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Options portfolio optimization of exotic options written on Mini S&P500 Index in an illiquid market with Conditional Value-at-Risk (CVaR)

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Abstract This paper studies portfolio optimization under Conditional Value-at-Risk (CVaR) in the derivatives markets in which the quotes come with bid and ask prices, as well as quantity constraints. Assuming that the distribution is known, the problem can be reduced to a linear programming problem using the method of Rockafellar and Uryasev (Journal of Risk 2, 3 (2000)). The expectation is approximated using the Gaussian Quadrature integral. To illustrate the technique, we computationally determine the optimal portfolios consisting of the standard put and call options written on the S&P500 Mini Index for various risk levels and modeling parameters. The index is modelled by the variance gamma (VG) process. The values of base-case parameters including VG parameters are equal to zero except for the standard deviation (σ) and the variance rate (ν). They are equal to 0.1206 and 0.0031, respectively. For this computation, the compounded interest rate is equal to zero. The market is assumed incomplete as the quotes come with bid and ask prices, and sizes. The results show the dependence of the optimized portfolio on the risk level and the investors probabilistic view.

MSC: 62P05; 90C05; 91G50

Keywords: Portfolio optimization; Conditional Value-at-Risk (CVaR); Derivatives markets; Gaussian Quadrature; Variance Gamma process

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1. Introduction

Portfolio optimization is concerned with the selection of investments, including financial assets, such as shares of a company, government bonds, derivatives (e.g., options and futures), etc. Since there are many assets, all investors attempt to invest their money in a variety of securities in order to minimize investment risks, while maximizing return on investment. Therefore, portfolio optimization is one of the most prevalent problems faced by various investors with different levels of capital, and is also one of the most difficult problems in the financial world [1].

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Furthermore, one of the most well-known methods for selecting a financial portfolio. is portfolio optimization. Markowitz [2] devised the primary method for solving the portfolio selection problem. The portfolio return is evaluated by the expected return of the portfolio, while the associated risk is measured by the variance of the portfolio return in the so-called mean-variance (MV) portfolio optimization model. Variance is one of the risk measures. However, it is not a coherent risk measure because the variance of a random variable is neither a sub-additive nor a sub-additive positive homogeneity [4, 5]. Therefore, variance has its weaknesses. Alternative risk measures such as Value-at-Risk (VaR) have been proposed to replace the variance. The risk that VaR considers is the possibility of an unfavorable event. VaR provides the possibility of the greatest loss with an investment time horizon and a given confidence level (e.g., 90%, 95%, and 99%).

VaR has proven to be a popular and an important tool for measuring risks. Many authors, however, have argued that VaR is a non convex function and lacks a sub-additive. This means, for any given portfolios X and Y, the VaR of the combinated portfolios X and Y is not less than the sum of the VaR of portfolio X and the VaR of portfolio Y. Therefore, VaR is not a coherent risk measure [3, 4]. Rockafellar and Uryasev [7] then suggested another alternative risk measure, that is, Conditional Value-at-Risk (CVaR). CVaR is the expected loss under the condition that it exceeds VaR. Therefore, VaR will never exceed CVaR. Although VaR has been widely used, CVaR has become a more popular risk measure since it is a convex function and the coherent risk measure [6, 10]. Furthermore, Rockafellar and Uryasev minimized the CVaR of portfolios with CVaR for stocks. For this paper, we apply this model to minimize the CVaR of the derivatives portfolio since derivative contracts have become popular and increasingly important as an investing strategy for maximizing returns, while reducing funding costs. Hence, we will focus on the optimization of portfolios for derivatives by using CVaR.

In this paper, we apply this approach to the options markets. We consider the standard call and put options, which are written on the S&P 500 Mini Index. The quotes come with bid and ask prices, as well as sizes. We essentially determine the optimal portfolio, the portfolio with given a required return and the smallest CVaR. We also investigate the changes in optimized portfolios, which are subject to various modeling parameters. We would like to know what parameters affect the CVaR value because we know that if the CVaR value is small, the riskiness level of the optimization portfolio also decreases in the same way.

The paper is organized as follows. Section 2 introduces the market of derivatives, and the description of the CVaR approach. Afterward, we will use the CVaR measure for optimizing the portfolio and for considering the minimization of the portfolio problem in the form of linear programming in solving for CVaR optimization. Next, we will explain the Variance-Gamma (VG) distribution for simulating the stock index. In section 3, we represent an optimal CVaR derivative investment portfolio and illustrate multiple results after changing the parameters. Finally, we present the discussions and conclusions in the last section.

2. THE PORTFOLIO OPTIMIZATION MODEL

2.1. THE MARKET

We used quotes for the S&P500 Mini Index options with the common maturity time T and payouts that only depend on the underlying value at maturity time T. In general, the payout on cash depends on the interest rate (r), while the options payout is determined by the value of the underlying (S_T) at maturity T and the strike price (K) of an option. The quotes were obtained from Bloomberg on 26th December 2020 at 2:55:00 PM, when the value of the S&P500 Mini

Index was 295.42 and the maturity time T is one month or $T = \frac{1}{12}$ years. The trading assets include a bank account and the standard call and put option whose strikes are from 260 to 330, and written on the S&P500 Mini Index, allowing us to use over 140 options. Therefore, the payoffs for holding units $x \in \mathbb{R}$ of an asset are shown in Table 1.

TABLE 1. The payoffs as functions of the number of units x holds.

Asset	Payoff as a function of the position x
Cash	$e^{rT}x$
Call option	$\max\{(S_T - K), 0\}x$
Put option	$\max\{(K - S_T), 0\}x$

The market is an incomplete market as the quotes come with bid and ask prices and sizes, where the bid and ask price are the best potential price for the buyers and sellers in the marketplace, respectively. The market quotes come with finite quantities. The given examples of quotes available are on the 26th of December 2020 at 2:55:00 PM for derivatives, which are demonstrated in Table 2.

TABLE 2. Examples of market quotes on 26th December 2020 at 2:55:00 PM for options.

Options	Strike (K)	Bid price	Bid size($\times 100$)	Ask price	Ask size ($\times 100$)
Call	294	8.85	50	9.04	128
Put	294	7.99	50	8.23	50
Call	295	8.24	50	8.42	128
Put	295	8.37	50	8.61	50
Call	296	7.65	50	7.82	128
Put	296	8.77	50	9.02	50

2.2. CONDITIONAL VALUE-AT-RISK (CVAR)

In this paper, we consider the European options exercised only at the maturity date. These options are traded at two times, the initial time $t = 0$ and the maturity time $t = T$. For n assets, at the initial date $t = 0$, we know that $S_0 = (S_0^i)_{i \in n} \in \mathbb{R}^n$ is a vector of the initial underlying prices at $t = 0$. Additionally, we suppose that the vector of the uncertain underlying prices $S_T = (S_T^i)_{i \in n}$ at maturity time T is a random vector on the probability space (Ω, F, P). Then, for each S_T^i there is an F- measurable function on Ω, which is mapped onto real number $(\mathbb{R})$. That is, $S_T^i \in L^0(\Omega, F, P)$ becomes a linear space of collection of random variables that are mapped onto real numbers [16].

As we mentioned above, we will define other variables such as the return $(R_i(S_T^i))$ for each asset i as $(R_i(S_T^i))_{i \in n}$ and portfolio return. Let's start with the random vector of returns $(R(S_T))$ collecting the component of $R_i(S_T^i) = \dfrac{S_T^i - S_0^i}{S_0^i}$ for $i = 1, 2, \cdots, n$. It is also a random vector on the same probability space (Ω, F, P) and an F- measurable function on Ω. The portfolio return for each asset is shown in Table 3.

Next, we will determine the loss corresponding to a decision vector x associated with a feasible set of portfolio $X \subseteq \mathbb{R}^n$ and the random vector S_T called $f(x, S_T)$. Because S_T is a

TABLE 3. The portfolio return $(R(S_T))$ for each asset

Assets	Position	Portfolio return of the position
Cash	-	$(e^{rT} - 1)/1$
Call	Long	$\max\{(S_T - K), 0\}/$Ask price
	Short	$\max\{(S_T - K), 0\}/$Bid price
Put	Long	$\max\{(K - S_T), 0\}/$Ask price
	Short	$\max\{(K - S_T), 0\}/$Bid price

random vector, the loss function is also a real-valued random variable. Therefore, we can describe $f(x, S_T) : L^0(\Omega, F, P) \to \mathbb{R}$. For instance, since the negative return of this portfolio is a loss for this portfolio [7, 9], then it can be defined by

$$f(x, S_T) = -x^T R(S_T) = -\left[x_1 R_1(S_T^1) + \cdots + x_n R_n(S_T^n)\right]. \tag{2.1}$$

Meanwhile, we have the loss function $f(x, S_T)$ and the random vector S_T, which are associated by a probability measure P and are independent of parameter x. Thus, for a fixed x and $\alpha \in \mathbb{R}$, we can define the probability function $\Psi(x, \alpha) \in \mathbb{R}$, which is shown in the equation below.

$$\Psi(x, \alpha) = P[f(x, S_T) \le \alpha].$$

Moreover, for a continuous function with x and a measurable function with S_T of the uncertain loss function $f(x, S_T)$, we have

$$\mathbb{E}[f(x, S_T)] < \infty, \text{for each } x \in X.$$

Afterwards, we can change it to an integration form based on Definition 2.1 from a stochastic problem.

Definition 2.1. For each x, the loss $f(x, S_T)$ is a random variable in $\mathbb{R}$ as S_T is a random variable and $p(S_T)$ denotes the probability density function for S_T in $\mathbb{R}$. The probability of the loss not exceeding a threshold α is determined by the cumulative distribution function [7],

$$\Psi(x, \alpha) = \int_{f(x,S_T) \le \alpha} p(S_T)\, dS_T. \tag{2.2}$$

Note that the region of integration $f(x, S_T) \le \alpha$ is the set $A = \{\, \omega \in \Omega \mid f(x, S_T) \le \alpha \,\}$.

Definition 2.2. The β-VaR value for the loss random variable associated with portfolio x and probability level β in $(0, 1)$ will be represented by $\alpha_\beta(x)$, which can be defined as [7]:

$$\alpha_\beta(x) = \min\{\, \alpha \in \mathbb{R} \mid \Psi(x, \alpha) \ge \beta \,\}. \tag{2.3}$$

Definition 2.3. The β-CVaR value for the loss random variable associated with x and probability level β in $(0, 1)$ will be represented by $\phi_\beta(x)$, which can be defined as [7]:

$$\phi_\beta(x) = \mathbb{E}[f(x, S_T) \mid f(x, S_T) \le \alpha_\beta(x)] \tag{2.4}$$

$$= \frac{\mathbb{E}[\mathbf{1}_{f(x,S_T) \le \alpha_\beta(x)} f(x, y)]}{P(f(x, S_T) \le \alpha_\beta(x))} \tag{2.5}$$

$$= (1 - \beta)^{-1} \int_{f(x,S_T) \ge \alpha_\beta(x)} f(x, S_T) p(S_T)\, dS_T. \tag{2.6}$$

Lemma 2.4. *(Rockafellar and Uryasev, [7]) The minimization of β-CVaR for the loss associated with x over all $x \in X$ is equivalent to minimizing $F_\beta(x,\alpha)$ over all $(x,\alpha) \in X \times \mathbb{R}$ as follows;*

$$\min_{x\in X} \phi_\beta(x) = \min_{(x,\alpha)\in X\times\mathbb{R}} F_\beta(x,\alpha), \tag{2.7}$$

where

$$F_\beta(x,\alpha) = \alpha + (1-\beta)^{-1} \int_{S_T\in\mathbb{R}^m} [f(x,S_T)-\alpha]^+ p(S_T)\, dS_T, \tag{2.8}$$

where

$$[f(x,S_T)-\alpha]^+ = \begin{cases} f(x,S_T)-\alpha, & f(x,S_T)-\alpha > 0 \\ 0, & f(x,S_T)-\alpha \le 0. \end{cases}$$

In Lemma 2.4, we use $F_\beta(x,\alpha)$ to optimize CVaR instead of the function $\phi_\beta(x)$. $F_\beta(x,\alpha)$ can be used to approximate CVaR or $\phi_\beta(x)$ because it is a convex function and from Lemma 2.4. Additionally, if X is a convex set, then the CVaR minimization problem

$$\min_{(x,\alpha)\in X\times\mathbb{R}} F_\beta(x,\alpha) \tag{2.9}$$

is a convex programming problem. Therefore, this function has the following principal properties that make it useful for the calculation of VaR and CVaR [12]:

(1) $F_\beta(x,\alpha)$ is a convex function of α.
(2) α_β is a minimum over α of $F_\beta(x,\alpha)$.
(3) ϕ_β is the minimum value over α of $F_\beta(x,\alpha)$.

Since the integral in Definition (2.4) of $F_\beta(x,\alpha)$ can be approximated in various ways, we use the gaussian legendre quadrature to approximate the integration term of Equation (2.8) for this paper.

Theorem 2.5. *The minimization of CVaR in derivatives market where the derivatives are written only on a single underlying can be written as:*

$$\begin{aligned} \min_{(x,\alpha)\in X\times\mathbb{R}} \tilde{F}_\beta(x,\alpha) = \min_{(x,\alpha)\in X\times\mathbb{R}} \; & \alpha \\ & + \frac{1}{(1-\beta)} \sum_{k=1}^{q} \left[-\sum_{i=1}^{n} [x_i R_i(S_T^k)] - \alpha \right]^+ p(S_T^k) w_k, \end{aligned} \tag{2.10}$$

where S_T^k is the solution of the gaussian polynomial on the interval $[0,c]$, where c is a large number, $p(S_T^k)$ is a probability density function of S_T^k and w_k is the corresponding weight of S_T^k that is defined as:

$$w_k = \frac{2}{(1-(S_T^k)^2)[P_q'(S_T^k)]^2},$$

for $k = 1,\cdots,q$ where q is the number of gaussian quadrature points and the related orthogonal polynomials are legendre polynomials, determined by $P_q(S_T)$. If we consider the q^{th} polynomial normalized given $P_q(1)=1$, the k^{th} gaussian node, S_T^k is the k^{th} root of P_q.

Lemma 2.4 determines the auxiliary function that is used to approximate the CVaR value, so we know that the term of integration in Equation (2.8) is a multiple integration, but for this paper we use only the underlying value (S_T) that the spot price is equal to 295.42. Then we will change the multiple integration to integration in one dimension. The auxiliary function that is used to approximate the CVaR value in Theorem 2.5 is defined as:

$$F_\beta(x,\alpha) = \alpha + (1-\beta)^{-1}\int_{S_T\in\mathbb{R}} [f(x,S_T)-\alpha]^+ p(S_T)\, dS_T, \tag{2.11}$$

where

$$[f(x,S_T)-\alpha]^+ = \begin{cases} f(x,S_T)-\alpha, & f(x,S_T)-\alpha > 0 \\ 0, & f(x,S_T)-\alpha \le 0. \end{cases}$$

Afterward, we reform the integral term in Equation (2.11) to the summation term which appear in the equation below.

$$\sum_{k=1}^{q} \left[f(x,S_T^k)-\alpha\right]^+ p(S_T^k) w_k,$$

where

$$f(x,S_T^k) = -x^T R(S_T^k) = -\left[\sum_{i=1}^{n}\left[x_i R_i(S_T^k)\right]\right].$$

Unlike Rockafellar and Uryasev, [7] who used the Monte Carlo method to simulate the value of the underlying at time T to estimate the expectation, our method uses the gaussian quadrature. This reduces the time to estimate the expectation and to solve the optimization problem greatly. As we know, it requires a large number of simulation to acquire accurate estimate for the expectation. This is even more true for derivative portfolios which need much more simulated paths. In addition to the great deal of time used for simulation, having many simulated paths makes the optimization bigger. This is because, to solve the optimization problem, we need to introduce dummy variables to transform the problem in Lemma 2.4 to a linear programming problem. The number of the dummy variables is equal to the number of simulation.

Another issue for using the simulation approach to estimate the expectation in the optimization problem is that the simulated values of the underlying at maturity time are not far enough from the spot. This means that some derivatives such as a call option whose strike is very high will never expire in the money. The optimal solution will consist of short selling that option as much as possible. However, the probability that call option will expire in the money is not zero.

The expectation is essentially an integral whose domain is a nonnegative real number. However, we can not evaluate such improper integral. Thus, we only estimate it on the domain where the underlying value at maturity is from zero to some large enough number.

2.3. Numerical Implementation

As you can see, the approximate function of the auxiliary function ($\tilde{F}_\beta(x,\alpha)$) in Equation (2.10) has an indicator function in terms of $[f(x,S_T)-\alpha]^+$ which is hard to minimize. We then operate this function by using a similar technique to Rockafellar and Uryasev, [7] to make it easy to simplify. Then our minimization portfolio now is as follows:

$$\min_{(x,\alpha)\in X\times\mathbb{R}} \alpha + \frac{1}{(1-\beta)}\sum_{i=1}^{q} u(S_T^i)p(S_T^i)w_i, \tag{2.12}$$

where

$$u(S_T^i) = \left[f(x,S_T^i)-\alpha\right]^+.$$

Subject to

$$
\begin{aligned}
u(S_T^i) &\geq 0, \\
u(S_T^i) + \left[x^T R(S_T^i) + \alpha\right] &\geq 0, \quad i = 1, 2, ..., n, \\
\sum x &= 1, \\
x^T \bar{R}(S_T) &= Q, \\
x, \alpha &\in \mathbb{R},
\end{aligned}
$$

where $\bar{R}(S_T)$ is the average return and Q is the required return.

Thus, we can consider the minimization portfolio problem (2.12) as a linear programming problem. We solve this problem using Linprog that is a built-in function in Matlab. Hence, we will explain Linprog's syntex to solve the linear programming problem.

To begin with, let $z^T = \left[x^1, x^2, \cdots, x^n, \alpha, u^1, u^2, \cdots, u^q\right]$, then the Linprog's syntex is as follows:

$$
\min f^T z, \tag{2.13}
$$

subject to

$$
\begin{aligned}
A \times z &\leq b, \\
Aeq \times z &= beq, \\
lb \leq z &\leq ub,
\end{aligned}
$$

where f, b, beq, lb, ub are vectors and A, Aeq are matrices.

The command for using the Linprog's syntex to solve the linear programming problem is $linprog(f, A, b, Aeq, beq, lb, ub)$. The solutions are the optimal portfolio z and the optimal value of $f^T z$.

After we change the minimization portfolio from the model (2.12) to the linear programming problem in the model (2.13), for the maturiy time T, our problem can be written as,

$$
\min f^T z, \tag{2.14}
$$

subject to

$$
\begin{aligned}
A \times z &\leq b, \\
Aeq \times z &= beq, \\
lb \leq z &\leq ub,
\end{aligned}
$$

where $f^T = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & \frac{1}{1-\beta} p(S_1) w_1 & \frac{1}{1-\beta} p(S_2) w_2 & \cdots & \frac{1}{1-\beta} p(S_q) w_q \end{bmatrix}$,

$$
A = -\begin{bmatrix} R(S_1^1) & R(S_1^2) & \cdots & R(S_1^n) & 1 & 1 & 0 & \cdots & 0 \\ R(S_2^1) & R(S_2^2) & \cdots & R(S_2^n) & 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ R(S_q^1) & R(S_q^2) & \cdots & R(S_q^n) & 1 & 0 & 0 & \cdots & 1 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},
$$

$$
Aeq = \begin{bmatrix} 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ \bar{R}(S^1) & \bar{R}(S^2) & \cdots & \bar{R}(S^q) & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, beq = \begin{bmatrix} 1 \\ Q \end{bmatrix},
$$

$R(S_i^j)$ is a return of the j^{th} asset of the i^{th} scenario, and $\bar{R}(S^j)$ is a mean return of the j^{th} asset.

2.4. VARIANCE-GAMMA (VG) DISTRIBUTION

Nowadays, there are many distributions that can be used to simulate stock market returns. For example, normal distribution, t distribution, and VG distribution. As we all know, the daily stock market returns are not normally distributed because the stock market return distributions seem to have tails that are much fatter than normal distribution, so it is more suitable to use it with other distributions that have fat tails. The authors suggested the VG distribution instead of normal distribution because VG is obtained by evaluating the Brownian motion with a constant drift at a gamma distributed time change [14, 15]. The parameters of VG provide control over the skewness and kurtosis of the return distribution.

We will define the options price of the Variance-Gamma (VG) distribution [15]. Let the option price be given by

$$S(t) = S(0)\exp(mt + X(t;\sigma_S,\nu_S,\theta_S) + \omega_S t), \tag{2.15}$$

where $\omega_S = \dfrac{1}{\nu_S}\ln(1-\theta_S\nu_S - \sigma_S^2\nu_S/2)$, m is average rate of return on the option, and the statistical parameters are denoted by the subscript S.

Afterwards, we change the rate of return from the average rate of return to the compound interest rate r. Then the risk neutral process be given by

$$S(t) = S(0)\exp(rt + X(t;\sigma_{RN},\nu_{RN},\theta_{RN}) + \omega_{RN} t), \tag{2.16}$$

where $\omega_{RN} = \dfrac{1}{\nu_{RN}}\ln(1-\theta_{RN}\nu_{RN} - \sigma_{RN}^2\nu_{RN}/2)$ and the risk neutral parameters are denoted by the subscript RN.

Theorem 2.6. *The density function for the price $z = S(t)$ at the exercise time (t) has a log-VG distribution dynamics of Equation (2.15) and is defined as [15]:*

$$f_{VG}(z) = \frac{2\exp(\theta x/\sigma^2)}{\nu^{(t/\nu)}\bar{2}\pi\sigma\Gamma(\frac{t}{\nu})}\left(\frac{x^2}{2\sigma^2/\nu+\theta^2}\right)^{\frac{t}{2\nu}-\frac{1}{4}} XK_{\frac{t}{\nu}-\frac{1}{2}}\frac{1}{\sigma^2}\sqrt{x^2(2\sigma^2/\nu+\theta^2)}, \tag{2.17}$$

where K is the modified bessel function of the second kind and

$$x = \ln(z) - \ln(S(0)) - mt - \frac{t}{\nu}\ln(1-\theta\nu-\sigma^2\nu/2).$$

As an example, the price of call option in case of long position, $c(S(0);K,T)$, for the strike K and maturity time T, is determined by

$$c(S(0);K,T) = e^{-rT}\mathbb{E}\left[\max(S_T-K,0)\right], \tag{2.18}$$

where r is the interest rate and this expectation is taken under the risk neutral process of Equation (2.16).

As explained above, we have a mathematical model that is used to minimize the CVaR for the options portfolio in Equation (2.12). Afterwards, we will change this situation of minimization to a numerical implement by adopting the model while solving a linear problem as shown in Equation (2.14). As you can see, the options prices S_T are random variables. Then, we will simulate the options prices by using VG distribution with only the underlying value. Lastly, we will minimize the CVaR value for this portfolio and discuss the results in the next section.

3. RESULTS

As the previous section, we have realized that the distribution that is suitable for simulating the vector of return needs to be a continuous distribution. In this paper, we assume that the log return of the S&P500 Mini Index is VG distributed. At the maturity time $T = \frac{1}{12}$ years, we use the gaussian quadrature with 500 points to estimate the integration shown in Equation (2.11). We also assume that the random vector of the underlying prices S_T does not depend on the decision vector x. Unless otherwise stated, the parameters used in computation are as in Table 4. They are computed using 10 years of historical data. We would like to optimize the portfolio, which has a minimum CVaR and a given expected return. In addition, the initial wealth that is used for the investment is $\$100,000$ and the percentage of required return is 400, which is a high value because the percentage of required return of options must be greater than the required return of shares for trading. Moreover, limited amount of buying and selling are allowed using bid and ask sizes and taking account of the bid and ask prices.

TABLE 4. Base-case parameters including VG parameters and the compounded interest rate(r).

μ	σ	ν	θ	r
0	0.1206	0.0031	0	0

TABLE 5. The expectation, the standard deviation, the VaR and CVaR of the CVaR minimization portfolio with any confidence levels

Confidence levels	Expectation ($)	Standard deviation ($)	VaR ($)	CVaR ($)
95	499,965.6332	213,504.6531	-233,244.3822	-111,693.2899
99	499,969.6131	350,159.1180	-30,841.4099	74,066.4235

From Table 5, if we compare the expectation and the standard deviation of the payoff for 95% and 99% confidence levels, we can see that the expectation and the standard deviation of the payoff with 99% confidence level are greater than the values at 95% confidence level. In addition, the 95% VaR and 99% VaR are -233,244.3822 and -30,841.4099, respectively. We know that the 95% VaR is less than 99% VaR and they are negative values, which means we will make a profit if we invest in assets as in Figure 3. In contrast, the 95% CVaR and 99% CVaR are different, since the 95% CVaR is a negative value, but the 99% CVaR is positive, then we can obtain a profit at the 95% confidence interval; however, for the 99% confidence interval, we found that we will lose money because of the positive value of the loss value or the CVaR value.

Moreover, Figure 1 demonstrates the histogram of the net payoff of the portfolio after 30 days with 95% confidence level. It shows that the cutoff point of this histogram is around $230,000, which is 95% VaR value, so the VaR cutoff point of this histogram is equal to 95% VaR value and the area after the VaR cutoff point is 95% of the histogram. Therefore, the average loss that exceed the VaR value in the left tail is the CVaR value for each confidence level. For example, the net payoff of the portfolio selected that exceed 95% VaR value in Figure 1 has many values. There are both positive and negative values, but the average of this area is positive since 95% CVaR value is equal to -111,693.2899. Similarly, Figure 2 demonstrates the histogram of the net payoff of the portfolio after 30 days, with 99% confidence level. The average of the right area that exceed VaR value is negative so that 99% CVaR is positive and equals to 74,066.4235. Therefore,

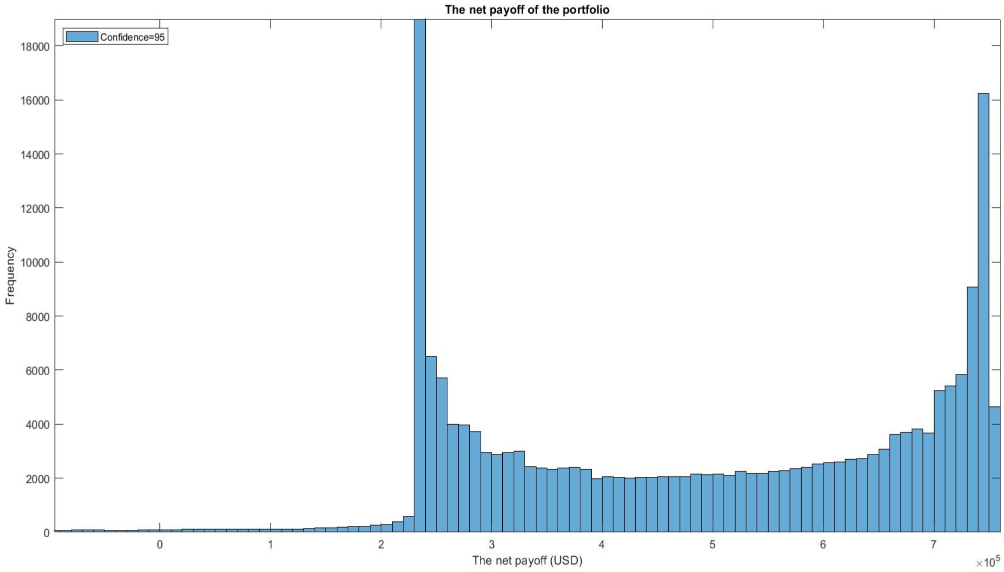

FIGURE 1. The histogram of net payoff of the optimal portfolio by the minimization of CVaR after 30 days with 95% confidence level obtained by using 300,000 out-of-sample simulated index value

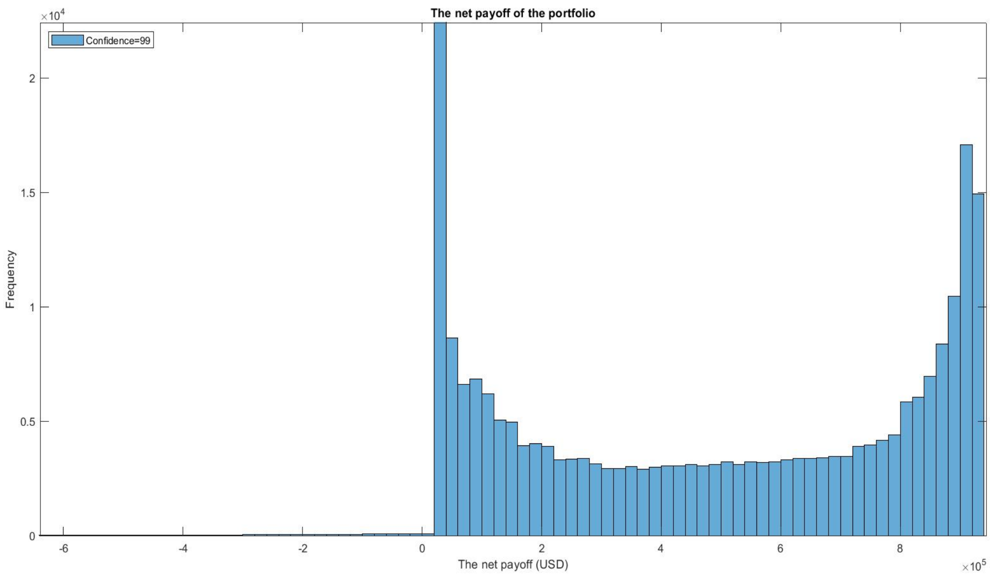

FIGURE 2. The histogram of net payoff of the optimal portfolio by the minimization of CVaR after 30 days with 99% confidence level obtained by using 300,000 out-of-sample simulated index value

we can confirm the theory that the 99%CVaR value must be greater than or equal to the 95%CVaR value.

To study the effect of optimal portfolio, we repeat the optimization and change the parameters (e.g., σ and ν) for simulating the index values by using the VG distribution. Table 6 and Table 7 exhibit the rise of σ and ν, followed similarly by the rise in value of CVaR. In addition, the

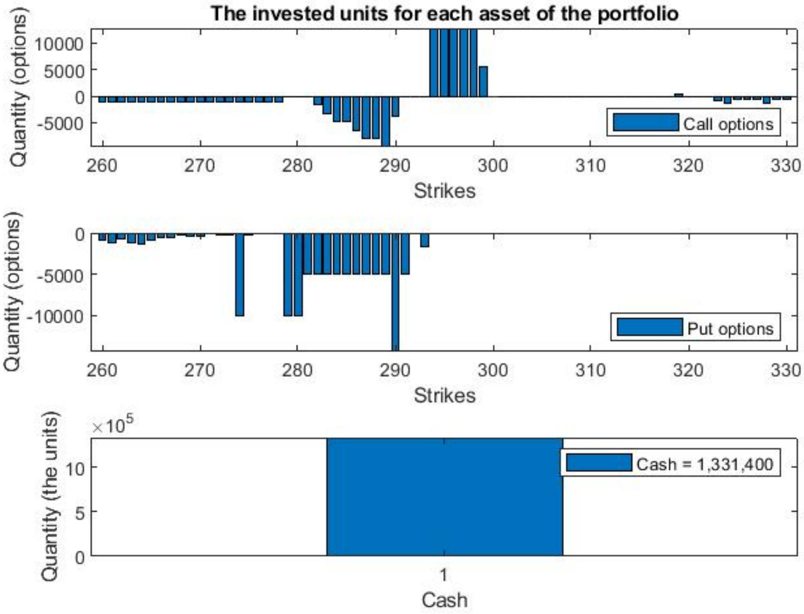

FIGURE 3. The proportion of cash and asset j from optimizing the portfolio selected after 30 days with 95% confidence level obtained by using 300,000 out-of-sample simulated index value

TABLE 6. The expectation, the standard deviation, the 95%VaR and 95%CVaR of the CVaR minimization portfolio after changing σ

σ	Expectation ($)	Standard deviation ($)	VaR ($)	CVaR ($)
0.01	500,000	0.012340856	-400,000	-399,999.9999
0.05	499,999.8871	3,185.094854	-399,783.0798	-398,670.0211
0.1206	499,965.6332	213,504.6531	-233,244.3822	-111,693.2899
0.2	499,133.7618	702,889.2016	346,956.6677	547,731.8037

TABLE 7. The expectation, the standard deviation, the 95%VaR and 95%CVaR of the CVaR minimization portfolio after changing ν

ν	Expectation ($)	Standard deviation ($)	VaR ($)	CVaR ($)
0.001	507,904.9454	218,008.4877	-234,286.3407	-118,346.7735
0.0031	499,965.6332	213,504.6531	-233,244.3822	-111693.2899
0.01	499,874.4617	219,069.9503	-232,618.4550	-87,185.3817
0.1	499,788.1843	276,296.0258	-140,107.6781	100,655.6834

effect from changing the standard deviation of the return on the stock (σ) is shown in Table 6. It has also been noticed that as the standard deviation increases, the CVaR value increases. Additionally, we have also observed that if $\sigma = 0.2000$, the portfolio can lose money because VaR and CVaR are positive numbers, which imply that the portfolio has a 95% chance of making

TABLE 8. The expectation, the standard deviation, the 95%VaR and 95%CVaR of the CVaR minimization portfolio with other percentages of the required return

Required return ($\times 100\%$)	Expectation (\$)	Standard deviation (\$)	VaR (\$)	CVaR (\$)
2	300,072.1339	72,463.0421	- 167,967.0981	- 117,323.8604
3	400,069.2447	145,477.4599	- 206,585.6946	- 122,445.0835
4	499,965.6332	213,504.6531	- 233,244.3822	- 111,693.2899
5	599,811.5596	285,164.2267	- 236,238.0140	- 86,660.8016
6	699,618.6478	348,141.1252	- 243,936.2769	- 51,196.4221
7	799,394.2615	418,366.1802	- 222,496.1260	- 6,400.5556
8	899,154.1257	486,551.4825	- 192,275.9600	46,715.9098
9	998,854.9645	552,089.7232	- 147,146.9564	112,810.1560
10	1,098,511.3814	595,452.9670	- 133,932.1352	196,000.8996

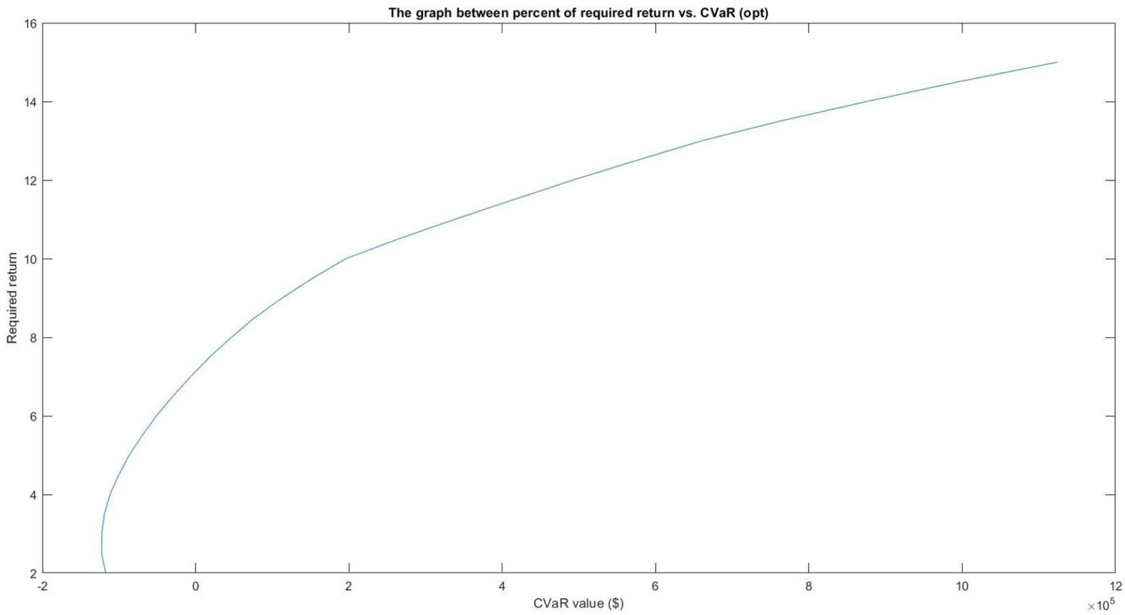

FIGURE 4. The graph of the CVaR value with other required returns

more than 346,956.6677, and the average amount of loss function that exceed the VaR value is 547,731.8037. In the same way, If ν increases, our CVaR value will increase, so ν slightly affects the CVaR value. Therefore, if the values of both parameters increase, our portfolios will be more vulnerable than the base-case in Table 5 because the CVaR values are greater than the 95% CVaR value of the base-case in Table 5. This means, the risky optimization portfolios grows higher if we compare them with the base-case.

In addition, as the required rate of return (Q) increases, the CVaR value also increases. From Table 8, it is observed that the required rate of return impacts all values such as the expectation, the standard deviation, the 95% VaR and the 95% CVaR. This effects the profit of the selected portfolio. Table 8 and Figure 4 show that we can make a higher profit if the required return is going down because the trend of the first period of this graph of the CVaR value with required returns is increasing rapidly in the range of the required return from 200% to 1,200%, and the trend of the whole graph is like a log curve. Therefore, the risky optimization portfolio rises up from the base-case at 95% confidence level in Table 5.

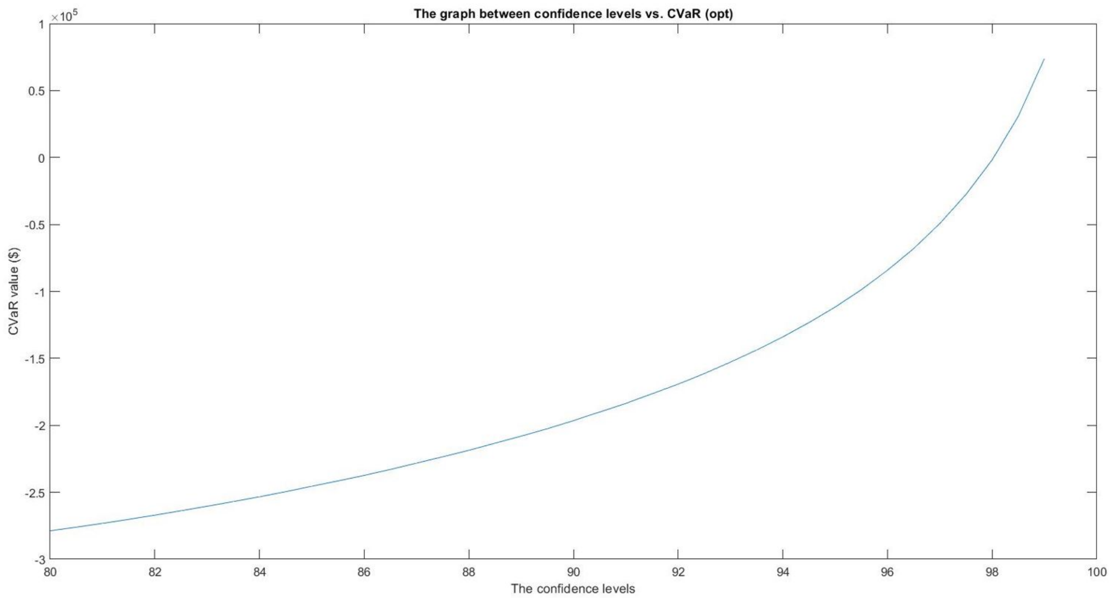

FIGURE 5. The graph of the CVaR value with other confidence levels

Moreover, we know that if we consider the optimization portfolio with CVaR at confidence levels 90%, 95%, and 99%, then all the values are very different. They are negative values, which means that this portfolio can earn a profit, except for when the value at 99% confidence level is positive. From Figure 5, we deduce that the expected return and the associated return risk increase as the confidence level β decreases. Furthermore, the trend of this graph is similar to an exponential curve. We notice that the CVaR value at 99% confidence level is the highest and approximately $75,000.

However, the linear programming problem is solved with the interior-point solver of MOSEK 9.2 that is suitable for convex optimization. Afterward, we set up MOSEK instead of the optimization problem in MATLAB. Then, we measure the CPU time used to run this minimization problem. The elapsed time is 2.8862 seconds on a PC with Intel(R) Core(TM) i7-6500U CPU @ 2.60 GHz processor and 8.00 GB memory. Therefore, the efficiency for the optimization problem using the VG distribution and the gaussian legendre quadrature is more efficient than when using the Monte Carlo techniques in the case of Rockafellar and Uryasev [7] as it took about 8 minutes with a Mathematica version of 400-800 iterations of the variable metric code on a 450 MHz Pentium II.

4. DISCUSSIONS AND CONCLUSIONS

In portfolio optimization, risk and return are the uncertain parameters; therefore, we additionally used CVaR as the risk measure to measure the estimated risk in expected return. We have illustrated this minimization problem on options portfolio, while we optimized the objective function as a continuously differentiable function in MATLAB. In particular, we have shown that, when the options values are computed through variance-gamma approximations and through the use of the gaussian legendre quadrature instead of the Monte Carlo techniques, the efficiency for solving this problem is better than the Monte Carlo techniques. Additionally, we know that our experiments are more diversified and the CVaR value depends on many parameters such as σ, ν, and Q. From Section 3, the results show that σ, ν, and Q can be more risky if their values increase. Moreover, the affect of the CVaR value from Figure 4 is shown as logarithmic growth, but for the confidence level β, the affect is shown as exponential growth in Figure 5. Lastly, the confidence levels reflect the situations of whether investors lose or make money.

In future research, we will apply the model for large data set, consider options portfolio hedging problems and investigate indifference in pricing of exotic options written on the index. Furthermore, we will simultaneously analyze the problem of minimizing both standard deviation and the CVaR as the results show that their value share similar trends.

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