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The Relative Rank of Orientation-preserving or Orientation-reversing Transformation Semigroups with Restricted Range on a Finite Chain

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Abstract Let S be a semigroup and let G be a subset of S. A set G is a generating set of S which is denoted by $\langle G\rangle = S$. The rank of S is the minimal size or cardinality of a generating set of S, i.e. $rank(S) := \min\{|G| : G \subseteq S, \langle G\rangle = S\}$. Then the idea of rank leads to a new definition of rank is called the relative rank of S modulo U is the minimal size of a subset G such that $G \cup U$ generates S, i.e. $rank(S : U) := \min\{|G| : G \subseteq S, \langle G \cup U\rangle = S\}$. A set $G \subseteq S$ with $\langle G \cup U\rangle = S$ is called a generating set of S modulo U. Let X be a finite chain and let Y be a subchain of X. Denote by $\mathcal{T}(X, Y)$ the set of all full transformations from X to Y which is so-called the full transformation semigroup with restricted range Y and it was firstly introduced and studied by Symons in 1975. In this work, we determine the relative rank of the semigroup $\mathcal{OPR}(X, Y)$ of all orientation-preserving or orientation-reversing transformations with restricted range modulo the semigroup $\mathcal{O}(X, Y)$ of all order-preserving transformations with restricted range. In addition, we also determine the relative rank of the semigroup $\mathcal{OD}(X, Y)$ of all order-preserving or order-reversing transformations with restricted range modulo the semigroup $\mathcal{O}(X, Y)$ of all order-preserving transformations with restricted range. Furthermore, we obtain that $\mathcal{O}(X, Y) \subseteq \mathcal{OD}(X, Y) \subseteq \mathcal{OPR}(X, Y)$ and they are subsemigroups of $\mathcal{T}(X, Y)$.

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1. Introduction and Preliminaries

Let $X = \{1 < 2 < \cdots < n\}$ be a finite chain with $|X| = n$ where $n \in \mathbb{N}$. We denote by $\mathcal{T}(X)$ the semigroup of all full transformations under the composition of functions. In this paper, we will compose functions from the left to the right, i.e. $x(\alpha\beta) = (x\alpha)\beta$ for all $x \in X$. Let $\alpha \in \mathcal{T}(X)$. We denote by $im(\alpha)$ the image of α, i.e. $im(\alpha) := \{x\alpha : x \in X\}$ and denote by $rank(\alpha)$ the cardinality of $im(\alpha)$, i.e. $rank(\alpha) := |im(\alpha)|$. The kernel of

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α is the set $\ker(\alpha) := \{(x, y) \in X \times X : x\alpha = y\alpha\}$. It is an equivalence relation on X and it is called $\ker(\alpha)$-classes or $\ker(\alpha)$-blocks. A set $T \subseteq X$ is called a transversal of $\ker(\alpha)$ if $|B \cap T| = 1$ for all $\ker(\alpha)$-classes B. For subsets B_1, B_2 of X, $B_1 < B_2$ means $x_1 < x_2$ for all $x_1 \in B_1$ and for all $x_2 \in B_2$. For a subset A of X, $\alpha|_A$ is a mapping from A to X with $x(\alpha|_A) := x\alpha$ for all $x \in A$. Then $\alpha|_A$ is so-called the mapping α restricted to A.

Let G be a subset of a semigroup S. Then a generating set G of S is denoted by $\langle G\rangle = S$. The rank of S is the minimal size of a generating set G, i.e. $rank(S) := \min\{|G| : G \subseteq S, \langle G\rangle = S\}$. The relative rank of S modulo U is the minimal size of a subset $G \subseteq S$ such that $G \cup U$ generates S, i.e. $rank(S : U) := \min\{|G| : G \subseteq S, \langle G \cup U\rangle = S\}$. Therefore, we obtain immediately that $rank(S : \emptyset) = rank(S)$, $rank(S : S) = 0$, $rank(S : A) = rank(S : \langle A\rangle)$ and $rank(S : A) = 0$ if and only if $\langle A\rangle = S$. In addition, a set $G \subseteq S$ with $\langle G \cup U\rangle = S$ is called a generating set of S modulo U. The relative rank generalizes the rank of a semigroup which was introduced by Howie, Ruškuc and Higgins [10].

A transformation $\alpha \in \mathcal{T}(X)$ is called orientation-preserving (orientation-reversing, respectively) if there is a decomposition $X = [\alpha]_1 \cup [\alpha]_2$ with $[\alpha]_1 < [\alpha]_2$, $y_1\alpha \geq y_2\alpha$ ($y_1\alpha \leq y_2\alpha$, respectively) for all $y_1 \in [\alpha]_1$ and $y_2 \in [\alpha]_2$, and $x\alpha \leq y\alpha$ ($x\alpha \geq y\alpha$, respectively) for all $x \leq y \in [\alpha]_1$ or $x \leq y \in [\alpha]_2$. If $[\alpha]_2 = \emptyset$ then α is called order-preserving. Moreover, if $[\alpha]_1 = \emptyset$ with $x\alpha \geq y\alpha$ for all $x \leq y \in [\alpha]_2$ then α is called order-reversing. Notice that the product of two orientation-preserving transformations is an orientation-preserving and the product of two orientation-reversing transformations is also an orientation-preserving. We denote by $\mathcal{O}(X)$, $\mathcal{OD}(X)$, $\mathcal{OP}(X)$, $\mathcal{OR}(X)$ and $\mathcal{OPR}(X)$ the semigroup of all order-preserving transformations, the semigroup of all order-preserving or order-reversing transformations, the semigroup of all orientation-preserving transformations, the set of all orientation-reversing transformations and the semigroup of all orientation-preserving or orientation-reversing transformations, respectively. It is clear that $\mathcal{O}(X)$ is a proper subsemigroup of $\mathcal{OD}(X)$, $\mathcal{OP}(X)$ and $\mathcal{OPR}(X)$. In addition, we also know that $\mathcal{OD}(X)$ is a proper subsemigroup of $\mathcal{OPR}(X)$. The semigroup $\mathcal{OP}(X)$ has been widely studied (see in [1], [2], [3], [4], [6] and [13]). The rank of $\mathcal{OP}(X)$, $\mathcal{O}(X)$ and $\mathcal{T}(X)$ are equal $2, n$ and 3, respectively (see [1], [4] and [10]). Moreover, we obtain that $rank(\mathcal{OP}(X) : \mathcal{O}(X)) = 1$, $rank(\mathcal{T}(X) : \mathcal{O}(X)) = 2$, and $rank(\mathcal{T}(X) : \mathcal{OP}(X)) = 1$ (see in [2] and [10]).

Let $Y = \{l_1 < l_2 < \cdots < l_m\}$ be a subchain of X with $|Y| = m$ and $1 < m < n$. Then we consider the following sets:

$$\begin{aligned}
\mathcal{T}(X, Y) &:= \{\alpha \in \mathcal{T}(X) : im(\alpha) \subseteq Y\},\\
\mathcal{O}(X, Y) &:= \{\alpha \in \mathcal{O}(X) : im(\alpha) \subseteq Y\},\\
\mathcal{OD}(X, Y) &:= \{\alpha \in \mathcal{OD}(X) : im(\alpha) \subseteq Y\},\\
\mathcal{OP}(X, Y) &:= \{\alpha \in \mathcal{OP}(X) : im(\alpha) \subseteq Y\},\\
\mathcal{OPR}(X, Y) &:= \{\alpha \in \mathcal{OPR}(X) : im(\alpha) \subseteq Y\}.
\end{aligned}$$

Then they are subsemigroups of $\mathcal{T}(X, Y)$ and $\mathcal{T}(X)$ under the composition of funtions. The semigroup $\mathcal{T}(X, Y)$ is defined by Symons and it is called the full transformation semigroup with restricted range [12]. The other semigroups are introduced by Fernandes et al. in [5] and [6]. Moreover, the transformation semigroups with restricted range have been widely investigated (see in [5], [7] and [11]). The rank of $\mathcal{T}(X, Y)$ is equal to $S(n, m)$ which is the stirling number of second kind [9]. In [5] and [6], the authors proved that $rank(\mathcal{O}(X,Y)) = \binom{n-1}{m-1} + |Y^\sharp|$ where $Y^\sharp$ is the set of captive elements and $rank(\mathcal{OP}(X,Y)) = \binom{n}{m}$. In [13], we obtained that $rank(\mathcal{T}(X,Y) : \mathcal{O}(X,Y))$ is equal to $S(n,m) - \binom{n-1}{m-1}$ or $S(n,m) - \binom{n-1}{m-1} + 1$ depends on set Y.

In this paper, we determine the relative rank of some subsemigroups of $\mathcal{T}(X,Y)$. In section 2.1, we calculate the relative rank $\mathcal{OD}(X,Y)$ modulo $\mathcal{O}(X,Y)$. In section 2.2, we describe the relative rank $\mathcal{OP}(X,Y)$ modulo $\mathcal{O}(X,Y)$. Finally, we determine the relative rank $\mathcal{OPR}(X,Y)$ modulo $\mathcal{O}(X,Y)$ in section 2.3.

2. Main Results

2.1. Reletive Rank of $\mathcal{OD}(X,Y)$ Modulo $\mathcal{O}(X,Y)$

In this section, we determine the relative rank of $\mathcal{OD}(X,Y)$ modulo $\mathcal{O}(X,Y)$. First, we define a mapping $\beta^* : X \to Y$ by

$$x\beta^* := \begin{cases} l_m & \text{if } x < l_1 \\ l_{m-i+1} & \text{if } l_i \leq x < l_{i+1},\ 1 \leq i < m \\ l_1 & \text{if } x \geq l_m\ . \end{cases}$$

It is clear that β^* is order-reversing, i.e. $\beta^* \in \mathcal{OD}(X,Y)$.

Proposition 2.1. $\mathcal{OD}(X,Y) = \langle \mathcal{O}(X,Y), \beta^* \rangle$.

Proof. Let $\alpha \in \mathcal{OD}(X,Y) \setminus \mathcal{O}(X,Y)$. Define a mapping $\theta : X \to Y$ by $x\theta := x(\alpha\beta^*)$ for all $x \in X$. Then we observe that $\theta \in \mathcal{O}(X,Y)$ because the product of two order-reversing transformations is an order-preserving transformation. Let $x \in X$. Therefore, $x(\theta\beta^*) = x(\alpha\beta^*)\beta^* = x\alpha(\beta^*\beta^*) = x\alpha(id|_Y) = x\alpha$, i.e. $\alpha = \theta\beta^*$. Hence, $\mathcal{OD}(X,Y) = \langle \mathcal{O}(X,Y), \beta^* \rangle$. ■

Proposition 2.2. $rank(\mathcal{OD}(X,Y) : \mathcal{O}(X,Y)) = 1$.

Proof. By Proposition 2.1, we obtain that $rank(\mathcal{OD}(X,Y) : \mathcal{O}(X,Y)) \leq 1$. Since $\mathcal{O}(X,Y)$ is a proper subsemigroup of $\mathcal{OD}(X,Y)$, we obtain that $rank(\mathcal{OD}(X,Y) : \mathcal{O}(X,Y)) \geq 1$. Altogether, we can conclude that $rank(\mathcal{OD}(X,Y) : \mathcal{O}(X,Y)) = 1$. ■

2.2. Reletive Rank of $\mathcal{OP}(X,Y)$ Modulo $\mathcal{O}(X,Y)$

In this section, we study and describe the relative rank of $\mathcal{OP}(X,Y)$ modulo $\mathcal{O}(X,Y)$ [3]. Define the set $\mathcal{P}'$ by

$$\mathcal{P}' := \{\ker(\alpha) : \alpha \in \mathcal{OP}(X,Y), rank(\alpha) = m\} \setminus \{\ker(\alpha) : \alpha \in \mathcal{O}(X,Y), rank(\alpha) = m\}.$$

Therefore, $\mathcal{P}'$ is the set of all partitions of X into $m-1$ intervals and one block, which is the union of two intervals B_1 and B_n such that $1 \in B_1$ and $n \in B_n$. For each $P' \in \mathcal{P}'$, we fix an $\alpha_{P'} \in \mathcal{OP}(X,Y) \setminus \mathcal{O}(X,Y)$ with $\ker(\alpha_{P'}) = P'$. Then we can compute the cardinality of $\mathcal{P}'$ as the following lemma.

Lemma 2.3. *[3]* $|\mathcal{P}'| = \binom{n-1}{m}$.

Next, we define a mapping $\eta^* : X \to Y$ by

$$x\eta^* := \begin{cases} l_{i+1} & \text{if } l_i \leq x < l_{i+1},\ 1 \leq i < m \\ l_1 & \text{if } l_m \leq x \text{ or } x < l_1. \end{cases}$$

It is easy to see that $\eta^* \in \mathcal{OP}(X,Y)$. Then we can state the main result as the following theorem.

Theorem 2.4. *[3] $\mathcal{OP}(X,Y) = \langle \mathcal{O}(X,Y), \{\alpha_{P'} : P' \in \mathcal{P}'\}, \eta^* \rangle$.*

Therefore, we get the relative rank of $\mathcal{OP}(X,Y)$ modulo $\mathcal{O}(X,Y)$ as follows:

Proposition 2.5. *[3] If $1 \notin Y$ or $n \notin Y$, then $rank(\mathcal{OP}(X,Y) : \mathcal{O}(X,Y)) = \binom{n-1}{m}$.*

Proposition 2.6. *[3] If $\{1, n\} \subseteq Y$, then $rank(\mathcal{OP}(X,Y) : \mathcal{O}(X,Y)) = 1 + \binom{n-1}{m}$.*

2.3. Reletive Rank of $\mathcal{OPR}(X,Y)$ Modulo $\mathcal{O}(X,Y)$

In this section, we determine the relative rank of $\mathcal{OPR}(X,Y)$ modulo $\mathcal{O}(X,Y)$. For $|Y| = 2$, we obtain that the semigroup $\mathcal{OPR}(X,Y)$ and the semigroup $\mathcal{OP}(X,Y)$ are coincide. Then we obtain immediately the following propositions.

Proposition 2.7. *If $|Y| = 2$ and $1 \notin Y$ or $n \notin Y$, then $rank(\mathcal{OPR}(X,Y) : \mathcal{O}(X,Y)) = \binom{n-1}{2}$.*

Proposition 2.8. *If $|Y| = 2$ and $\{1, n\} \subseteq Y$, then $rank(\mathcal{OPR}(X,Y) : \mathcal{O}(X,Y)) = 1 + \binom{n-1}{2}$.*

So, the rest of this section will consider a set Y is a subchain of X with $|Y| \geq 3$. Notice that

$$\{\ker(\alpha) : \alpha \in \mathcal{OP}(X,Y), rank(\alpha) = m\} = \{\ker(\alpha) : \alpha \in \mathcal{OPR}(X,Y), rank(\alpha) = m\}.$$

Define the set $\mathcal{P}$ by

$$\mathcal{P} := \{\ker(\alpha) : \alpha \in \mathcal{OPR}(X,Y), rank(\alpha) = m\} \setminus \{\ker(\alpha) : \alpha \in \mathcal{O}(X,Y), rank(\alpha) = m\}.$$

For each $P \in \mathcal{P}$, we fix an $\varphi_P \in \mathcal{OPR}(X,Y) \setminus \mathcal{OP}(X,Y)$ with $\ker(\varphi_P) = P$. Then we obtain that $|\mathcal{P}| = |\mathcal{P}'|$, i.e. $|\mathcal{P}| = \binom{n-1}{m}$. Next, we define a mapping $\beta_1^* : X \to Y$ with $\ker(\beta_1^*) = \ker(\eta^*)$ by

$$x\beta_1^* := \begin{cases} l_1 & \text{if } l_1 \leq x < l_2 \\ l_{m-i+1} & \text{if } l_{i+1} \leq x < l_{i+2},\ 1 \leq i < m-1 \\ l_2 & \text{if } l_m \leq x \text{ or } x < l_1. \end{cases}$$

It is easy to see that $\ker(\beta_1^*) = \ker(\eta^*)$ and $\beta_1^* \in \mathcal{OPR}(X,Y) \setminus \mathcal{OP}(X,Y)$. Next, we define a mapping $\beta_2^* : X \to Y$ by

$$x\beta_2^* := \begin{cases} l_2 & \text{if } l_1 \leq x < l_2 \\ l_1 & \text{if } l_2 \leq x < l_3 \\ l_{m-i+1} & \text{if } l_{i+2} \leq x < l_{i+3},\ 1 \leq i < m-2 \\ l_3 & \text{if } l_m \leq x \text{ or } x < l_1. \end{cases}$$

It is clear that $\beta_2^* \in \mathcal{OPR}(X,Y) \setminus \mathcal{OP}(X,Y)$. Since $\ker(\beta_1^*) = \ker(\eta^*)$ and $im(\beta_1^*)$ is a transversal of $\ker(\beta_2^*)$, we can compute that $\beta_1^*\beta_2^* = \eta^*$.

Then we can state the main proposition of this section to show that $\mathcal{OPR}(X,Y) = \langle \mathcal{O}(X,Y), \{\varphi_P : P \in \mathcal{P}\}, \beta_1^*, \beta_2^*\rangle$.

Proposition 2.9. $\mathcal{OPR}(X,Y) = \langle \mathcal{O}(X,Y), \{\varphi_P : P \in \mathcal{P}\}, \beta_1^*, \beta_2^*\rangle$.

Proof. Let $\beta \in \mathcal{OPR}(X,Y)$. Then we will consider two cases.

Case 1. $\beta \in \mathcal{OP}(X,Y)$. For each $P \in \mathcal{P}$, we put $\alpha_{P'} := \varphi_P\beta_1^*$, where $\varphi_P \in \mathcal{OPR}(X,Y) \setminus \mathcal{OP}(X,Y)$. Then $\alpha_{P'} \in \mathcal{OP}(X,Y) \setminus \mathcal{O}(X,Y)$ with $rank(\alpha_{P'}) = m$ and $\ker(\alpha_{P'}) = \ker(\varphi_P)$. Let $B := \{\alpha_{P'} : P' \in \mathcal{P}'\}$. By Theorem 2.4, we get that $\mathcal{OP}(X,Y) = \langle \mathcal{O}(X,Y), \{\alpha_{P'} : P' \in \mathcal{P}'\}, \eta^*\rangle$. Therefore, $\beta \in \mathcal{OP}(X,Y) = \langle \mathcal{O}(X,Y), \{\alpha_{P'} : P' \in \mathcal{P}'\}, \eta^*\rangle = \langle \mathcal{O}(X,Y), \{\alpha_{P'} : P' \in \mathcal{P}'\}, \beta_1^*\beta_2^*\rangle \subseteq \langle \mathcal{O}(X,Y), \{\varphi_P : P \in \mathcal{P}\}, \beta_1^*, \beta_2^*\rangle$.

Case 2. $\beta \in \mathcal{OPR}(X,Y) \setminus \mathcal{OP}(X,Y)$. Put $\theta := \beta\beta_1^*$. Then $\theta \in \mathcal{OP}(X,Y)$ because the product of two orientation-reversing transformations is orientation-preserving. From Case 1, we have $\theta \in \mathcal{OP}(X,Y) \subseteq \langle \mathcal{O}(X,Y), \{\varphi_P : P \in \mathcal{P}\}, \beta_1^*, \beta_2^*\rangle$. Let $x \in X$. Therefore, $x(\theta\beta_1^*) = x(\beta\beta_1^*\beta_1^*) = x\beta(\beta_1^*\beta_1^*) = x\beta(id|_Y) = x\beta$, i.e. $\beta = \theta\beta_1^*$. Hence, $\beta \in \langle \mathcal{O}(X,Y), \{\varphi_P : P \in \mathcal{P}\}, \beta_1^*, \beta_2^*\rangle$.

Altogerther, we obtain that $\mathcal{OPR}(X,Y) = \langle \mathcal{O}(X,Y), \{\varphi_P : P \in \mathcal{P}\}, \beta_1^*, \beta_2^*\rangle$. ■

Lemma 2.10. *Let $A \subseteq \mathcal{OPR}(X,Y) \setminus \mathcal{O}(X,Y)$ such that $\langle \mathcal{O}(X,Y), A\rangle = \mathcal{OPR}(X,Y)$. Then there is a set $A' \subseteq A$ with $\{\ker(\alpha) : \alpha \in A'\} = \mathcal{P}$.*

Proof. Assume that there is $P \in \mathcal{P}$ with $P \notin \{\ker(\alpha) : \alpha \in A\}$. Since $\varphi_P \in \mathcal{OPR}(X,Y) = \langle \mathcal{O}(X,Y) \cup A\rangle$, there are $\theta_1 \in \mathcal{O}(X,Y) \cup A$ and $\theta_2 \in \mathcal{OPR}(X,Y)$ sucht that $\varphi_P = \theta_1\theta_2$. Since $rank(\varphi_P) = m$, we obtain that $\ker(\varphi_P) = \ker(\theta_1)$, i.e. $\ker(\theta_1) = P$. Hence, $\theta_1 \notin A$ and $\theta_1 \notin \mathcal{O}(X,Y)$ because $P \notin \{\ker(\alpha) : \alpha \in \mathcal{O}(X,Y)\}$ that is a contradiction. Therefore, there is a set $A' \subseteq A$ with $\{\ker(\alpha) : \alpha \in A'\} = \mathcal{P}$. ■

To obtain the main results of section we will consider two possibilities. First, we consider the case $|X \setminus Y| = 1$, i.e. $|X| = m+1$. So, $|\mathcal{P}| = \binom{m+1-1}{m} = 1$ that means $\mathcal{P} = \{P\}$. Then we obtain the following results.

Theorem 2.11. *If $|X \setminus Y| = 1$ and $1 \notin Y$ or $n \notin Y$, then we have $rank(\mathcal{OPR}(X,Y) : \mathcal{O}(X,Y)) = 2$.*

Proof. Since $1 \notin Y$ or $n \notin Y$, we can assume without loss of generaltity that $\beta_1^* = \varphi_P$. By Proposition 2.9, we have $\mathcal{OPR}(X,Y) = \langle \mathcal{O}(X,Y), \varphi_P, \beta_2^*\rangle$, i.e. $rank(\mathcal{OPR}(X,Y) : \mathcal{O}(X,Y)) \leq 2$.

Let $A \subseteq \mathcal{OPR}(X,Y) \setminus \mathcal{O}(X,Y)$ such that $\langle \mathcal{O}(X,Y), A\rangle = \mathcal{OPR}(X,Y)$. By Lemma 2.10, there is a set $A' \subseteq A$ with $\{\ker(\alpha) : \alpha \in A'\} = \mathcal{P}$, i.e. $rank(\mathcal{OPR}(X,Y) : \mathcal{O}(X,Y)) \geq |A'| \geq |\mathcal{P}| \geq 1$. Assume that $\langle \mathcal{O}(X,Y), \varphi_P\rangle = \mathcal{OPR}(X,Y)$. We define a mapping $\beta : X \to Y$ by

$$x\beta := \begin{cases} l_2 & \text{if } x \leq l_1 \\ l_1 & \text{if } l_2 \leq x < l_3 \\ l_{m-i+1} & \text{if } l_{i+1} \leq x < l_{i+2},\ 1 \leq i < m-2 \\ l_3 & \text{if } l_m \leq x\ . \end{cases}$$

So, we can verify that $\beta \in \mathcal{OPR}(X,Y) \setminus \mathcal{O}(X,Y)$ and $(1,n) \notin \ker(\beta)$. Since $\beta \in \mathcal{OPR}(X,Y) \setminus \mathcal{O}(X,Y) \subseteq \mathcal{OPR}(X,Y) = \langle \mathcal{O}(X,Y), \varphi_P\rangle$, there are $\theta_1, \theta_2, \ldots, \theta_k \in \mathcal{O}(X,Y) \cup$

$\{\varphi_P\}$ such that $\beta = \theta_1\theta_2\cdots\theta_k$. Since $rank(\beta) = m$ and $(1, n) \in \ker(\varphi_P)$, we obtain that $(1, n) \notin \ker(\theta_i)$ for all $i \in \{2, 3, \ldots, l\}$ that implies $\theta_2\cdots\theta_l \in \mathcal{O}(X, Y)$. Since $rank(\beta) = m$, we get that $\ker(\beta) = \ker(\theta_1)$. If $\theta_1 \in \mathcal{O}(X, Y)$, then we have $\theta_1\theta_2\cdots\theta_k \in \mathcal{O}(X, Y)$ that is a contradiction. If $\theta_1 = \varphi_P$, then we have $(1, n) \in \ker(\beta)$ that contradicts with $(1, n) \notin \ker(\beta)$. Then $rank(\mathcal{OPR}(X, Y) : \mathcal{O}(X, Y)) \geq 2$. Altogether, we obtain that $rank(\mathcal{OPR}(X, Y) : \mathcal{O}(X, Y)) = 2$. ∎

Theorem 2.12. *If $|X \setminus Y| = 1$ and $\{1, n\} \subseteq Y$, then $rank(\mathcal{OPR}(X, Y) : \mathcal{O}(X, Y)) = 3$.*

Proof. By Proposition 2.9, we obtain that $\mathcal{OPR}(X, Y) = \langle \mathcal{O}(X, Y), \varphi_P, \beta_1^*, \beta_2^* \rangle$, i.e. $rank(\mathcal{OPR}(X, Y) : \mathcal{O}(X, Y)) \leq 3$.

Let $A \subseteq \mathcal{OPR}(X, Y) \setminus \mathcal{O}(X, Y)$ such that $\langle \mathcal{O}(X, Y), A \rangle = \mathcal{OPR}(X, Y)$. By Lemma 2.10, there is a set $A' \subseteq A$ with $\{\ker(\alpha) : \alpha \in A'\} = \mathcal{P}$, i.e. $rank(\mathcal{OPR}(X, Y) : \mathcal{O}(X, Y)) \geq |A'| \geq |\mathcal{P}| = 1$. Assume that $\langle \mathcal{O}(X, Y), \varphi_P \rangle = \mathcal{OPR}(X, Y)$. By the definition of η^*, we have $\eta^* \in \mathcal{OP}(X, Y) \setminus \mathcal{O}(X, Y) \subseteq \mathcal{OPR}(X, Y)$, where $\ker(\eta^*) \notin \mathcal{P}$ because $(1, n) \notin \ker(\eta^*)$. Since $\eta^* \in \mathcal{OP}(X, Y) \setminus \mathcal{O}(X, Y) \subseteq \mathcal{OPR}(X, Y) = \langle \mathcal{O}(X, Y), \varphi_P \rangle$, there are $\theta_1, \ldots, \theta_l \in \mathcal{O}(X, Y) \cup \{\varphi_P\}$ such that $\eta^* = \theta_1\cdots\theta_l$. Since $rank(\eta^*) = m$ and $\{1, n\} \subseteq Y$, we obtain that $(1, n) \notin \ker(\theta_i)$ for all $i \in \{2, 3, \ldots, l\}$ that implies $\theta_2\cdots\theta_l \in \mathcal{O}(X, Y)$. Since $rank(\eta^*) = m$, we get $\ker(\eta^*) = \ker(\theta_1)$. If $\theta_1 \in \mathcal{O}(X, Y)$, then we have $\theta_1\theta_2\cdots\theta_k \in \mathcal{O}(X, Y)$ that is a contradiction. If $\theta_1 = \varphi_P$, then we have $(1, n) \in \ker(\eta^*)$ that contradicts with $(1, n) \notin \ker(\eta^*)$. That means $\eta^* \notin \langle \mathcal{O}(X, Y), \varphi_P \rangle$, i.e. $rank(\mathcal{OPR}(X, Y) : \mathcal{O}(X, Y)) \geq 2$. Next, we assume that $\langle \mathcal{O}(X, Y), \varphi_P, \eta^* \rangle = \mathcal{OPR}(X, Y)$. Then we define a mapping $\beta : X \to Y$ by

$$x\beta := \begin{cases} l_2 & \text{if } x \leq l_1 \\ l_1 & \text{if } l_2 \leq x < l_3 \\ l_{m-i+1} & \text{if } l_{i+1} \leq x < l_{i+2},\ 1 \leq i < m-2 \\ l_3 & \text{if } l_m \leq x\ . \end{cases}$$

So, we can verify that $\beta \in \mathcal{OPR}(X, Y) \setminus \mathcal{OP}(X, Y)$ and $(1, n) \notin \ker(\beta)$. Since $\beta \in \mathcal{OPR}(X, Y) \setminus \mathcal{OP}(X, Y) \subseteq \mathcal{OPR}(X, Y) = \langle \mathcal{O}(X, Y), \varphi_P, \eta^* \rangle$, there are $\xi_1, \xi_2, \ldots, \xi_k \in \mathcal{O}(X, Y) \cup \{\varphi_P, \eta^*\}$ such that $\beta = \xi_1\xi_2\cdots\xi_k$. Since $rank(\beta) = m$ and $\{1, n\} \subseteq Y$, we obtain that $(1, n) \notin \ker \xi_i$ for all $i \in \{2, 3, \ldots, k\}$ that implies $\xi_2, \ldots, \xi_k \in \mathcal{O}(X, Y) \cup \{\eta^*\}$. Therefore, $\xi_2\cdots\xi_k \in \mathcal{OP}(X, Y)$. Since $rank(\beta) = m$, we get that $\ker(\beta) = \ker(\xi_1)$. If $\xi_1 \in \mathcal{O}(X, Y) \cup \{\eta^*\}$, then we have $\beta = \xi_1\xi_2\cdots\xi_k \in \mathcal{OP}(X, Y)$ that is a contradiction because $\beta \in \mathcal{OPR}(X, Y) \setminus \mathcal{OP}(X, Y)$. If $\xi_1 = \varphi_P$, then we have $(1, n) \in \ker \beta$ that contradicts with $(1, n) \notin \ker(\beta)$. Altogether, we obtain that $\beta \notin \langle \mathcal{O}(X, Y), \varphi_P, \eta^* \rangle$, i.e. $rank(\mathcal{OPR}(X, Y) : \mathcal{O}(X, Y)) \geq 3$. ∎

From Theorem 2.11 and Theorem 2.12, we obtain the immediately two corollaries as show the following:

Corollary 2.13. *If $|X \backslash Y| = 1$ and $1 \notin Y$ or $n \notin Y$, then $\mathcal{OPR}(X, Y) = \langle \mathcal{O}(X, Y), \beta_1^*, \beta_2^* \rangle$.*

Corollary 2.14. *If $|X \backslash Y| = 1$ and $\{1, n\} \subseteq Y$, then $\mathcal{OPR}(X, Y) = \langle \mathcal{O}(X, Y), \varphi_P, \beta_1^*, \beta_2^* \rangle$.*

Finally, we consider $|X \backslash Y| \geq 2$ and we can consider two cases as the following theorems.

Theorem 2.15. *If $|X \setminus Y| \geq 2$ and $1 \notin Y$ or $n \notin Y$, then we have $rank(\mathcal{OPR}(X, Y) : \mathcal{O}(X, Y)) = \binom{n-1}{m}$.*

Proof. Since $1 \notin Y$ or $n \notin Y$, we can assume without loss of generaltity that $\beta_1^*, \beta_2^* \in \{\varphi_P : P \in \mathcal{P}\}$. By Proposition 2.9, we have $\mathcal{OPR}(X,Y) = \langle \mathcal{O}(X,Y), \{\varphi_P : P \in \mathcal{P}\}\rangle$, i.e. $rank(\mathcal{OPR}(X,Y) : \mathcal{O}(X,Y)) \leq |\{\varphi_P : P \in \mathcal{P}\}| = \binom{n-1}{m}$.

Let $A \subseteq \mathcal{OPR}(X,Y) \setminus \mathcal{O}(X,Y)$ such that $\langle \mathcal{O}(X,Y), A\rangle = \mathcal{OPR}(X,Y)$. By Lemma 2.10, there is a set $A' \subseteq A$ with $\{\ker(\alpha) : \alpha \in A'\} = \mathcal{P}$, i.e. $rank(\mathcal{OPR}(X,Y) : \mathcal{O}(X,Y)) \geq |A'| \geq |\mathcal{P}|$.

Altogether, we have $rank(\mathcal{OPR}(X,Y) : \mathcal{O}(X,Y)) = |\mathcal{P}| = \binom{n-1}{m}$. ■

Theorem 2.16. *If $|X \setminus Y| \geq 2$ and $\{1,n\} \subseteq Y$, then $rank(\mathcal{OPR}(X,Y) : \mathcal{O}(X,Y)) = 2 + \binom{n-1}{m}$.*

Proof. By Proposition 2.9, we have $\mathcal{OPR}(X,Y) = \langle \mathcal{O}(X,Y), \{\varphi_P : P \in \mathcal{P}\}, \beta_1^*, \beta_2^*\rangle$, i.e. $rank(\mathcal{OPR}(X,Y) : \mathcal{O}(X,Y)) \leq 2 + |\{\varphi_P : P \in \mathcal{P}\}| = 2 + \binom{n-1}{m}$.

Let $A \subseteq \mathcal{OPR}(X,Y) \setminus \mathcal{O}(X,Y)$ such that $\langle \mathcal{O}(X,Y), A\rangle = \mathcal{OPR}(X,Y)$. By Lemma 2.10, there is a set $A' \subseteq A$ with $\{\ker(\alpha) : \alpha \in A'\} = \mathcal{P}$, i.e. $rank(\mathcal{OPR}(X,Y) : \mathcal{O}(X,Y)) \geq |A'| \geq |\mathcal{P}| = \binom{n-1}{m}$. Assume that $\langle \mathcal{O}(X,Y), \{\varphi_P : P \in \mathcal{P}\}\rangle = \mathcal{OPR}(X,Y)$. By the definition of η^*, we have $\eta^* \in \mathcal{OP}(X,Y) \setminus \mathcal{O}(X,Y) \subseteq \mathcal{OPR}(X,Y)$, where $\ker(\eta^*) \notin \mathcal{P}$ because $(1,n) \notin \ker(\eta^*)$. Since $\eta^* \in \mathcal{OP}(X,Y) \setminus \mathcal{O}(X,Y) \subseteq \mathcal{OPR}(X,Y) = \langle \mathcal{O}(X,Y), \{\varphi_P : P \in \mathcal{P}\}\rangle$, there are $\theta_1, \ldots, \theta_l \in \mathcal{O}(X,Y) \cup \{\varphi_P : P \in \mathcal{P}\}$ such that $\eta^* = \theta_1 \cdots \theta_l$. Since $rank(\eta^*) = m$ and $\{1,n\} \subseteq Y$, we obtain that $(1,n) \notin \ker(\theta_i)$ for all $i \in \{2,3,\ldots,l\}$ that implies $\theta_2 \cdots \theta_l \in \mathcal{O}(X,Y)$. Since $rank(\eta^*) = m$, we get that $\ker(\eta^*) = \ker(\theta_1)$. If $\theta_1 \in \mathcal{O}(X,Y)$, then we have $\theta_1\theta_2 \cdots \theta_k \in \mathcal{O}(X,Y)$ that is a contradiction. If $\theta_1 = \varphi_P$ for some $P \in \mathcal{P}$, then we have $(1,n) \in \ker(\eta^*)$ that contradicts with $(1,n) \notin \ker(\eta^*)$. That means $\eta^* \notin \langle \mathcal{O}(X,Y), \{\varphi_P : P \in \mathcal{P}\}\rangle$, i.e. $rank(\mathcal{OPR}(X,Y) : \mathcal{O}(X,Y)) \geq 1 + \binom{n-1}{m}$. Next, assume that $\langle \mathcal{O}(X,Y), \{\varphi_P : P \in \mathcal{P}\}, \eta^*\rangle = \mathcal{OPR}(X,Y)$. By the definition of β_1^*, we get that $\beta_1^* \in \mathcal{OPR}(X,Y) \setminus \mathcal{OP}(X,Y)$ and $\ker(\beta_1^*) \notin \mathcal{P}$ because $(1,n) \notin \ker(\beta_1^*)$. Since $\beta_1^* \in \mathcal{OPR}(X,Y) \setminus \mathcal{OP}(X,Y) \subseteq \mathcal{OPR}(X,Y) = \langle \mathcal{O}(X,Y), \{\varphi_P : P \in \mathcal{P}\}, \eta^*\rangle$, there are $\xi_1, \xi_2, \ldots, \xi_k \in \mathcal{O}(X,Y) \cup \{\varphi_P : P \in \mathcal{P}\} \cup \{\eta^*\}$ such that $\beta_1^* = \xi_1\xi_2 \cdots \xi_k$. Since $rank(\beta_1^*) = m$ and $\{1,n\} \subseteq Y$, we obtain that $(1,n) \notin \ker(\xi_i)$ for all $i \in \{2,3,\ldots,k\}$ that implies $\xi_2, \ldots, \xi_k \in \mathcal{O}(X,Y) \cup \{\eta^*\}$. Therefore, $\xi_2 \cdots \xi_k \in \mathcal{OP}(X,Y)$. Since $rank(\beta) = m$, we get that $\ker(\beta_1^*) = \ker(\xi_1)$. If $\xi_1 \in \mathcal{O}(X,Y) \cup \{\eta^*\}$, then we have $\beta_1^* = \xi_1\xi_2 \cdots \xi_k \in \mathcal{OP}(X,Y)$ that is a contradiction because $\beta_1^* \in \mathcal{OPR}(X,Y) \setminus \mathcal{OP}(X,Y)$. If $\xi_1 = \varphi_P$ for some $P \in \mathcal{P}$, then we have $(1,n) \in \ker(\beta_1^*)$ that contradicts with $(1,n) \notin \ker(\beta_1^*)$. Altogether, we get that $\beta_1^* \notin \langle \mathcal{O}(X,Y), \{\varphi_P : P \in \mathcal{P}\}, \eta^*\rangle$, i.e. $rank(\mathcal{OPR}(X,Y) : \mathcal{O}(X,Y)) \geq 2 + \binom{n-1}{m}$.

Consequently, we obtain that $rank(\mathcal{OPR}(X,Y) : \mathcal{O}(X,Y)) = 2 + \binom{n-1}{m}$. ■

3. Conclusion

In this paper, we study transformation semigroup with restricted range $\mathcal{T}(X,Y)$ and its subsemigroups. We also calculate the relative rank of subsemigroups of $\mathcal{T}(X,Y)$. In

Section 1, we introduce some notation and some definition of transformation semigroups to use through this paper. In section 2.1, we obtain the relative rank of $\mathcal{OD}(X,Y)$ modulo $\mathcal{O}(X,Y)$ as shown in Proposition 2.1 and Proposition 2.2. In section 2.2, we study and describe the relative rank of $\mathcal{OP}(X,Y)$ modulo $\mathcal{O}(X,Y)$ as shown in Theorem 2.4 and Proposition 2.5-2.6. In section 2.3, we calculate the relative rank of $\mathcal{OPR}(X,Y)$ modulo $\mathcal{O}(X,Y)$ as shown in Proposition 2.7-2.8, Theorem 2.11-2.12 and Theorem 2.15-2.16. In future work, we can study other kind structure of transformation semigroup with restricted range.

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