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Lipschitz Range in an Elementary Proof of the Brouwer Fixed Point Theorem, and a Proof of the Nash Equilibrium

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Abstract The Brouwer fixed point theorem is well known and useful, but considered quite advanced. We present a short elementary proof of it, mainly following Howard [1] but avoiding the reference to inverse function theorem. This is based on using Lipschitz range, instead of only a Lipschitz constant. Once the Brouwer theorem is available, it suffices for reaching the existence of Nash equilibrium by exponentiating the sectionally convex goal functions to positive and sectionally convex, and cycling continuous projections to the respective epigraphs one coordinate at a time. The continuity of these mappings follows from use of a simplified version of the maximum theorem, which guarantees parametric continuity of the minimizer (of projection distance) when it is unique. In this way, actually three powerful theorems (Brouwer, simplified maximum, and Nash equilibrium) are presented in combination, in such length that fits in a single lecture.

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1. Introduction

In this presentation the concept of Lipschitz range is utilized together with the fact that identity perturbed by a contraction is an open mapping, to find a short and (for students of calculus or mathematical analysis) convenient proof of the Brouwer fixed point theorem. The authors believe this proof is not only short but also novel in using the open mapping property.

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As an application, a similarly short proof of existence of the Nash equilibrium is provided. To make this application intuitively appealing and concise in notation, we introduce mappings that are here called epishifts. Their continuity is required, and follows from a simplified maximum theorem – a further topic covered here that we would like to see included in courses on analysis.

Overall the approach is a composition of several important, even celebrated topics, seeking to make them accessible to the widest possible audience. However, in particular those with interest in mathematical economics, game theory, or optimization, can benefit from this composition.

2. Preliminaries

In this section, we recall some basic definitions and facts from mathematical analysis. We also define a mapping, here called *epishift*, for which we don't have a prior reference although it appears a natural tool.

Throughout this paper we focus for simplicity on subsets of the Euclidean space $\mathbb{R}^n$ and either self-maps or real-valued functions on them. However, some background theorems are presented more naturally for generic metric spaces, or for complete normed spaces (i.e., Banach spaces).

2.1. Basic definitions

Consider metric spaces (X, d_X) and (Y, d_Y), and a mapping $f : X \to Y$.

Definition 2.1. If for some nonnegative real k_f and K_f

$$k_f d_X(a, b) \le d_Y(f(a), f(b)) \le K_f d_X(a, b) \quad \text{for all } a, b \in X \tag{2.1}$$

then the interval $[k_f, K_f]$ is the *Lipschitz range* of f.

Commonly K_f is called the *Lipschitz constant* of f while the lower bound, trivially satisfied by $k_f = 0$, is ignored. However, when a positive k_f exists, obviously f is one-to-one and its inverse function has the reciprocal value $1/k_f$ as its Lipshitz constant; having a Lipschitz constant implies uniform continuity of that function. For a sum of functions some care is needed at the lower bound, while the upper bound obeys a simple rule. The best general bounds are $k_{f+g} = \max\{0, k_f - K_g, k_g - K_f\}$ and $K_{f+g} = K_f + K_g$.

When strictly $K_f < 1$ the function f is a *contraction mapping*, while if $K_f \le 1$ then f can be called *nonexpansive*. These concepts are particularly useful with self-maps from X to itself. We recall that when X is a complete metric space where Cauchy sequences must converge, then the fixed point (abbreviated FP from now on) principle of Banach applies to a contraction self-map f and there exists a unique x_0 satisfying $x_0 = f(x_0)$. Further, iterative application of $f(\cdot)$ gives a sequence converging to x_0 regardless of choice of starting point.

Definition 2.2. The mapping $f : X \to Y$ is an *open mapping*, if the image of any open set in X is open in Y.

Finally, we define the epishift mapping that will play a key role in our proof of Nash equilibrium.

Definition 2.3. Let the arguments of positive real valued continuous f be partitioned to a Euclidean finite dimensional x, constrained in a compact convex set X, and to the rest as parameters p. Assume f is convex in x for a constant p, so that we can find a unique x^* that minimizes the Euclidean distance to the graph:

$$x^* = \arg\min_{z \in X} d((x,0),(z,f(z,p))) = \arg\min_{z \in X} \sqrt{(x-z)^2 + f(z,p)^2} \tag{2.2}$$

In other words, the point $(x,0)$ is projected onto the epigraph of a convex function, giving $(x^*, f(x^*,p))$. Such projection is known to be unique, i.e. single-valued. Given p, the mapping $x \mapsto x^*$ is here called the *epishift* and is denoted by $x^* = s(x; f(\cdot,p))$.

Note that the epishift consists of two consecutive projections: first project $(x,0)$ to its closest point on the convex positive valued epigraph, then project this back onto the x-domain to x^*. Both projections are nonexpansive functions (that is, continuous single-valued functions with Lipschitz constant 1), and the minimizer x^* is unique when x and p are given. Moreover, if f is continuous jointly in (z,p), the distance minimized is jointly continuous in (z,x,p) because its expression in the definition above is composed of continuous operations. Later on, this will allow applying the simplified maximum theorem to infer joint continuity in the parameters of the epishift, i.e. that $s(x; f(\cdot,p))$ is jointly continuous in (x,p). This epishift will later play a key role, and it only maps the vector argument x to a new Euclidean value x^*; the rest of the coordinates denoted here by p only define a context. It is left as an exercise to the reader to show that x_0 is a fixed point of the epishift, $x_0 = s(x_0; f(\cdot,p))$, if and only if it minimizes the convex and positive $f(x,p)$ over $x \in X$ with convex compact Euclidean set X.[1]

Finally, requiring f to be positive is not a serious restriction, since transforming it to the positive $\exp f$ conserves convexity as well as location of minimum. The former because $\exp(\cdot)$ is convex and increasing so in a composition as the outer mapping it preserves convexity, and the latter because it is strictly increasing. Therefore we will assume later, without restriction, that sectionally convex real valued functions (i.e. convex in some input variables to be minimized along these only) are positive valued.

2.2. Some elementary results

We denote the identity mapping that leaves every point fixed by I, overloading the notation also used for the unit matrix. The Lipschitz range of this function is precisely 1. Now perturbing this with a self-map contraction f will give a strictly positive Lipschitz range $[1-K_f, 1+K_f]$, so that $I - f$ is one-to-one with continuous inverse. However, we need to strengthen this result slightly, towards $I - f$ having a 'large' set of images – it does appeal to intuition that a homeomorphism (bijection continuous in both directions) should not be able to change dimensionality. For this we will show that $I - f$ is an open mapping. The choice of minus sign here is only for notational convenience in the next proof.

Our interest in this perturbation of identity is, that a self-map can be represented naturally in this form, with f giving for each argument the deviation from original position; and Brouwer's FP theorem is about self-maps. The following is Theorem 8.1 in Khamsi and Kirk [2] with minor changes of notation and proof.

[1]Hint: Make a drawing that shows the distance sphere touching the convex graph at the closest point, and show that tangent line to the sphere at the touch point keeps the graph above it, due to convexity. Namely, if there were a graph point below the tangent, its line of sight to the touching point would intersect with interior of the sphere; and this line segment is above the convex graph. This means the touching point was not the closest on the graph, a contradiction. Those familiar with convex analysis can apply a 'separating hyperplane theorem' for the same effect.

Lemma 2.4. *Let G be an open subset of a Banach space X and $f : G \to X$ a contraction mapping. Then $(I - f)(G)$ is an open subset of X.*

Proof. Let $x_0 \in G$ map to $y_0 = x_0 - f(x_0)$. When y is sufficiently close to y_0 we have to show that it is also in the image of G, so $y = x - f(x)$ for some $x \in G$. This is written as a fixed point problem for x, namely $x = y + f(x)$. As f is a contraction with Lipschitz constant $K < 1$, we can apply Banach's FP principle, provided some neighborhood of x_0 is self-mapped by $y + f(\cdot)$. To check we try a direct calculation.

$$\begin{aligned} \|y + f(x) - x_0\| &= \|f(x) - f(x_0) + f(x_0) - x_0 + y\| \\ &\leq \|f(x) - f(x_0)\| + \|y - y_0\| \\ &\leq K\|x - x_0\| + \|y - y_0\| \end{aligned} \tag{2.3}$$

If the ball $B(x_0; r)$ is in G and $\|y - y_0\| \leq (1 - K)r$, then this shows that the ball is mapped into itself by the contraction $y + f(\cdot)$ and the fixed point of this self-map gives $y = x - f(x)$. The ball $B(y_0; (1 - K)r)$ is thereby included in the image set, and since y_0 has an open neighborhood in that set, the proof is complete. ∎

3. Proofs of the main statements

We are now ready to prove the main results. The theorems are not necessarily presented in their widest generality, instead the approach chosen seeks transparency and 'low cost' to the interested readers, both in terms of background and effort required.

The full version of this theorem is called the maximum theorem, or theorem of the maximum, or Berge's maximum theorem. We simplify it and its proof by requiring that we get a unique solution function $x^*(t)$, instead of a set of solutions for each given t. Also, we present the substance for minimizing, not for maximizing, and call doing either of these extremizing. This theorem suffices for our purposes later, and is by itself a useful result on parametric continuity of optimal solutions.

Theorem 3.1 (Berge's maximum theorem, simplified version)**.** *Assume $f(x, t)$ is jointly continuous for $(x, t) \in C \subset \mathbb{R}^n \times T$, with Euclidean x and compact C, and T a metric space. We say that t is feasible if the set*

$$C_t := \{x \in \mathbb{R}^n \mid (x, t) \in C\} \tag{3.1}$$

is nonempty, and assume all of T is feasible by restricting it if necessary. Since C_t is compact, the sets

$$\arg\min_{x \in C_t} f(x, t) \tag{3.2}$$

are well-defined and nonempty for each $t \in T$. If they are singletons, they define a solution function $x^*(t)$ that is continuous *in $t \in T$, and also the value function* $\min_{x \in C_t} f(x, t) = f(x^*(t), t)$ *is continuous in t.*

Proof. Continuity and compactness imply uniform continuity in (x, t), from which it immediately follows that $f(x^*(t), t)$ is continuous in t.

Assume now that the extremizers are unique, defining the function $x^*(t)$. We need to prove its continuity in the 'parameters' t.

Assume to the contrary a discontinuity. Then there exist an $\epsilon > 0$ and a sequence $t_n \to t$ with a subsequence such that $\|x^*(t_{n_k}) - x^*(t)\| > \epsilon$. Due to compactness $(x^*(t_{n_k}), t_{n_k}) \in C$ has a convergent subsequence, which we relabel to have $t_n \to t$ and $x^*(t_n) \to x_0 \neq$

$x^*(t)$. By joint continuity of f, $f(x^*(t_n),t_n) \to f(x_0,t) > f(x^*(t),t) \leftarrow f(x^*(t),t_n)$, the inequality following from shared t and the uniqueness as minimizer of x^*. Thus

$$\lim_{n\to\infty} f(x^*(t_n),t_n) - f(x^*(t),t_n) > 0 \tag{3.3}$$

contradicting that $x^*(t_n)$ are minimizers, and a discontinuity is not possible. ∎

Clearly this theorem applies equally to any extremization, and essentially says that if there was *joint* continuity to begin with, then continuity is preserved when an argument is eliminated (by minimizing over it), this being the more obvious claim. Further the extremizers have a locus continuous in the parameters t *if* this locus is a single-valued function; this being the far from obvious claim. Having such continuous locus also implies the more obvious claim of continuous optimal values.

A direct application of the theorem above to epishift gives that $s(x; f(\cdot,p))$ is jointly continuous in (x,p), since all the conditions are satisfied ascertaining this parametric continuity (when p has locally a compact neighborhood, or is overall constrained within a compact set).

We now prove a theorem equivalent to Brouwer's FP theorem for a unit ball in finite dimensional Euclidean space. We denote the open unit ball by $B^n := \{x \in \mathbb{R}^n : \|x\| < 1\}$, while its closure is $\bar{B}^n$, and the unit sphere boundary of these is $S^{n-1} := \{x : \|x\| = 1\}$. The superscripts show dimensionality of each item, and for the sphere this is not the dimensionality of the underlying Euclidean space but one less.

Theorem 3.2. *There is no smooth map $I + g : \bar{B}^n \to S^{n-1}$ holding all points of S^{n-1} fixed.*

Note that the mapping is given in deviation form, so g has to vanish at a fixed point. The proof we give is modified from Howard [1], specifically to avoid reference to the inverse function theorem. Twice continuous differentiability suffices for the proof below as smoothness requirement, but we choose to not distract from the essence with such specification.

Proof. Assume to the contrary that the points on the unit sphere are held fixed. For $t \in [0,1]$ define $f_t(x) := x + tg(x) = (1-t)x + t(I+g)(x)$, which is a self-map on $\bar{B}^n$ by the triangle inequality applied to the latter expression. Since the deviations g on the unit sphere are zero, they remain zero with this scaling: the unit sphere is fixed for f_t also. Based on smoothness the deviation g has a Lipschitz constant on the compact domain, and $tg(\cdot)$ is a contraction for small enough t. Then f_t is an open one-to-one mapping based on a preceding lemma applied to $I - (-tg)$. In particular, the open unit ball is mapped to an open set within the closed unit ball, while the boundary sphere is held fixed, and no other point can get mapped into this bounding sphere by the one-to-one map. If some point in the unit ball were not an image, the closed compact image of the compact unit ball would include some boundary inside the unit ball. On the other hand, only the open unit ball contributes to image points in the open unit ball, and its total image is open, not including any boundary. This contradiction shows that there is no boundary for the image inside the unit ball, instead the mapping must be onto the whole unit ball. The above deduction holds for all small enough t to give contractions tg. To show that it holds for all t, we note the following. The volume covered by the image is, with obvious shorthand for the Jacobian determinant,

$$\int_{\bar{B}^n} \det f_t'(x)\ dx = \int_{\bar{B}^n} \det(I + tg'(x))\ dx \tag{3.4}$$

The determinant is clearly a polynomial in t and so is the volume. Since the volume stays constant (that of the unit ball) for small values of t, that latter polynomial is a constant and the volume is fixed for all real t. In particular, it will not go to zero for $t = 1$, which it would if the image was the boundary sphere. Therefore a smooth retraction of the unit ball to its boundary, holding the boundary fixed, is not possible. ∎

The Brouwer FP theorem (for unit ball) is implied by the above as follows (we only sketch the reasoning). Assume a smooth self-map of the unit ball has no fixed point. Then the image point can always be connected to the source point with a line segment, which can be continued to the bounding sphere. This mapping to the bounding sphere would provide a smooth retraction, holding the boundary fixed, a contradiction. Therefore at least one fixed point must exist.

Note that unlike in the Banach FP theorem, a FP based on Brouwer's theorem need not be unique. Similarly, the Nash equilibrium below need not be unique either. Simple counterexamples are immediate with the use of identity or constant functions.

The proof above assumed smoothness, and any continuous function on a compact domain can be approximated arbitrarily closely with a smooth function. By this means Brouwer's FP theorem can be extended to not require smoothness. Further extension to any set homeomorphic to a unit ball then follows, and this includes convex compact Euclidean sets.

We will next prove the existence of a Nash equilibrium, and to avoid notational complications we give the proof for only two players. No substantial changes enter the argument with any finite number of players.

Let the first player have control of some action variables x, and seek to minimize $f(x, y)$ that is convex in x and jointly continuous in (x, y). Similarly, the second player manipulates y to minimize the jointly continuous $g(x, y)$ that is convex in y. Our goal is to show that the players can find a steady 'equilibrium' where both can hold their action variables constant and the sectional minima are simultaneously achieved.

To ensure that the sectional minima exist, we assume that each action variable is constrained within a convex and compact set in Euclidean space.

It is noted that we can without restriction assume f and g to be positive valued, as discussed earlier in the context of epishifts. For chosen actions, only the locations of the minima are of importance, not the values of the goal functions.

Assume our initial state is (x, y) and apply the epishifts sequentially as follows.

$$x_1 \quad := \quad s(x; f(\cdot, y)) \tag{3.5}$$

$$y_1 \quad := \quad s(y; g(x_1, \cdot)) \tag{3.6}$$

Because all operations are jointly continuous, the mapping $(x, y) \mapsto (x_1, y_1)$ is also such. Further this is a self-map on a compact convex Euclidean set, therefore Brouwer's FP theorem applies. But at a fixed point, both the above epishifts must also have their fixed points, so f and g are both at their sectional minima, respectively along x and along y. This joint minimum is a Nash equilibrium point.

Since with multiple players we would simply add more epishifts to the self-map, always using the most recent available updates of the action variables in epishifts that follow, the above reasoning confirms the following theorem.

Theorem 3.3. *Assume that each of the players $i = 1, \dots, n$ controls its action variables x_i within the bounds of a compact convex finite dimensional Euclidean set. Given continuous functions f_i that can depend on all action variables, such that each f_i is convex in x_i when the other action variables are held fixed, then there exists at least one Nash equilibrium $(x_i^*)_{i=1}^n$ at which each f_i is minimized among the actions allowed to player i.*

Effectively this implies that if each player follows faithfully their own goal function, that is collaborative enough for possibility of steady coexistence, regardless of whether those goal functions are objectionable to some other players. Following the rules in a game is already a form of collaboration.

Without a proof or even any equations, we note the following further connection. The celebrated von Neumann's minimax theorem can be considered a very special case of Nash equilibrium. Instead of two distinct functions, the minimax theorem has a single jointly continuous function that is jointly convex in variables labeled x (so over these we can minimize) and jointly concave in variables y (over which we can maximize). The difficulty in proving the minimax theorem is to show that a so-called saddle point exists, one that sectionally minimizes over x and maximizes over y simultaneously. Taking the negative of the given function to be the goal function of the second player, to be minimized over y, gives the situation in above theorem. Now the Nash equilibrium is equivalent to having a saddle point. So, the minimax theorem is easily accessible once the Nash theorem has been reached (subject to some compactness conditions and not considering extended real values).

4. Discussion

An account of the history of Brouwers FP theorem, as regards its precedents, prior alternative proofs, and extensions, is available with ample references online in the MS-thesis of Stuckless [3].

Less emphasis on the priors and more on the multitude of extensions and applications during the past century is given in the extensive review by Park [4] from 1999, citing nearly 300 references.

The approach taken here may be novel, to the best knowledge of the authors, and is based on personal preferences with emphasis on the ease of access: for background, standard calculus with some metric space topology suffice.

The continuing importance of Brouwer's FP theorem in its base form, for a compact convex set in finite dimensional Euclidean space, is evidenced by recent publications pursuing elementary proofs. In 2015 Dhompongsa and Nantadilok [5] published a proof using induction on the finite dimension, with the Tietze extension theorem in background requirements for readers. In contrast, the more recent 2018 approach in [6] is combinatorial, with Bolzano-Weierstrass theorem needed in a limit process. While these approaches can also be considered elementary, they remain burdened by especially notational complications – so offering the alternative in this paper can still be considered well motivated, despite the long history and also other recent work.

The simplified maximum theorem has not been included in conventional mathematical analysis syllabi, or at least not in the standard textbooks, even though it is also accessible to a wide readership and eminently useful. The full version, available for example in the comparatively modern books by Ok [7] and by Sundaram [8], employs much more demanding machinery of set-valued functions (also called correspondencies) whose upper or lower hemicontinuity replace the familiar continuity concept of single-valued functions. It is appealing to have a widely accessible simplified version that guarantees parametric continuity of the optimizer when it is unique at fixed parameters, provided some fairly nonrestrictive conditions.

Combining the simplified maximum theorem with our epishift mappings, and applying Brouwers FP theorem, provided a rather straightforward proof of existence of the Nash equilibrium. While the authors chose to call a specific mapping 'epishift', it is possible (perhaps even likely) that this mapping has been introduced earlier elsewhere for convex positive functions with another naming convention. Epishifts do make the Nash equilibrium appear 'natural' once the Brouwer FP theorem is familiar. Extensive reading on Nash equilibria is available for example in the book edited by Chinchuluun [9].

The epishift for $f > 0$ can be expressed in terms of the proximal mapping from convex analysis as $prox_{(1/2)f^2}$, or it could be viewed as Tikhonov regularization applied when minimizing f^2.

The celebrated von Neumann minimax theorem can be viewed as a nearly trivial consequence of Nash equilibrium, but of course these two emerged historically in the reversed order neither was trivial at its time.

The overall approach presented is hopefully pedagogically useful, and while no claims of originality can be made since all actual results are 'classic', the approach taken may still have been entertaining to some readers. The authors believe this approach allows access to several key results at a comparatively low effort, and with relatively modest demands on background knowledge.

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