Thai Journal of Mathematics
Special Issue (Annual Meeting in Mathematics, 2008) : 1-13

www.math.science.cmu.ac.th/thaijournal
Online ISSN 1686-0209

Stability and Robust Stability of Discrete-Time Switched Systems with Delays**

J. Thipcha and P. Niamsup*

Abstract : In this paper, we study stability and roust stability of the zero solution of discrete-time switched systems with delays. And we give sufficient conditions which guarantee that the zero solution of discrete-time switched systems with delays is asymptotically stable and robustly stable. Numerical simulations are also given to illustrate the results.

Keywords : Switched system, Asymptotically stable, Robustly stable, Lyapunov function, Multiple Delay.
2000 Mathematics Subject Classification : Applied Mathematics.

1 Introduction

An important class of hybrid dynamical systems is a class of switched systems with delays, which compose of a family of continuous-time or discrete-time subsystems and a rule orchestrating the switching between these subsystems. A discrete-time switched system can be described by a difference equation of the form

$$x_{k+1} = f_{i_k}(x_k, x_{k-h}), \qquad k \in \mathbb{Z}^+ = \{0, 1, 2, ...\}$$

where $h \geq 1$ is the state delay, $\{f_{i_k}(\cdot) : i_k \in \mathcal{M}\}$ is a family of functions from $\mathbb{R}^{2n}$ to $\mathbb{R}^n$ that is parameterized by the index set $\mathcal{M}$, and $i_k \in \mathcal{M}$ is a switching signal. The set $\mathcal{M}$ is typically a finite set. Some examples of such switched systems: Automobile with a manual gearbox,see [5]. The motion of a car that travels along a fixed path can be characterised by two continuous states: velocity v and position s. The system has two input: the throttle angle (u) and the engaged gear (g). It is evident that the manner in which the velocity of the car responds to the throttle input depends on the engaged gear. Recently, there have been many studies of

∗Corresponding author:
**This paper is a part of Jenjira's thesis from Chiang Mai University.

stability analysis in switched systems such as: In 2005, V.N. Phat [4] studied stability and robust stability of the system given by a difference equation of the form

$$x_{k+1} = [A_{i_k} + \Delta A_{i_k}(k)]x_k + [B_{i_k} + \Delta B_{i_k}(k)]x_{k-h} \tag{1.1}$$

where $k \in \mathbb{Z}^+, i_k \in \mathcal{M} = 1, 2, \ldots, N, h \geq 1, x_k \in \mathbb{R}^n, A_{i_k}, B_{i_k} \in \mathbb{R}^{n\times n}$ are given constant matrices, $\Delta A_{i_k}(k), \Delta B_{i_k}(k)$ are the parameter uncertainties matrices which are assumed to be of the form

$$\Delta A_{i_k}(k) = E_{i_k}F(k)F_{i_k}, \Delta B_{i_k}(k) = G_{i_k}F(k)H_{i_k}$$

where $E_{i_k}, F_{i_k}, G_{i_k}, H_{i_k}$ are given constant matrices of appropriate dimensions and $F(k)$ is the uncertain matrix such that $F(k)^T F(k) \leq I$. V.N. Phat gave new sufficient conditions for asymptotic stability and robust stability of the zero solution of system (1). In 2004, G. Xie and L. Wang[6] studied the asymptotic stability of the zero solution of an autonomous switched systems with constant delay. Moreover, in 2005, M. Yu, L. Wang and T. Chu [7] studied discrete-time system with time-varying delays and they gave sufficient condition for exponential stability.

2 Preliminaries

The following notations will be used throughout. $\mathbb{Z}^+$ denotes the set of all nonnegative integers. $\mathbb{R}^n$ denotes the n - finite-dimensional Euclidean space with the Euclidean norm $\|\cdot\|$. A matrix Q is positive definite, denoted by $Q > 0$ if $x^TQx > 0$ for any $x \neq 0$. A matrix Q is positive definite iff $\exists\beta > 0 : x^TQx \geq \beta\|x\|^2$. A matrix Q is negative definite, denoted by $Q < 0$ if $x^TQx < 0$ for any $x \neq 0$. A matrix Q is negative definite iff $\exists\beta > 0 : x^TQx \leq -\beta\|x\|^2$.

Consider system of difference equations of the form

$$x_{k+1} = f(k, x_k), \qquad x(k_0) = x_0 \tag{2.1}$$

where $k \in \mathbb{Z}^+ = \{0, 1, \ldots\},\ k \geq k_0,\ x_k \in \mathbb{R}^n$ and $f(\cdot) :\ \mathbb{Z}^+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$.

Definition 2.1. [3] *A point $\bar{x}$ is called an equilibrium point of equation (2.1) if $\bar{x} = f(k, \bar{x})$ for all $k \geq k_0$. For all purposes of the stability theory we can assume that the equilibrium point is the zero solution.*

Definition 2.2. [3] *The zero solution of equation (2.1) is called stable if given $\epsilon > 0$ and $k_0 \geq 0$, there exists $\delta = \delta(\epsilon, k_0)$ (if δ is independent of k_0) such that $\|x_{k_0}\| < \delta$ implies $\|x_k\| < \epsilon$ for all $k \geq k_0$.*

Definition 2.3. [3] *The zero solution of equation (2.1) is called asymptotically stable if the zero solution $\bar{0}$ is stable and $x_k \longrightarrow 0$ as $k \longrightarrow +\infty$.*

Definition 2.4. [3] *The zero solution of equation (2.1) is called unstable if it is not stable.*

The function $V : \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be a Lyapunov function on a subset $\mathcal{D}$ of $\mathbb{R}^n$ if
(1) $V(x_k)$ is continuous on $\mathcal{D}$.
(2) $V(x_k)$ is a positive definite if $V(\bar{0}) = 0$ and $V(x_k) > 0$.
(3) $\Delta V(x)_k = V(x_{k+1}) - V(x_k)$ is negative semi definite if $\Delta V(\bar{0}) = 0$ and $\Delta V(x_k) \leq 0$ for any $x_k \neq \bar{0}$, see [2].

Definition 2.5. [4] *The zero solution of a linear discrete-time switched systems (1.1) when $F(k) = 0$ is asymptotically stable if there exists a positive definite scalar function $V(x) : \mathbb{R}^n \longrightarrow \mathbb{R}^+$, and a switching law $i_k \in \{1, 2, ..., N\}$ such that*

$$\Delta V(x_k) = V(x_{k+1}) - V(x_k) < 0$$

along the solution of the system.

Definition 2.6. [4] *A linear uncertain discrete-time switched system (1.1) is robustly stable if there exists a positive definite scalar function $V(x) : \mathbb{R}^n \longrightarrow \mathbb{R}^+$, and a switching law $i_k \in \{1, 2, ..., N\}$ such that*

$$\Delta V(x_k) = V(x_{k+1}) - V(x_k) < 0$$

along the solution of the system for all uncertainties.

Definition 2.7. [4] *A system of symmetric matrices $\{L_i\}, i = 1, 2, ..., \mathrm{N}$, is said to be strictly complete if for every nonzero $x \in \mathbb{R}^n \setminus \{0\}$ there exists $i \in \{1, 2, ..., N\}$ such that $x^T L_i x < 0$.*

Remark 2.1. [4] *: Let us define the sets*

$$\Omega_i = \{x \in \mathbb{R}^n : x^T L_i x < 0\}.$$

It is easy to show that the system $\{L_i\}, i = 1, 2, ..., N$, is strictly complete if and only if

$$\bigcup_{i=1}^{N} \Omega_i = \mathbb{R}^n \setminus \{0\}.$$

A sufficient condition for the strictly completeness of $\{L_i\}$ is that there exist numbers $\tau_i \geq 0, i = 1, 2, ..., p$ such that $\sum_{i=1}^{p} \tau_i > 0$ and $\sum_{i=1}^{p} \tau_i L_i < 0$. This condition is also necessary if $N = 2$.

Lemma 2.2. [4] *Let $P > 0$, $F^T(k)F(k) \leq I$, and M, N are constant matrices. If there is a number $\varepsilon > 0$ such that $\varepsilon I - M^T P M > 0$, then*

$$[A + MF(k)N]^T P[A + MF(k)N] \leq A^T R^{-1} A + \varepsilon N^T N$$

where $R = P^{-1} - (\frac{1}{\varepsilon})MM^T$.

3 Stability Analysis of Discrete-time Switched Systems with Delays

In this section, we present main results of this paper. Condition the following the discrete-time switched systems with delays.

$$x_{k+1} = [A_{i_k} + \Delta A_{i_k}(k)]x_k + [B_{i_k} + \Delta B_{i_k}(k)]x_{k-h_k} \tag{3.1}$$

where $h_k \in \mathbb{Z}^+$ are state delays satisfying $0 < h_1 \leq h_k \leq h_2 < +\infty$, $k \in \mathbb{Z}^+$, $x_k \in \mathbb{R}^n$,$i_k \in \{1, 2, ..., \mathrm{N}\}$ is a switching signal and $A_{i_k}, B_{i_k} \in \mathbb{R}^{n\times n}$ are given constant matrices and $\Delta A_{i_k}(k),\ \Delta B_{i_k}(k)$ are uncertain matrices which are assumed to be of the form

$$\Delta A_{i_k}(k) = E_{i_k}F(k)F_{i_k}$$
$$\Delta B_{i_k}(k) = H_{i_k}F(k)G_{i_k}$$

where $E_{i_k},\ F_{i_k},\ H_{i_k},\ G_{i_k}$ are given constant matrices of appropriate dimensions and $F(k)$ is a given matrix which $F^T(k)F(k) \leq I$. We will investigate the stability of the zero solution of the discrete-time switched system with delays (3.1) without uncertainties, namely $\Delta A_{i_k}(k) = \Delta B_{i_k}(k) = 0$. The sufficient condition is given in the following theorem.

Theorem 3.1. *Let h_1 and h_2 be positive integers such that $0 < h_1 < h_2$. The zero solution of the discrete-time system (3.1)without uncertainties is asymptotically stable if for any time-varying delay h_k satisfying $0 < h_1 \leq h_k \leq h_2 < +\infty$ there exist symmetric positive definite matrices P_i and Q_i such that the following condition holds:*
The system of symmetric matrices $\{L_i(P_i, O_i)\}$ is strictly complete where $L_i(P_i, O_i)$ is defined by

$$L_i(P_i, Q_i) = \begin{bmatrix} A_i^T P_i A_i - P_i + (h_2 - h_1 + 1)Q_i & A_i^T P_i B_i \\ B_i^T P_i A_i & B_i^T P_i B_i - Q_i \end{bmatrix} \tag{3.2}$$

or there exist numbers $\lambda_i \geq 0$, $i = 1, 2, ..., \mathrm{N}$ such that $\sum_{i=1}^N \lambda_i > 0$ and

$$\sum_{i=1}^{N} \lambda_i L_i(P_i, Q_i) < 0.$$

The switching signal is chosen as $i_k = i$ whenever $y_k \in \Omega_i = \{y \in \mathbb{R}^{2n} : y^T L_i(P_i, Q_i)y < 0, y_k = [x_k\ ,\ x_{k-h_k}]\}$.

Proof Let $\Omega_i = \{y \in \mathbb{R}^{2n} : y^T L_i y < 0\}$ where $y_k = [x_k\ ,\ x_{k-h_k}]$. Let P_i and Q_i be symmetric positive definite matrices. From Remark 2.1 , $\bigcup_{i=1}^N \Omega_i = \mathbb{R}^{2n} \setminus \{0\}$. From assumption, as the system $\{L_i(P_i, Q_i)\}$ is strictly complete, there exist $\lambda_i \geq 0,\ i = 1, 2, \ldots, N$ such that $\sum_{i=1}^N \lambda_i > 0$ and

$$\sum_{i=1}^{N} \lambda_i L_i(P_i, Q_i) < 0. \tag{3.3}$$

Since $\lambda_i \geq 0$ and $\sum_{i=1}^{N} \lambda_i > 0$, there is always a number $\varepsilon > 0$ such that for any $y_k \in \mathbb{R}^{2n} \setminus \{0\}$,we obtain that

$$\sum_{i=1}^{N} \lambda_i y_k^T L_i(P_i, Q_i) y_k \leq -\varepsilon y_k^T y_k.$$

Thus, there is at least an index $i_m \in \{1, 2, ..., \mathrm{N}\}$ satisfying

$$y_k^T L_{i_m}(P_{i_m}, Q_{i_m}) y_k < -\sigma y_k^T y_k \tag{3.4}$$

where $\sigma = \frac{\varepsilon}{\sum_{i=1}^{N} \lambda_i} > 0$, namely there exist an index $i_m \in \{1, 2, \ldots, N\}$ such that $y_k \in \Omega_{i_m}$. We choose switching law as $i_k = i_m$. Consider the Lyapunov function

$$V_{i_m}(x_k) = V_{i_{m1}}(x_k) + V_{i_{m2}}(x_k) + V_{i_{m3}}(x_k)$$

where $V_{i_{m1}}(x_k) = x_k^T P_{i_m} x_k$, $V_{i_{m2}}(x_k) = \sum_{l=k-h_k}^{k-1} x_l^T Q_{i_m} x_l$, $V_{i_{m3}}(x_k) = \sum_{j=-h_2+2}^{-h_1+1} \sum_{l=k+j-1}^{k-1} x_l^T Q_{i_m} x_l$. We obtain that the difference of the Lyapunov function satisfies

$$\begin{aligned}
\Delta V_{i_{m1}}(x_k) &= V_{i_{m1}}(x_{k+1}) - V_{i_{m1}}(x_k) \\
&= x_{k+1}^T P_{i_m} x_{k+1} - x_k^T P_{i_m} x_k \\
&= \Big[A_{i_m} x_k + B_{i_m} x_{k-h_k}\Big]^T P_{i_m} \Big[A_{i_m} x_k + B_{i_m} x_{k-h_k}\Big] - x_k^T P_{i_m} x_k \\
&= x_k^T A_{i_m}^T P_{i_m} A_{i_m} x_k + x_k^T A_{i_m}^T P_{i_m} B_{i_m} x_{k-h_k} + x_{k-h_k}^T B_{i_m}^T P_{i_m} A_{i_m} x_k \\
&\quad + x_{k-h_k}^T B_{i_m}^T P_{i_m} B_{i_m} x_{k-h_k} - x_k^T P_{i_m} x_k \\
&= x_k^T \Big[A_{i_m}^T P_{i_m} A_{i_m} - P_{i_m}\Big] x_k + x_k^T A_{i_m}^T P_{i_m} B_{i_m} x_{k-h_k} + x_{k-h_k}^T B_{i_m}^T P_{i_m} A_{i_m} x_k \\
&\quad + x_{k-h_k}^T B_{i_m}^T P_{i_m} B_{i_m} x_{k-h_k} \\
\Delta V_{i_{m2}}(x_k) &= V_{i_{m2}}(x_{k+1}) - V_{i_{m2}}(x_k) \\
&= \sum_{l=k+1-h_{k+1}}^{k} x_l^T Q_{i_m} x_l - \sum_{l=k-h_k}^{k-1} x_l^T Q_{i_m} x_l \\
&= \sum_{l=k+1-h_{k+1}}^{k-h_1} x_l^T Q_{i_m} x_l + x_k^T Q_{i_m} x_k - x_{k-h_k}^T Q_{i_m} x_{k-h_k} \\
&\quad + \sum_{l=k+1-h_1}^{k-1} x_l^T Q_{i_m} x_l - \sum_{l=k+1-h_k}^{k-1} x_l^T Q_{i_m} x_l.
\end{aligned}$$

Since $h_k \geq h_1$, we have

$$\sum_{l=k+1-h_1}^{k-1} x_l^T Q_{i_m} x_l - \sum_{l=k+1-h_k}^{k-1} x_l^T Q_{i_m} x_l \leq 0.$$

Thus $\Delta V_{i_{m2}} \leq \sum_{l=k+1-h_{k+1}}^{k-h_1} x_l^T Q_{i_m} x_l + x_k^T Q_{i_m} x_k - x_{k-h_k}^T Q_{i_m} x_{k-h_k}$.
For the third term of the Lyapunov function, we obtain the difference Lyapunov function

$$\begin{aligned}
\Delta V_{i_{m3}}(x_k) &= V_{i_{m3}}(x_{k+1}) - V_{i_{m3}}(x_k)\\
&= \sum_{j=-h_2+2}^{-h_1+1} \sum_{l=k+j}^{k} x_l^T Q_{i_m} x_l - \sum_{j=-h_2+2}^{-h_1+1} \sum_{l=k+j-1}^{k-1} x_l^T Q_{i_m} x_l\\
&= \sum_{j=-h_2+2}^{-h_1+1} \Big[x_k^T Q_{i_m} x_k - x_{k+j-1}^T Q_{i_m} x_{k+j-1} \Big]\\
&= (h_2 - h_1) x_k^T Q_{i_m} x_k - \sum_{l=k+1-h_2}^{k-h_1} x_l^T Q_{i_m} x_l.
\end{aligned}$$

Since $h_k \leq h_2$, we have

$$\sum_{l=k+1-h_{k+1}}^{k-h_1} x_l^T Q_{i_m} x_l - \sum_{l=k+1-h_2}^{k-h_1} x_l^T Q_{i_m} x_l \leq 0.$$

Then it follows that

$$\Delta V_{i_{m2}}(x_k) + \Delta V_{i_{m3}}(x_k) \leq (h_2 - h_1 + 1) x_k^T Q_{i_m} x_k - x_{k-h_k}^T Q_{i_m} x_{k-h_k}.$$

Thus, we obtain that

$$\begin{aligned}
\Delta V_{i_m}(x_k) \leq\ & x_k^T \Big[A_{i_m}^T P_{i_m} A_{i_m} - P_{i_m} + (h_2 - h_1 + 1) Q_{i_m} \Big] x_k + x_k^T A_{i_m}^T P_{i_m} B_{i_m} x_{k-h_k}\\
& + x_{k-h_k}^T B_{i_m}^T P_{i_m} A_{i_m} x_k + x_{k-h_k}^T \Big[B_{i_m}^T P_{i_m} B_{i_m} - Q_{i_m} \Big] x_{k-h_k}.
\end{aligned}$$

By completing the square, we obtain

$$\Delta V_{i_m}(x_k) \leq y_k^T \begin{bmatrix} A_{i_m}^T P_{i_m} A_{i_m} - P_{i_m} + (h_2 - h_1 + 1) Q_{i_m} & A_{i_m}^T P_{i_m} B_{i_m} \\ B_{i_m}^T P_{i_m} A_{i_m} & B_{i_m}^T P_{i_m} B_{i_m} - Q_{i_m} \end{bmatrix} y_k$$

where $y_k = \begin{bmatrix} x_k \\ x_{k-h_k} \end{bmatrix}$.
Let $L_{i_m}(P_{i_m}, Q_{i_m}) = \begin{bmatrix} A_{i_m}^T P_{i_m} A_{i_m} - P_{i_m} + (h_2 - h_1 + 1) Q_{i_m} & A_{i_m}^T P_{i_m} B_{i_m} \\ B_{i_m}^T P_{i_m} A_{i_m} & B_{i_m}^T P_{i_m} B_{i_m} - Q_{i_m} \end{bmatrix}$.
Thus

$$\Delta V_{i_m}(x_k) \leq y_k^T L_{i_m}(P_{i_m}, Q_{i_m}) y_k.$$

From (3.4), we obtain

$$\Delta V_{i_m}(x_k) \leq y_k^T L_{i_m}(P_{i_m}, Q_{i_m}) y_k < -\sigma y_k^T y_k.$$

Therefore, by Lyapunov stability theorem, the zero solution of switched system (3.1) without uncertainties is asymptotically stable. □

Example 3.1 Consider the discrete-time linear switched system (3.1) without uncertainties where $N = 2$, $A_1 = \begin{bmatrix} -\frac{1}{5} & -\frac{1}{10} \\ \frac{1}{10} & 0 \end{bmatrix}$, $B_1 = \begin{bmatrix} \frac{1}{4} & -\frac{1}{10} \\ 0 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} \frac{1}{20} & -\frac{1}{5} \\ \frac{1}{10} & 0 \end{bmatrix}$, $B_2 = \begin{bmatrix} \frac{1}{10} & -\frac{1}{4} \\ 0 & 0 \end{bmatrix}$, $h_1 = 1$ and $h_2 = 2$.
The positive definite matrices P_i and Q_i ; $i = 1, 2$ are chosen as
$P_1 = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$, $Q_1 = \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & 1 \end{bmatrix}$, $P_2 = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$ and $Q_2 = \begin{bmatrix} \frac{4}{5} & 0 \\ 0 & \frac{1}{5} \end{bmatrix}$.
We have

$$L_1 = \begin{bmatrix} -\frac{171}{50} & \frac{2}{25} & -\frac{1}{5} & \frac{2}{25} \\ \frac{2}{25} & \frac{1}{25} & -\frac{1}{10} & \frac{1}{25} \\ -\frac{1}{5} & -\frac{1}{10} & \frac{1}{20} & -\frac{1}{10} \\ \frac{2}{25} & \frac{1}{25} & -\frac{1}{10} & -\frac{24}{25} \end{bmatrix}, \quad L_2 = \begin{bmatrix} -\frac{237}{100} & -\frac{1}{25} & \frac{1}{50} & -\frac{1}{20} \\ -\frac{1}{25} & -\frac{36}{25} & -\frac{2}{25} & \frac{1}{5} \\ \frac{1}{50} & -\frac{2}{25} & -\frac{19}{25} & -\frac{1}{10} \\ -\frac{1}{20} & \frac{1}{5} & -\frac{1}{10} & \frac{1}{20} \end{bmatrix}$$

such that $\Omega_1 = \{x \in \mathbb{R}^4 : x^T L_1 x = -\frac{21}{50}x_1^2 + \frac{79}{625}x_2^2 + \frac{11}{10}x_3^2 - \frac{471}{500}x_4^2 - (x_1 - \frac{2}{25}x_2)^2 - (x_1 + \frac{1}{5}x_3)^2 - (x_1 - \frac{2}{25}x_4)^2 - (x_2 + \frac{1}{10}x_3)^2 - (x_2 - \frac{1}{25}x_4)^2 - (x_3 + \frac{1}{10}x_4)^2 < 0\}$
and
$\Omega_2 = \{x \in \mathbb{R}^4 : x^T L_2 x = \frac{63}{100}x_1^2 - \frac{849}{625}x_2^2 + \frac{617}{2500}x_3^2 + \frac{51}{400}x_4^2 - (x_1 + \frac{1}{25}x_2)^2 - (x_1 - \frac{1}{50}x_3)^2 - (x_1 - \frac{1}{20}x_4)^2 - (x_2 + \frac{2}{25}x_3)^2 - (x_2 - \frac{1}{5}x_4)^2 - (x_3 + \frac{1}{10}x_4)^2 < 0\}$, respectively.
Note that L_1 and L_2 are not negative definite and $\Omega_1 \bigcup \Omega_2 = \mathbb{R}^4 \setminus \{(0, 0, 0, 0)\}$.
For $\lambda_1 = \frac{1}{2}, \lambda_2 = \frac{1}{4}$ we have $\lambda_1 + \lambda_2 = \frac{3}{4} > 0$ and

$$\tfrac{1}{2}L_1 + \tfrac{1}{4}L_2 = \begin{bmatrix} -\frac{921}{400} & \frac{3}{100} & -\frac{19}{200} & \frac{11}{400} \\ \frac{3}{100} & -\frac{17}{50} & -\frac{7}{100} & \frac{7}{100} \\ -\frac{19}{200} & -\frac{7}{100} & -\frac{33}{200} & -\frac{3}{40} \\ \frac{11}{400} & \frac{7}{100} & -\frac{3}{40} & -\frac{187}{400} \end{bmatrix} < 0$$

Therefore, by theorem 3.1 the zero solution of switched system is asymptotically stable where the switching law is chosen as $i_k = i$ whenever $y_k \in \Omega_i$. □

Numerical Simulation

Let $N = 2$, we consider the following switched system. Let A_1, A_2, B_1 and B_2 be as in Example 3.1. If we let the delays $h_0 = h_2 = h_4 = \ldots = 1$ and $h_1 = h_3 = h_5 = \ldots = 2$ and let initial conditions be $x_{-1} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$ and $x_0 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$. Under the switching law given in theorem 3.1, we have the following numerical simulation of trajectory of solution of the swiched system.

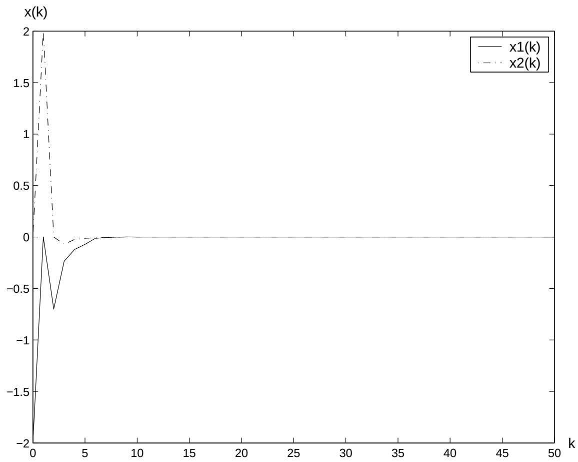

Figure 1: The trajectories $x_1(k)$ and $x_2(k)$ of the discrete-time linear switched systems in Example 3.1

4 Robust Stability of Discrete-time Switched Systems with Delays

In this section, we will investigate the robust stability of the zero solution of discrete-time switched systems with delays (3.1) and we give a sufficient condition for robust stability of this system. Let

$$\begin{aligned} R_i &= P_i^{-1} - \frac{1}{\varepsilon}(E_i^T E_i + H_i^T H_i) \\ \mathcal{X}_i &= \{y \in \mathbb{R}^{2n} : y^T W_i(P_i, Q_i) y < 0\} \\ W_{i_k}(P_{i_k}, Q_{i_k}) &= \begin{bmatrix} J_i & A_i^T R_i^{-1} B_i + \varepsilon F_i^T G_i \\ B_i^T R_i^{-1} A_i + \varepsilon G_i^T F_i & K_i \end{bmatrix} \end{aligned}$$

where

$$\begin{aligned} J_i &= (h_2 - h_1 + 1) Q_i - P_i + A_i^T R_i^{-1} A_i + \varepsilon F_i^T F_i \\ K_i &= B_i^T R_i^{-1} B_i + \varepsilon F_i^T G_i - Q_i. \end{aligned}$$

Theorem 4.1. *Let h_1 and h_2 be positive integers such that $0 < h_1 < h_2$. Assume that for any time-varying delay h_k satisfying $0 < h_1 \le h_k \le h_2 < +\infty$ there exist symmetric positive definite matrices P_i and Q_i and a number $\varepsilon > 0$ such that the following two conditions holds:*
(i) $\varepsilon I - (E_i^T P_i E_i + H_i^T P_i H_i) \; > \; 0$
(ii) the system matrices $W_i(P_i, Q_i)$ is strictly complete, or there exist numbers $\lambda_i \ge 0,\ i = 1, 2, \ldots, N$ such that $\sum_{i=1}^N \lambda_i > 0$ and $\sum_{i=1}^N \lambda_i W_i(P_i, Q_i) < 0$.

Then, the zero solution of uncertain discrete-time linear switched system (3.1) is robustly stable.

Proof Let $\mathcal{X}_i = \{y \in \mathbb{R}^{2n} : y^T W_i(P_i, Q_i) y < 0\}$. From Remark 2.1, $\bigcup_{i=1}^N \mathcal{X}_i = \mathbb{R}^{2n} \setminus \{0\}$. From assumption, as the system $W_i(P_i, Q_i)$ is stictly complete, there exist $\lambda_i \geq 0,\ i = 1, 2, \ldots, N$ such that $\sum_{i=1}^N \lambda_i > 0$ and

$$\sum_{i=1}^{N} \lambda_i W_i(P_i, Q_i) < 0. \tag{4.1}$$

Since $\lambda_i \geq 0$ and $\sum_{i=1}^N \lambda_i > 0$, there is always a number $\varepsilon > 0$ such that for any $y_k \in \mathbb{R}^{2n} \setminus \{0\}$, we obtain that

$$\sum_{i=1}^{N} \lambda_i y_k^T W_i(P_i, Q_i) y_k \leq -\varepsilon y_k^T y_k.$$

Thus, there is at least an index $i_m \in \{1, 2, \ldots, N\}$ satisfying

$$y_k^T W_{i_m}(P_{i_m}, Q_{i_m}) y_k < -\varepsilon y_k^T y_k \tag{4.2}$$

where $\sigma = \frac{\varepsilon}{\sum_{i=1}^N \lambda_i} > 0$, namely there exist an index $i_m \in \{1, 2, \ldots, N\}$ such that $y_k \in \mathcal{X}_{i_m}$. We choose switching law $i_k = i_m$. Let $\bar{A}_{i_k}$ and $\bar{B}_{i_k}$ by given by

$$\bar{A}_{i_k} = A_{i_k} + \Delta A_{i_k}(k)\ ,\quad \bar{B}_{i_k} = B_{i_k} + \Delta B_{i_k}(k).$$

Consider the following Lyapunov function

$$V_{i_m}(x_k) \quad = \quad V_{i_{m1}}(x_k) + V_{i_{m2}}(x_k) + V_{i_{m3}}(x_k)$$

where $V_{i_{m1}}(x_k) = x_k^T P_{i_m} x_k$, $V_{i_{m2}}(x_k) = \sum_{l=k-h_k}^{k-1} x_l^T Q_{i_m} x_l$, $V_{i_{m3}}(x_k) = \sum_{j=-h_2+2}^{-h_1+1} \sum_{l=k+j-1}^{k-1} x_l^T Q_{i_m} x_l$. We obtain that the difference of the Lyapunov function along the trajectory of solution of (3.1) like that Theorem 3.1, i.e.

$$\Delta V_{i_m}(x_k) \quad \leq \quad y_k^T \begin{bmatrix} \bar{A}_{i_m}^T P_{i_m} \bar{A}_{i_m} - P_{i_m} + (h_2 - h_1 + 1) Q_{i_m} & \bar{A}_{i_m}^T P_{i_m} \bar{B}_{i_m} \\ \bar{B}_{i_m}^T P_{i_m} \bar{A}_{i_m} & \bar{B}_{i_m}^T P_{i_m} \bar{B}_{i_m} - Q_{i_m} \end{bmatrix} y_k$$

where $y_k = \begin{bmatrix} x_k \\ x_{k-h_k} \end{bmatrix}$.

Let $L_{i_m}(P_{i_m}, Q_{i_m}) = \begin{bmatrix} \bar{A}_{i_m}^T P_{i_m} \bar{A}_{i_m} - P_{i_m} + (h_2 - h_1 + 1) Q_{i_m} & \bar{A}_{i_m}^T P_{i_m} \bar{B}_{i_m} \\ \bar{B}_{i_m}^T P_{i_m} \bar{A}_{i_m} & \bar{B}_{i_m}^T P_{i_m} \bar{B}_{i_m} - Q_{i_m} \end{bmatrix}$.

Thus

$$\Delta V_{i_m}(x_k) \leq y_k^T L_{i_m}(P_{i_m}, Q_{i_m}) y_k.$$

Now $L_{i_m}(P_{i_m},Q_{i_m}) = \begin{bmatrix} (h_2-h_1+1)Q_{i_m}-P_{i_m} & 0 \\ 0 & -Q_{i_m} \end{bmatrix} + \begin{bmatrix} \bar{A}_{i_m}^T \\ \bar{B}_{i_m}^T \end{bmatrix} P_{i_m} \begin{bmatrix} \bar{A}_{i_m} & \bar{B}_{i_m} \end{bmatrix}$.

We have $\begin{bmatrix} \bar{A}_{i_m}^T \\ \bar{B}_{i_m}^T \end{bmatrix} P_{i_m} \begin{bmatrix} \bar{A}_{i_m} & \bar{B}_{i_m} \end{bmatrix} = \begin{bmatrix} \hat{A}_{i_m} & \hat{B}_{i_m} \\ \hat{C}_{i_m} & \hat{D}_{i_m} \end{bmatrix}$ where

$$\begin{aligned}
\hat{A}_{i_m} &= (A_{i_m}^T + F_{i_m}^T F_{i_m}^T(k) E_{i_m}^T) P_{i_m} (A_{i_m} + E_{i_m} F_{i_m}(k) F_{i_m}) \\
\hat{B}_{i_m} &= (A_{i_m}^T + F_{i_m}^T F_{i_m}^T(k) E_{i_m}^T) P_{i_m} (B_{i_m} + H_{i_m} F_{i_m}(k) G_{i_m}) \\
\hat{C}_{i_m} &= (B_{i_m}^T + G_{i_m}^T F_{i_m}^T(k) H_{i_m}^T) P_{i_m} (A_{i_m} + E_{i_m} F_{i_m}(k) F_{i_m}) \\
\hat{D}_{i_m} &= (B_{i_m}^T + G_{i_m}^T F_{i_m}^T(k) H_{i_m}^T) P_{i_m} (B_{i_m} + H_{i_m} F_{i_m}(k) G_{i_m}).
\end{aligned}$$

Since there is a number $\varepsilon > 0$ such that $\varepsilon I - (E_{i_m}^T P_{i_m} E_{i_m} + H_{i_m}^T P_{i_m} H_{i_m}) > 0$, by Lemma 2.2, we have

$$\begin{aligned}
\hat{A}_{i_m} &\leq A_{i_m}^T R_{i_m}^{-1} A_{i_m} + \varepsilon F_{i_m}^T F_{i_m} \\
\hat{B}_{i_m} &\leq A_{i_m}^T R_{i_m}^{-1} B_{i_m} + \varepsilon F_{i_m}^T G_{i_m} \\
\hat{C}_{i_m} &\leq B_{i_m}^T R_{i_m}^{-1} A_{i_m} + \varepsilon G_{i_m}^T F_{i_m} \\
\hat{D}_{i_m} &\leq B_{i_m}^T R_{i_m}^{-1} B_{i_m} + \varepsilon G_{i_m}^T G_{i_m}.
\end{aligned}$$

Thus

$$\begin{bmatrix} \bar{A}_{i_m}^T \\ \bar{B}_{i_m}^T \end{bmatrix} P_{i_m} \begin{bmatrix} \bar{A}_{i_m} & \bar{B}_{i_m} \end{bmatrix} \leq \begin{bmatrix} A_{i_m}^T R_{i_m}^{-1} A_{i_m} + \varepsilon F_{i_m}^T F_{i_m} & A_{i_m}^T R_{i_m}^{-1} B_{i_m} + \varepsilon F_{i_m}^T G_{i_m} \\ B_{i_m}^T R_{i_m}^{-1} A_{i_m} + \varepsilon G_{i_m}^T F_{i_m} & B_{i_m}^T R_{i_m}^{-1} B_{i_m} + \varepsilon G_{i_m}^T G_{i_m} \end{bmatrix}.$$

So, we obtain

$$\begin{aligned}
L_{i_m}(P_{i_m},Q_{i_m}) &\leq \begin{bmatrix} J_{i_m} & A_{i_m}^T R_{i_m}^{-1} B_{i_m} + \varepsilon F_{i_m}^T G_{i_m} \\ B_{i_m}^T R_{i_m}^{-1} A_{i_m} + \varepsilon G_{i_m}^T F_{i_m} & K_{i_m} \end{bmatrix} \\
&= W_{i_m}(P_{i_m},Q_{i_m}).
\end{aligned}$$

where

$$\begin{aligned}
J_{i_m} &= (h_2-h_1+1)Q_{i_m} - P_{i_m} + A_{i_m}^T R_{i_m}^{-1} A_{i_m} + \varepsilon F_{i_m}^T F_{i_m} \\
K_{i_m} &= B_{i_m}^T R_{i_m}^{-1} B_{i_m} + \varepsilon G_{i_m}^T G_{i_m} - Q_{i_m}.
\end{aligned}$$

From $\Delta V_{i_m}(x_k) \leq y_k^T L_{i_m}(P_{i_m},Q_{i_m}) y_k$ and arguments above, we have

$$\Delta V_{i_m}(x_k) \leq y_k^T W_{i_m}(P_{i_m},Q_{i_m}) y_k$$

From (4.2), we obtain

$$\Delta V_{i_m}(x_k) \leq y_k^T W_{i_m}(P_{i_m},Q_{i_m}) y_k < -\sigma y_k^T y_k.$$

Thus, the zero solution of switched system (3.1) is robustly stable. □

Example 4.1 Consider the uncertain discrete-time linear switched system with

delays (3.1) where $N = 2$, $A_1 = \begin{bmatrix} -\frac{1}{5} & -\frac{1}{10} \\ \frac{1}{10} & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} \frac{1}{20} & -\frac{1}{5} \\ \frac{1}{10} & 0 \end{bmatrix}$, $B_1 = \begin{bmatrix} \frac{1}{4} & -\frac{1}{10} \\ 0 & 0 \end{bmatrix}$, $B_2 = \begin{bmatrix} \frac{1}{10} & -\frac{1}{4} \\ 0 & 0 \end{bmatrix}$, $E_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$, $E_2 = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$, $F_1 = \begin{bmatrix} 0 & \frac{1}{4} \\ \frac{1}{20} & 0 \end{bmatrix}$, $F_2 = \begin{bmatrix} 0 & \frac{3}{10} \\ \frac{1}{10} & 0 \end{bmatrix}$, $G_1 = \begin{bmatrix} 0 & -\frac{1}{10} \\ \frac{1}{20} & 0 \end{bmatrix}$, $G_2 = \begin{bmatrix} 0 & \frac{1}{4} \\ \frac{3}{10} & 0 \end{bmatrix}$, $H_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$, $H_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $h_1 = 1$, $h_2 = 2$ and $\varepsilon = 2$.
The positive definite matrices P_i and Q_i ; $i = 1, 2$ are chosen as
$P_1 = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$, $P_2 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $Q_1 = \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & 1 \end{bmatrix}$ and $Q_2 = \begin{bmatrix} \frac{4}{5} & 0 \\ 0 & \frac{1}{5} \end{bmatrix}$. We have
$\varepsilon I - (E_1^T P_1 E_1 + H_1^T P_1 H_1) = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$, $\varepsilon I - (E_2^T P_2 E_2 + H_2^T P_2 H_2) = \begin{bmatrix} \frac{8}{5} & 0 \\ 0 & \frac{2}{5} \end{bmatrix}$.
We have $W_1 = \begin{bmatrix} -\frac{2101}{600} & \frac{1}{25} & -\frac{19}{200} & \frac{1}{25} \\ \frac{1}{25} & \frac{29}{200} & -\frac{1}{20} & -\frac{3}{100} \\ -\frac{19}{200} & -\frac{1}{20} & -\frac{7}{100} & -\frac{1}{20} \\ \frac{1}{25} & -\frac{3}{100} & -\frac{1}{20} & -\frac{24}{25} \end{bmatrix}$, $W_2 = \begin{bmatrix} -\frac{671}{1800} & -\frac{2}{225} & \frac{29}{450} & -\frac{1}{90} \\ -\frac{2}{225} & -\frac{173}{450} & -\frac{4}{225} & \frac{7}{36} \\ \frac{29}{450} & -\frac{4}{225} & -\frac{11}{18} & -\frac{1}{45} \\ -\frac{1}{90} & \frac{7}{36} & -\frac{1}{45} & -\frac{7}{360} \end{bmatrix}$
such that
$\mathcal{X}_1 = \{x \in \mathbb{R}^4 : x^T W_1 x = -\frac{301}{600}x_1^2 + \frac{10733}{5000}x_2^2 + \frac{37661}{40000}x_3^2 - \frac{191}{200}x_4^2 - (x_1 - \frac{1}{25}x_2)^2 - (x_1 + \frac{19}{200}x_3)^2 - (x_1 - \frac{1}{25}x_4)^2 - (x_2 + \frac{1}{20}x_3)^2 - (x_2 + \frac{3}{100}x_4)^2 - (x_3 + \frac{1}{20}x_4)^2\}$
and
$\mathcal{X}_2 = \{x \in \mathbb{R}^4 : x^T W_2 x = \frac{4729}{1800}x_1^2 + \frac{163583}{101250}x_2^2 + \frac{15931}{40500}x_3^2 + \frac{41}{2160}x_4^2 - (x - 1 + \frac{2}{225}x_2)^2 - (x_1 - \frac{29}{450}x_3)^2 - (x_1 + \frac{1}{90}x_4)^2 - (x_2 + \frac{4}{225}x_3)^2 - (x_2 - \frac{7}{36}x_4)^2 - (x_3 + \frac{1}{45}x_4)^2\}$,
respectively.
Note that W_1 and W_2 are not negative definite and $\mathcal{X}_1 \bigcup \mathcal{X}_2 = \mathbb{R}^4 \setminus \{(0, 0, 0, 0)\}$.
For $\lambda_1 = \frac{1}{2} = \lambda_2$, we have $\lambda_1 + \lambda_2 = 1 > 0$ and

$$\frac{1}{2}W_1 + \frac{1}{2}W_2 = \begin{bmatrix} -\frac{3487}{1800} & \frac{7}{450} & -\frac{11}{720} & \frac{13}{900} \\ \frac{7}{450} & -\frac{431}{3600} & -\frac{61}{1800} & \frac{37}{450} \\ -\frac{11}{720} & -\frac{61}{1800} & -\frac{613}{1800} & -\frac{13}{360} \\ \frac{13}{900} & \frac{37}{450} & -\frac{13}{360} & -\frac{1763}{3600} \end{bmatrix} < 0.$$

Therefore, by theorem 4.1, the zero solution of system is robustly stable where the switching law is chosen as $i_k = i$ whenever $y_k \in \mathcal{X}_i$. $\square$

Numerical Simulation

Let $N = 2$, we consider the switched system. From Example 4.1, let $A_1, A_2, B_1, B_2, E_1, E_2$,
$F_1, F_2, G_1, G_2, H_1, H_2$ be in Example 4.1. If we let the delays $h_0 = h_2 = h_4 = \ldots = 1$ and $h_1 = h_3 = h_5 = \ldots = 2$ and let $F(k) = \begin{bmatrix} \frac{1}{k^2+2} & \frac{1}{k^2+2} \\ \frac{1}{k+2} & \frac{1}{k+2} \end{bmatrix}$. And let initial conditions be $x_{-1} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$, $x_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Under the switching law given in Theorem 4.1, we have the following numerical simulation of trajectory of solution

of the switched system. □

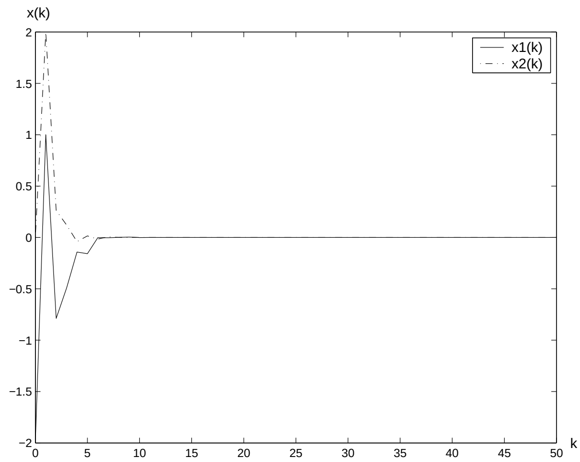

Figure 2: The trajectories $x_1(k)$ and $x_2(k)$ of the discrete-time switched systems in Example 4.1

5 Conclusion

In this work, we give sufficient conditions for asymptotic stability and robust stability of switched system. And we use Lyapunov theory to consider stability and robust stability of the zero solution of switched system (3.1). In Theorem 3.1 and 4.1, the Lyapunov function we used is a common Lyapunov function.

Acknowledgements : we wish to express our sincere thanks to Assoc. Prof. Dr. P. Niamsup for advising and improving.

References

[1] S.Barnett and R.G.Cameron, *Introduction to Mathematical Control Theory*, Oxford, Clarendon Press,1985.

[2] H.K.Khalil, *Nonlinear Systems*, Prentice Hall, Upper Saddle River, New Jersey, second edition, 1996.

[3] Mustafa R.S. Kulenović and Orlando Merino, *Discrete Dynamical systems and Difference Equations with Mathematica*, Chapman and Hall/CRC, U.S., 2002.

[4] V.N.Phat, *Robust Stability and Stabilizability of Uncertain Linear Hybrid Systems With State Delays*, IEEE Transactions on Circuit and Systems, Vol.52, No. 2, pp. 94-98, 2005.

[5] R.Shorten, F.Wirth, O.Mason, K.Wulff and C.King, *Stability Criteria for switched and Hybrid Systems*, preprint.

[6] G.Xie and L.Wang, *Quadratic Stability and Stabilization of Discrete-time Switched Systems with State Delay*, 43rd IEEE Conference on Decision and Control, 2004.

[7] M.Yu, L.Wang, and T.Chu, *Robust stabilization of discrete-time systems with time-varying delay*, American Control Conference, 2005.

(Received 13 May 2008)

Jenjira Thipcha
Department of Mathematics and Statistics,
Dhurakij Pundit University, THAILAND.
e-mail : `jen_0612_cmu@hotmail.com`

Piyapong Niamsup
Department of Mathematics,
Chiang Mai University, THAILAND.
e-mail :`scipnmsp@chiangmai.ac.th`