differential equation

$$(D^2 - 1)u'' = au + bu^3.$$

The representation of invariant solution for the subalgebra $\{Y_1\}$ is

$$u = u(t).$$

In this case the dynamics equation is reduced to the equation

$$u'' = au + bu^3.$$

The representation of invariant solution for the subalgebra $\{Y_3\}$ is

$$u = u(\xi),\ \xi = x^2 - t^2.$$

In this case the dynamics equation is reduced to the equation

$$\xi u'' + u' = -(au + bu^3)/4.$$

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