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Analysis of Low Order Universal Portfolios Generated by Two and Three Parameters Distributions on Malaysia's Stocks During the Covid-19 Pandemic.

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Abstract Universal Portfolio is an investment strategy that enables investors to reallocate their wealth by updating the portfolio daily to achieve maximum wealth. There is no assumption of a stochastic model for the stock price in a universal portfolio. Instead, this paper will study the low order universal portfolio generated by selected two and three parameters' distributions such as Pareto, Loglogistic, Paralogistic, Burr, and Transformed Gamma. Three portfolios, each containing three different companies selected from Kuala Lumpur Stock Exchange (KLSE), will be used throughout the study. In addition, opening and closing stock prices for 500, 1000, 1500, 2000, 2500, and 3000 trading days. In order to study the performance of the universal portfolio generated with selected distributions during the economic downturn, the selected trading period included the Covid-19 pandemic. Different parameter values were applied in this study to identify the significant parameter for each distribution, affecting the maximum wealth for each portfolio. The results obtained from each portfolio generated by selected distributions are compared with the best constant rebalanced portfolio (BCRP). The result shows that the Pareto distribution could generate wealth comparable to the BCRP with specific parameter values.

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1. Introduction

Markowitz [1] introduced a portfolio selection technique based on the mean and variance of a portfolio's return. Then, Sharpe [2] further studied the technique introduced by Markowitz. He suggested a diagonal model that can simplify the complex portfolio analysis computation task. Finally, Kelly [3]introduced the theory of rebalance portfolio. She suggested that the gambler maximize his wealth if he bet with the same money every time.

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Cover [4] introduces a universal portfolio, a portfolio investment strategy that enables investors to reallocate their wealth invested for each stock in the portfolio every trading day to maximize wealth. Then, the study of the universal portfolio was extended by Cover and Ordenlich [5] by including side information. However, the universal portfolio introduced by Cover did not include transaction costs which may affect the performance of the portfolio. Blum and Kalai [6] further studied on universal portfolio by including transaction cost. They suggested that transactions in the portfolio should not be done too often.

Tan [7] studied universal portfolio with finite-order due to the long implementation time and enormous computer storage needed by the moving order universal portfolio. A finite-order universal portfolio can generate a result with lesser implementation time compared to a moving-order universal portfolio. Pang, Liew, and Chang [8] generate finite-order universal with Ornstein-Uhlenback and Brownian Motion. They were able to generate a result that is comparable to a constant rebalance portfolio (CRP). Tan and Lim [9] extended the study of the finite-order universal portfolio with the Mixture-Current-Run (MCR) universal portfolio. They argued that the parameter of the distribution would affect the wealth generated by a finite-order universal portfolio. Therefore, MCR can identify the best parameter for selected distribution to generate the highest wealth.

This paper focused on generating low order universal portfolio with selected distributions. First, we identified the best parameter for selected distributions with parameter sensitivity testing. Then, we studied the performance of the universal portfolio in the short and long run, including the time during the Covid-19 pandemic. Last but not least, we compared the performance of low-order universal portfolios generated by diversified and non-diversified portfolios.

2. Methodology

In a universal portfolio, no stochastic model is being assumed for the stock price of m stocks. Transaction cost exists since a universal portfolio will reallocate the wealth invested in each stock every trading day. However, this paper assumed no transaction cost for every transaction and distributed dividend for each stock was not being considered. The portfolio vector is denoted by $\boldsymbol{b} = (\boldsymbol{b_1}, \boldsymbol{b_2}, \ldots, \boldsymbol{b_m})^{\boldsymbol{t}}$, where $\sum_{\boldsymbol{i}=\boldsymbol{1}}^{\boldsymbol{m}} \boldsymbol{b_i} = 1$, $\boldsymbol{b_i} \geq 0$. The proportion of current wealth invested in each stock on n^{th} trading day is denoted by $\boldsymbol{b_{ni}}$. The stock market vector is denoted by $\boldsymbol{X} = (\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots \boldsymbol{x_m})^t$, where $\boldsymbol{X} \geq 0$. The ratio of closing price to the opening price of stock i on n^{th} trading day is described by x_{ni}. With these two pieces of information, the wealth of universal portfolio on $(n+1)^{th}$ trading day is as follows:

$$
\begin{aligned}
S_{n+1} &= \prod_{j=1}^{n+1} \boldsymbol{b}_j^t \boldsymbol{x}_j \\
&= (\boldsymbol{b}_{n+1}^t \boldsymbol{x}_{n+1}) \prod_{j=1}^{n} \boldsymbol{b}_j^t \boldsymbol{x}_j \\
&= (\boldsymbol{b}_{n+1}^t \boldsymbol{x}_{n+1}) S_n
\end{aligned}
\tag{2.1}
$$

where

$$b_{n+1}^{t}x_{n+1} = \sum_{k=1}^{m} b_{n+1,k}x_{n+1,k} \tag{2.2}$$

The identical and independent random variable, $Y_1, Y_2, \ldots, Y_m$ with probability density function $g(y_1)$, $g(y_2), \ldots, g(y_m)$, respectively. The joint probability density function can be written as:

$$g(y_1, y_2, \ldots, y_m) = g(y_1)g(y_2), \ldots, g(y_m)$$

According to Tan [7], the order v universal portfolio generated with the identical and independent random variable $Y_1, Y_2, \ldots, Y_m$ can be written as:

$$b_{n+1,k} = \frac{\int_D y_k \left(y^t x_n\right)\ldots\left(y^t x_{n-(v-1)}\right)\boldsymbol{g}(\boldsymbol{y})\,\boldsymbol{dy}}{\int_D (y_1 + y_2 + \cdots + y_m)(y^t x_n)\ldots\left(y^t x_{n-(v-1)}\right)\boldsymbol{g}(\boldsymbol{y})\,\boldsymbol{dy}} \tag{2.3}$$

where $k = 1, 2, \ldots, m$.

From equation 2.2 and 2.3, the wealth function of order v universal portfolio on $(n+1)^{th}$ day can be derived as:

$$\begin{aligned} b_{n+1}^{t}x_{n+1} &= \frac{\sum_{k=1}^{m} x_{n+1,k}\int_D y_k \left(y^t x_n\right)\ldots.(y^t x_{n-(v-1)})\boldsymbol{g}(\boldsymbol{y})\,\boldsymbol{dy}}{\int_D (y_1 + y_2 + \cdots + y_m)(y^t x_n)\ldots\left(y^t x_{n-(v-1)}\right)\boldsymbol{g}(\boldsymbol{y})\,\boldsymbol{dy}} \\ &= \frac{\int_D \left(y^t x_{n+1}\right)\left(y^t x_n\right)\ldots.(y^t x_{n-(v-1)})\boldsymbol{g}(\boldsymbol{y})\,\boldsymbol{dy}}{\int_D (y_1 + y_2 + \cdots + y_m)(y^t x_n)\ldots\left(y^t x_{n-(v-1)}\right)\boldsymbol{g}(\boldsymbol{y})\,\boldsymbol{dy}} \end{aligned} \tag{2.4}$$

This paper focused on generating order 1, order 2, and order 3 universal portfolios. From equation 2.4 and assuming $E[Y_i^j]$ to be positive for $j = 1, 2, \ldots v+1$, $i = 1, 2, \ldots m$, the wealth function of order 1 universal portfolio can be written as:

$$\sum_{k=1}^{3} b_{n+1,k}x_{n+1,k} = \frac{\sum_{k=1}^{3} x_{n+1,k}\left\{x_{n,1}E[Y_k Y_1] + x_{n,2}E[Y_k Y_2] + \cdots + x_{n,m}E[Y_k Y_m]\right\}}{\sum_{k=1}^{3}\left\{x_{n,1}E[Y_k Y_1] + x_{n,2}E[Y_k Y_2] + \cdots + x_{n,m}E[Y_k Y_m]\right\}} \tag{2.5}$$

where

$$E[Y_k Y_i] = \begin{cases} E[Y_k]E[Y_i]\ , & if\ k \neq i \\ E\left[Y_k^2\right] & ,\ if\ k = i \end{cases}$$

From equation 2.4, the wealth function of order 2 universal portfolio can be written as:

$$\begin{aligned} &\sum_{k=1}^{3} b_{n+1,k}x_{n+1,k} \\ &= \frac{\sum_{k=1}^{3} x_{n+1,k}\left\{x_{n,1}x_{n-1,1}E[Y_k Y_1 Y_1] + x_{n,2}x_{n-1,2}E[Y_k Y_1 Y_2] + \cdots + x_{n,m}x_{n-1,m}E[Y_k Y_m Y_m]\right\}}{\sum_{k=1}^{3}\left\{x_{n,1}x_{n-1,1}E[Y_k Y_1 Y_1] + x_{n,2}x_{n-1,2}E[Y_k Y_1 Y_2] + \cdots + x_{n,m}x_{n-1,m}E[Y_k Y_m Y_m]\right\}} \end{aligned} \tag{2.6}$$

where

$$E\left[Y_k Y_{i_1} Y_{i_2}\right] = \begin{cases} E\left[Y_k\right] E\left[Y_{i_1}\right] E\left[Y_{i_2}\right], & if\ k \neq i_1 \neq i_2 \\ E\left[Y_k^2\right] E\left[Y_{i_2}\right], & if\ k = i_1,\ k \neq i_2 \\ E\left[Y_k^2\right] E\left[Y_{i_1}\right], & if\ k = i_2,\ k \neq i_1 \\ E\left[Y_{i_2}^2\right] E\left[Y_k\right], & if\ k \neq\ i_1,\ i_2 = i_1 \\ E\left[Y_k^3\right], & if k =\ i_1 = i_2 \end{cases}$$

From equation 2.4, the wealth function of order 3 universal portfolio can be written as:

$$\sum_{k=1}^{3} b_{n+1,k} x_{n+1,k}$$
$$= \frac{\sum_{k=1}^{3} x_{n+1,k} \left\{ \begin{array}{c} x_{n,1} x_{n-1,1} x_{n-2,1} E\left[Y_k Y_1 Y_1 Y_1\right] + x_{n,2} x_{n-1,1} x_{n-2,2} E\left[Y_k Y_1 Y_1 Y_2\right] + \dots \\ + x_{n,m} x_{n-1,m} x_{n-2,m} E\left[Y_k Y_m Y_m Y_m\right] \end{array} \right\}}{\sum_{k=1}^{3} \left\{ \begin{array}{c} x_{n,1} x_{n-1,1} x_{n-2,1} E\left[Y_k Y_1 Y_1 Y_1\right] + x_{n,2} x_{n-1,1} x_{n-2,2} E\left[Y_k Y_1 Y_1 Y_2\right] + \dots \\ + x_{n,m} x_{n-1,m} x_{n-2,m} E\left[Y_k Y_m Y_m Y_m\right] \end{array} \right\}} \tag{2.7}$$

where

$$E\left[Y_k Y_{i_1} Y_{i_2} Y_{i_3}\right] = \begin{cases} E\left[Y_k\right] E\left[Y_{i_1}\right] E\left[Y_{i_2}\right] E\left[Y_{i_3}\right], & if\ k \neq i_1 \neq i_2 \neq i_3 \\ E\left[Y_k^2\right] E\left[Y_{i_2}\right] E\left[Y_{i_3}\right], & if\ k = i_1,\ k \neq i_2,\ k \neq i_3, i_2 \neq i_3 \\ E\left[Y_k^2\right] E\left[Y_{i_1}\right] E\left[Y_{i_3}\right], & if\ k = i_2,\ k \neq i_1,\ k \neq i_3, i_1 \neq i_3 \\ E\left[Y_k^2\right] E\left[Y_{i_1}\right] E\left[Y_{i_2}\right], & if\ k = i_3,\ k \neq i_1,\ k \neq i_2, i_1 \neq i_2 \\ E\left[Y_{i_1}^2\right] E\left[Y_{i_3}\right] E\left[Y_k\right], & if\ \ i_2 = i_1, k \neq\ i_1,\ k \neq i_2, k \neq\ i_3 \\ E\left[Y_{i_1}\right] E\left[Y_{i_3}^2\right] E\left[Y_k\right], & if\ \ i_2 = i_3, k \neq\ i_1,\ k \neq i_2, k \neq\ i_3 \\ E\left[Y_{i_2}\right] E\left[Y_{i_3}^2\right] E\left[Y_k\right], & if\ \ i_1 = i_3, k \neq\ i_1,\ k \neq i_2, k \neq\ i_3 \\ E\left[Y_{i_1}^2\right] E\left[Y_k^2\right], & if\ k = i_3, i_1 = i_2, k \neq i_1 \\ E\left[Y_{i_3}^2\right] E\left[Y_k^2\right], & if\ k = i_1, i_2 = i_3, k \neq i_2 \\ E\left[Y_{i_2}^2\right] E\left[Y_k^2\right], & if\ k = i_2, i_1 = i_3, k \neq i_3 \\ E\left[Y_{i_3}\right] E\left[Y_k^3\right], & if\ k = i_1 = i_2,\ k \neq i_3 \\ E\left[Y_{i_2}\right] E\left[Y_k^3\right], & if\ k = i_1 = i_3,\ k \neq i_2 \\ E\left[Y_{i_1}\right] E\left[Y_k^3\right], & if\ k = i_2 = i_3,\ k \neq i_1 \\ E\left[Y_k^4\right], & if k =\ i_1 = i_2 = i_3 \end{cases}$$

Five distributions are selected from two and three-parameter distributions in generating the low-order universal portfolios and the respective k^{th} moment function is shown in Tables 1 and 2:

Table 1: Moment Function of Two-Parameters Distribution

Two-Parameters Distribution	$\mathbf{k^{th}}$ **moment function,** $\mathbf{E(X^k)}$
Pareto – α, θ $\quad 0 < k < \alpha$	$\frac{\theta^k \Gamma(k+1)\Gamma(\alpha-k)}{\Gamma(\alpha)}$
Loglogistic – γ, θ $\ -\gamma < k < \gamma$	$\theta^k \Gamma\left(1+\frac{k}{\gamma}\right)\Gamma\left(1-\frac{k}{\gamma}\right)$
Paralogistic – α, θ $-\alpha < k < \alpha^2$	$\frac{\theta^k \Gamma(1+\frac{k}{\alpha})\Gamma(\alpha-\frac{k}{\alpha})}{\Gamma(\alpha)}$

Table 2: Moment Function of Three-Parameters Distribution

Three-Parameters Distribution	$\mathbf{k^{th}}$ **moment function,** $\mathbf{E(X^k)}$
Burr - α, θ, γ $\quad -\gamma < k < \alpha\gamma$	$\frac{\theta^k \Gamma\left(1+\frac{k}{\gamma}\right)\Gamma\left(\alpha-\frac{k}{\gamma}\right)}{\Gamma(\alpha)}$
Transformed Gamma - α, θ, $\tau$$-\alpha\tau < k$	$\frac{\theta^k \Gamma\left(\alpha+\frac{k}{\tau}\right)}{\Gamma(\alpha)}$

3. Preliminaries Result

Five companies were chosen from Kuala Lumpur Stock Exchange (KLSE) to form three sets of portfolios. Table 3 shows the portfolios that were used to generate universal portfolios throughout this paper. The opening and closing prices of the companies were collected from Yahoo Finance (https://finance.yahoo.com/) for 1500 trading days, from 5^{th} January 2015 to 27^{th} January 2021.

Table 3: List of Malaysia companies for portfolio X, Y and Z.

Portfolio	**Companies**
X	Malayan Banking Berhad, Hong Leong Bank Berhad, Public Bank berhad
Y	Malayan Banking Berhad, Top Glove Corporation Berhad, Fraser & Neave Holding Berhad
Z	Malayan Banking Berhad, Hong Leong Bank Berhad, Top Glove Corporation Berhad

The portfolio returns on the first day, S_1 is calculated by assuming the initial wealth of the portfolio, $S_0 = 1$ and the initial proportion of the wealth of the portfolio, $b_0 = (0.3333, 0.3333, 0.3334)$. The parameter of the selected distributions was randomly generated for 500 combinations to generate 500 samples of terminal wealth, S_{1500}. The highest wealth was recorded for each distribution and benchmark, with the best constant rebalance portfolio (BCRP) and CRP. Algorithm 1 shows how Pareto distribution generates a wealth of order 1 universal portfolio with 500 randomly generated parameter value combinations.The highest wealth achieved will be selected among these 500 randomly generated parameter value combinations, and the results were grouped in tables to have a quick analysis and observation. From the observation, we may have a brief idea of which parameters combination will significantly impact the wealth achieved in portfolios X, Y, and Z, respectively. Tables 4, 5, and 6 show the highest wealth generated by each distribution with randomly generated parameter values, BCRP and CRP for portfolios X, Y, and Z, respectively.

From the result obtained, it can be observed that the Pareto distribution able to outperform the other distribution by generating the highest wealth with order 1 and order 2 universal portfolios. However, most of the selected distributions with randomly generated parameter values can outperform CRP but not BCRP; This indicates the necessity to carry out parameter sensitivity tests to determine the suitable parameter values for the selected distributions to generate wealth comparable to BCRP.

Algorithm 1: Low-order universal portfolio generated by Pareto distribution with the randomly generated parameter value.

(1) Initialization: $S_0 = 1$, $b_0 = (0.3333, 0.3333, 0.3334)$
(2) **for i = 1 to 499**
(3) Randomly generate value of $\alpha_1, \alpha_2, \alpha_3, \theta_1, \theta_2$ and $\theta_3$which fulfill the parameter condition stated in Tables 1 and 2.
(4) **for t = 1 to n**
(5) Generate $b_{n+1}^{t} x_{n+1}$
(6) Update universal portfolio return: $S_{t+1} = \left(b_{t+1}^{t} x_{t+1}\right) S_t$
(7) **Next t**
(8) **Next i**

Table 4: The highest wealth achieved of the universal portfolio generated by selected distributions, CRP, and BCRP for portfolio X with the randomly generated parameter value.

Distribution	**Portfolio X**		
	Parameter	**Terminal Wealth**	**Timer (Second)**
Pareto (Order 1)	(3, 54, 18, 63, 20, 3)	2.7637	1.05
Loglogistic (Order 3)	(50, 72, 67, 86, 11, 1)	2.5998	11.06
Paralogistic (Order 1)	(93, 96, 100, 85, 6, 2)	2.6389	1.09
Burr (Order 1)	(99, 65, 34, 94, 7, 5, 50, 74, 37)	2.5799	1.45
Transformed Gamma (Order 1)	(72, 54, 33, 41, 25, 17, 1, 83, 84)	2.7466	1.46
	Terminal Wealth		
CRP	2.4399		
BCRP	2.7740		

Table 5: The highest wealth achieved of the universal portfolio generated by selected distributions, CRP, and BCRP for portfolio Y with the randomly generated parameter value.

Distribution	**Portfolio Y**		
	Parameter	**Terminal Wealth**	**Timer (Second)**
Pareto (Order 2)	(94, 67, 4, 17, 13, 40)	2.7637	4.59
Loglogistic (Order 3)	(7, 35, 28, 5, 1, 40)	2.5998	11.47
Paralogistic (Order 1)	(4, 32, 25, 5, 1, 40)	2.6389	1.67
Burr (Order 1)	(59, 12, 68, 18, 1, 76, 31, 7, 66)	2.5799	1.19
Transformed Gamma (Order 2)	(70, 53, 76, 13, 28, 73, 57, 66, 1)	2.7466	4.77
	Terminal Wealth		
CRP	5.6782		
BCRP	6.7750		

Table 6: The highest wealth achieved of the universal portfolio generated by selected distributions, CRP, and BCRP for portfolio Z with the randomly generated parameter value.

Distribution	**Portfolio Z**		
	Parameter	**Terminal Wealth**	**Timer (Second)**
Pareto (Order 2)	(6, 34, 27, 5, 1, 40)	3.2732	2.62
Loglogistic (Order 3)	(37, 15, 34, 55, 2, 89)	3.2456	12.46
Paralogistic (Order 1)	(94, 43, 61, 54, 1, 64)	3.2533	1.22
Burr (Order 1)	(61, 91, 95, 66, 2, 67, 99, 3, 98)	3.2548	1.33
Transformed Gamma (Order 3)	(64, 20, 81, 36, 1, 83, 97, 95, 25)	3.2570	11.09
	Terminal Wealth		
CRP	3.0889		
BCRP	3.2570		

A parameter sensitivity test was carried out in this study; each parameter value of the selected distribution was tested by selecting 10 to 100. The parameter sensitivity test for each distribution was carried out according to the pattern of the parameter values combination that can generate the highest wealth from the previous section. Only order 1 universal portfolio with 1500 trading days was generated in this section. First, the parameter that had an apparent influence on the wealth generated by the universal portfolio was recorded. Next, those parameters that did not significantly affect the wealth generated are fixed with a selected parameter value, 50. Then, the significant parameter values combination for the selected distributions was selected according to the result obtained from the parameter sensitivity test. With the selected parameter values combination, the highest wealth generated by order 1, order 2, and order 3 universal portfolio was recorded and compared with BCRP and CRP. Algorithm 2 shows the method to perform the parameter sensitivity test.

Algorithm 2: Sensitivity testing on the distribution's parameter in Low-order universal portfolio.

(1) **Input**: Increase the parameter value for the selected distribution parameter from 50 to 100 and decrease the parameter value from 50 to 10, increment or decrement by 10, and fix the other parameter at 50.
(2) **Output:** Terminal Wealth, S_n
(3) Initialization: $S_0 = 1$, $b_0 = (0.3333, 0.3333, 0.3334)$
(4) **for t = 1 to n**
(5) Generate $b^t_{n+1} x_{n+1}$
(6) Update universal portfolio return: $S_{t+1} = \left(b^t_{t+1} x_{t+1}\right) S_t$
(7) **Next t**

The result obtained from the parameter sensitivity test for each distribution was shown in Figures 1, 2, 3, 4, and 5. The result showed that each parameter of the selected distributions reacted differently for each portfolio. Some parameters of the selected distribution need to stay as small as possible, while some need to stay as large as possible. Therefore, to achieve the highest possible wealth of the universal portfolio, the selected parameter value for each distribution needs to fulfill the parameter condition found from the result obtained. For example, in algorithm 1, 500 parameter combinations were randomly generated and tested for the Pareto distribution. From the result, it can be observed that the most impactful parameters that influence the wealth achieved for portfolio X are $\alpha_2, \alpha_3, \theta_1$. Next, we can proceed to algorithm 2. Those insignificant parameters, $\alpha_1, \theta_2, \theta_3$ will be assigned with a constant value of 50, while α_2,α_3,θ_1 will be tested with a given range from 10 to 100. The increment or decrement by 10 will be applied to the significant parameter to observe the terminal wealth achieved. All the results are being recorded and presented in the figure. From figure 1, it can observe that the changing of single parameters $\alpha_2, \alpha_3, \theta_1$ will increase the terminal wealth for portfolio X. However, the best wealth can be achieved when the three parameters simultaneously increase.

Table 7 compares the highest possible wealth achieved by each distribution with the selected parameter value for each portfolio. It could be observed that the wealth generated by each distribution was comparable to the other and the wealth generated by BCRP and CRP. Figures 6, 7, and 8 showed that Pareto distribution and Transformed Gamma distribution could generate higher wealth than the other distributions. Besides, both could generate wealth that has only a slight difference. However, for portfolio Z, Pareto distribution showed a better performance than Transformed Gamma distribution. It even outperformed BCRP; This indicates Pareto distribution is a better distribution in generating a high performance of universal portfolio than the other selected distributions.

Figure 1: Parameter sensitivity test of Pareto distribution with the selected parameter that provides significant changes in portfolios X, Y, and Z.

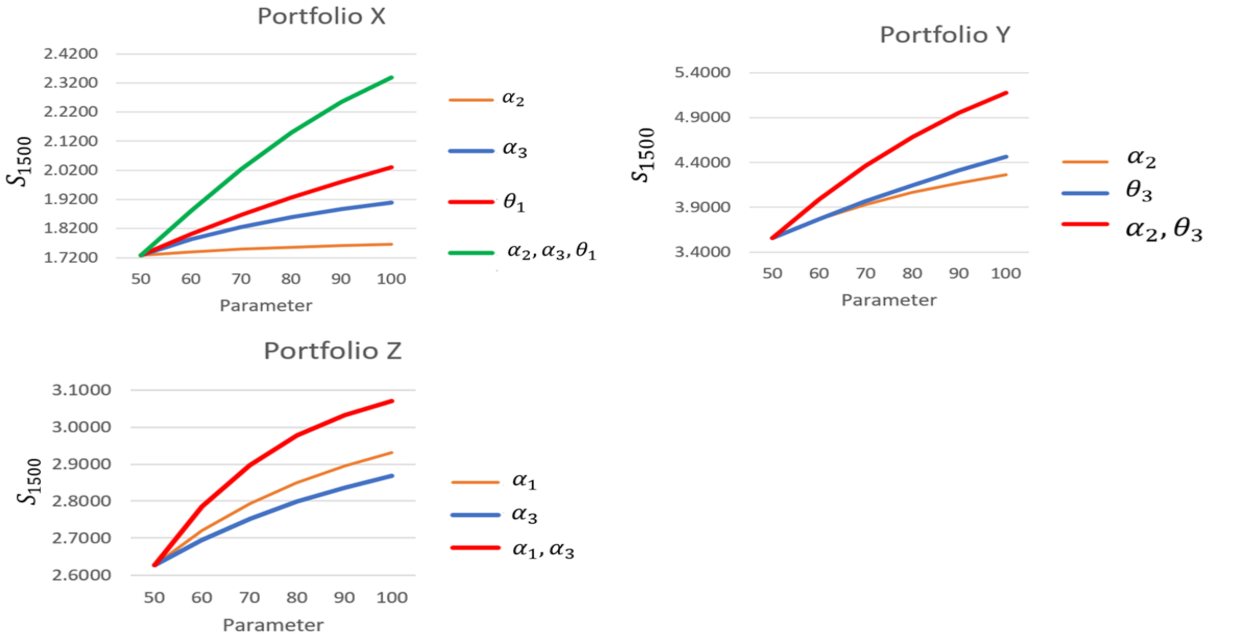


Figure 2: Parameter sensitivity test of Loglogistic distribution with the selected parameter that provides significant changes in portfolios X, Y, and Z.

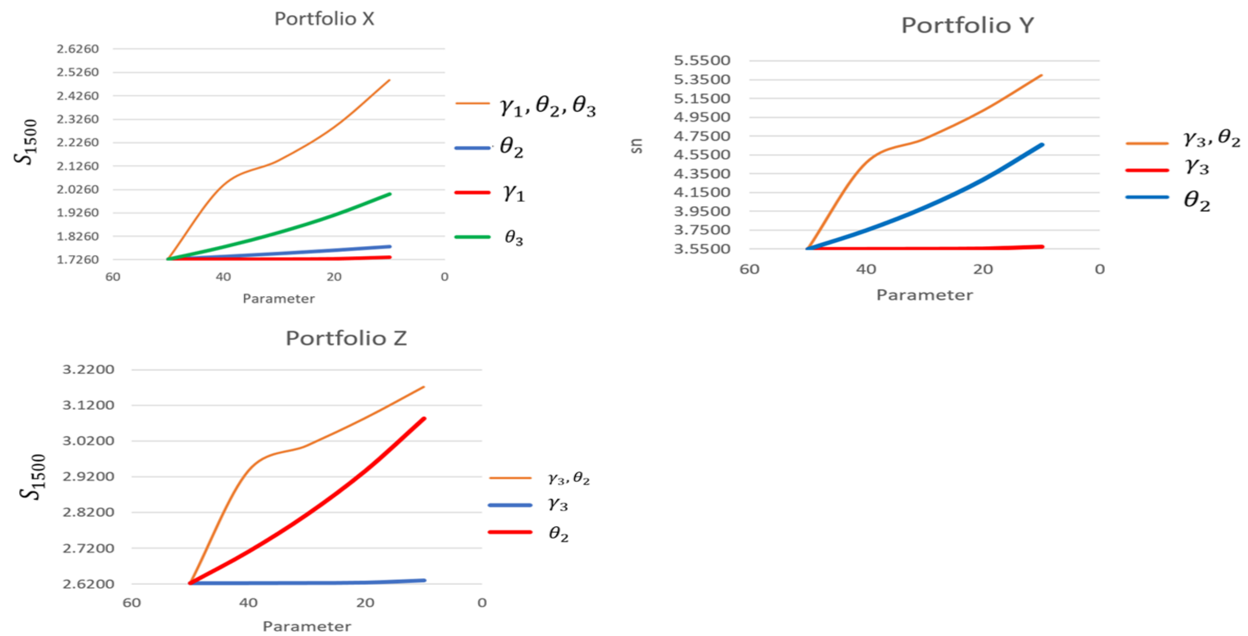


Figure 3: Parameter sensitivity test of Paralogistic distribution with the selected parameter that provides significant changes in portfolios X, Y, and Z.

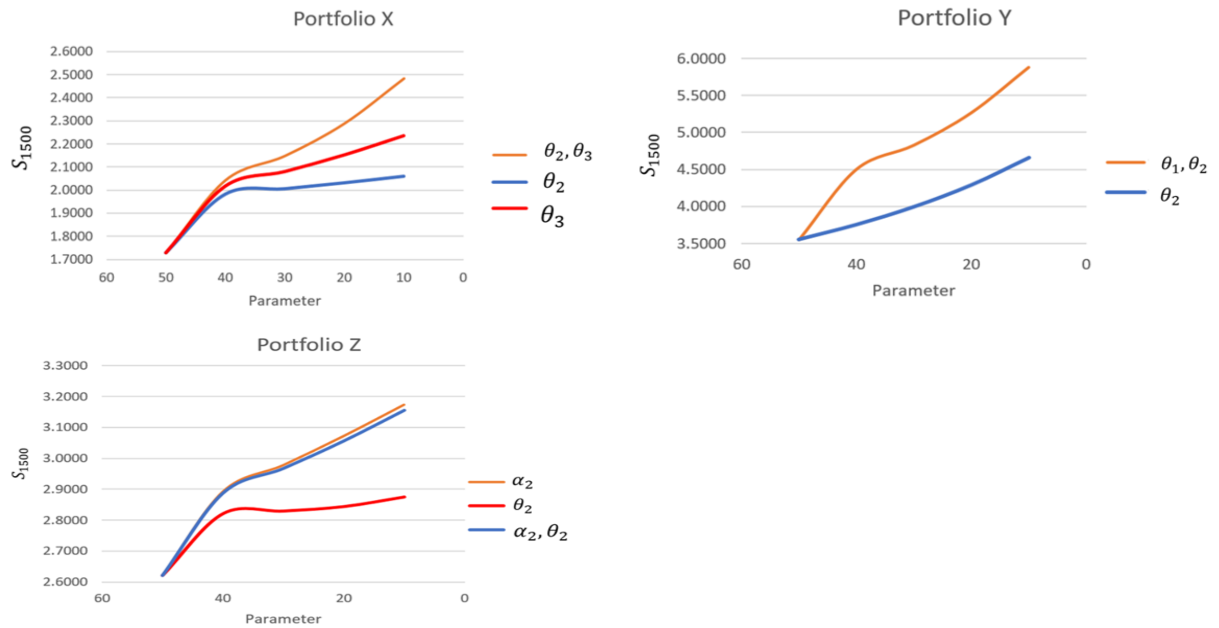


Figure 4: Parameter sensitivity test of Burr distribution with the selected parameter that provides significant changes in portfolios X, Y, and Z.

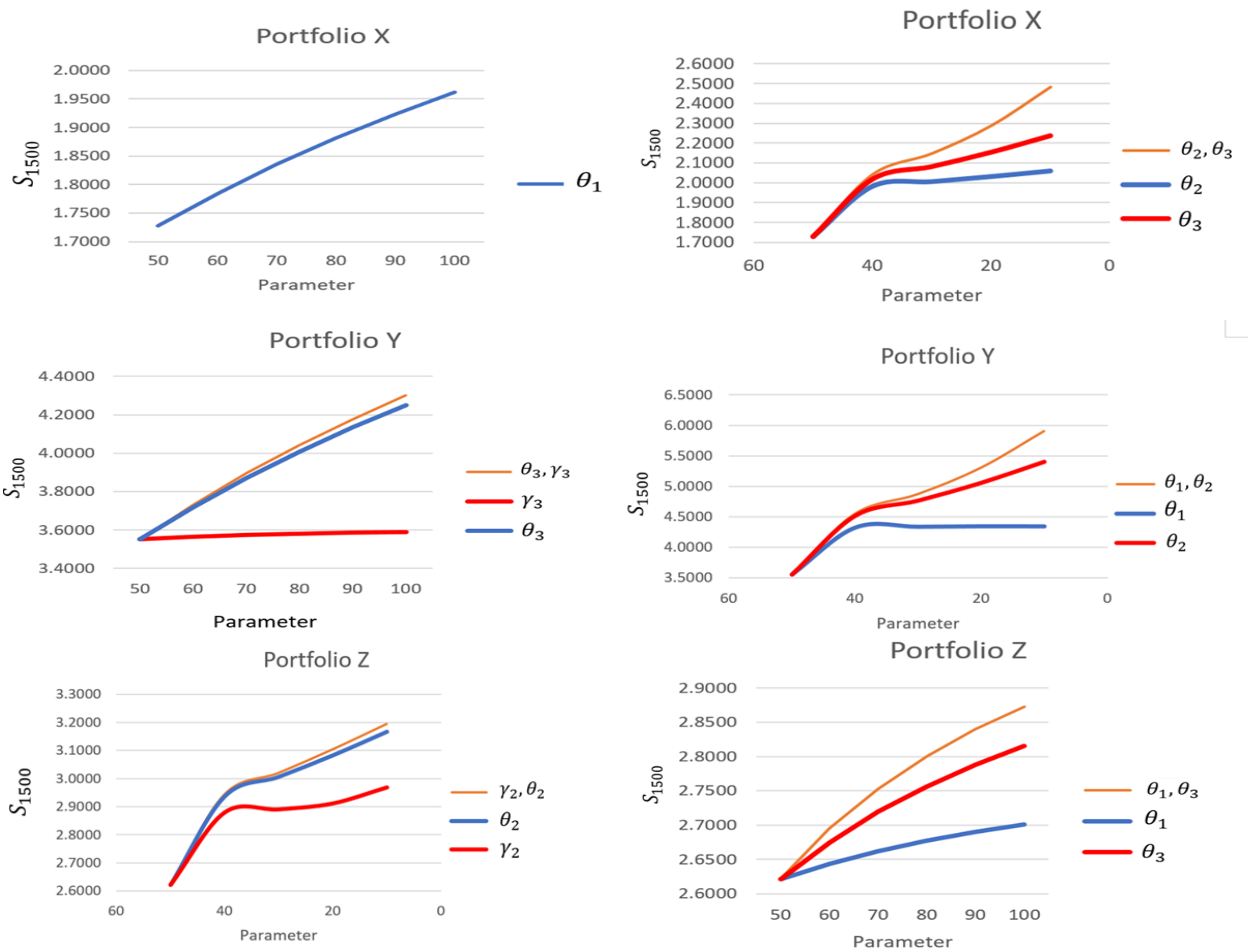


Figure 5: Parameter sensitivity test of Transformed Gamma distribution with the selected parameter that provides significant changes in portfolios X, Y, and Z.

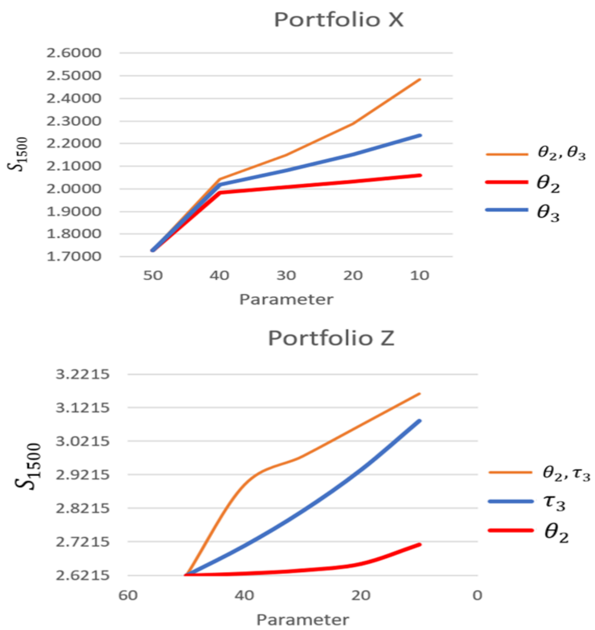


Table 7: The highest wealth achieved of the universal portfolio generated by selected distributions, CRP and BCRP for portfolios X, Y, and Z with selected parameter value in 1500 trading days.

Distribution	Portfolio	Order of Universal Portfolio	Parameter	S_{1500}
Pareto $(\alpha_1, \alpha_2, \alpha_3, \theta_1, \theta_2, \theta_3)$	X	3	(5, 170, 170, 170, 1, 1)	2.7708
	Y	3	(100, 100, 5, 1, 1, 100)	6.7857
	Z	3	(5, 170, 170, 1, 1, 100)	4.5595
Loglogistic $(\gamma_1, \gamma_2, \gamma_3, \theta_1, \theta_2, \theta_3)$	X	3	(5, 100, 100, 100, 1, 1)	2.7544
	Y	3	(5, 100, 5, 1, 1, 100)	6.7328
	Z	3	(100, 100, 5, 100, 1, 100)	3.2619
Paralogistic $(\alpha_1, \alpha_2, \alpha_3, \theta_1, \theta_2, \theta_3)$	X	2	(4, 4, 4, 3500, 1, 1)	2.7702
	Y	2	(4, 4, 4, 1, 1, 500)	6.7710
	Z	1	(5, 5, 150, 100, 1, 100)	3.2666
Burr $(\alpha_1, \alpha_2, \alpha_3, \theta_1, \theta_2, \theta_3, \gamma_1, \gamma_2, \gamma_3)$	X	2	(1, 1, 1, 100, 1, 1, 5, 100, 100)	2.7475
	Y	1	(1, 1, 1, 1, 1, 500, 5, 5, 100)	6.7621
	Z	1	(150, 150, 150, 200, 1, 200, 100, 5, 100)	3.2584
Transformed Gamma $(\alpha_1, \alpha_2, \alpha_3, \theta_1, \theta_2, \theta_3, \tau_1, \tau_2, \tau_3)$	X	2	(150, 150, 1, 500, 1, 1, 1, 100, 100)	2.7708
	Y	2	(1, 1, 150, 1, 1, 1000, 1, 1, 1)	6.7859
	Z	1	(100, 1, 1, 1, 100, 1, 100, 1)	3.2721

Figure 6: Comparison of 1500 trading days universal portfolio's wealth generated by selected distributions with selected parameter values with BCRP and CRP for portfolio X.

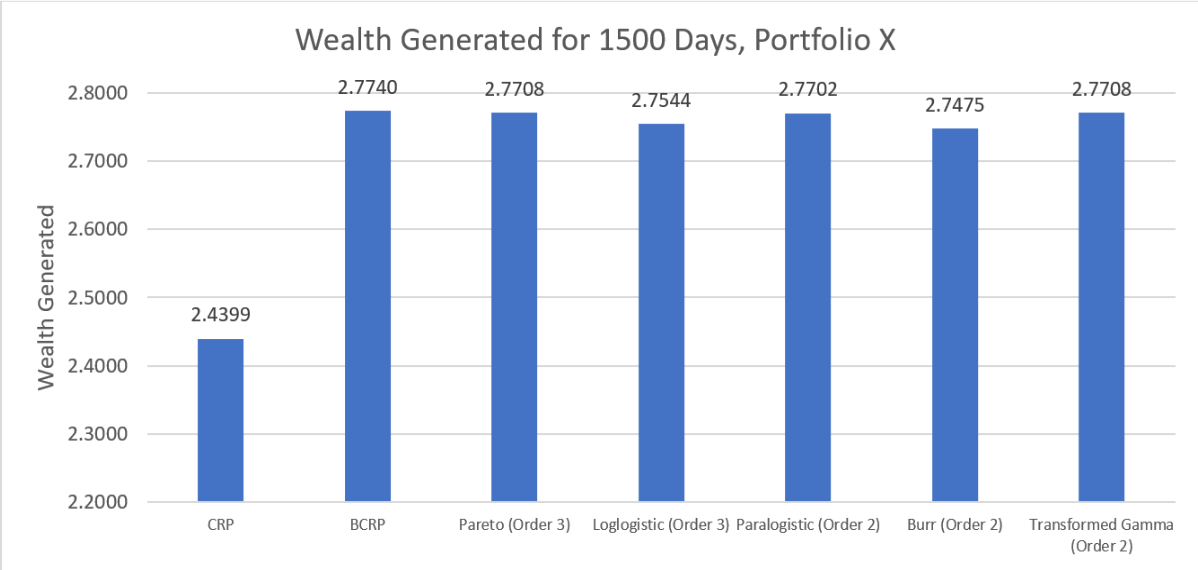


Figure 7: Comparison of 1500 trading days universal portfolio's wealth generated by selected distributions with selected parameter values with BCRP and CRP for portfolio Y.

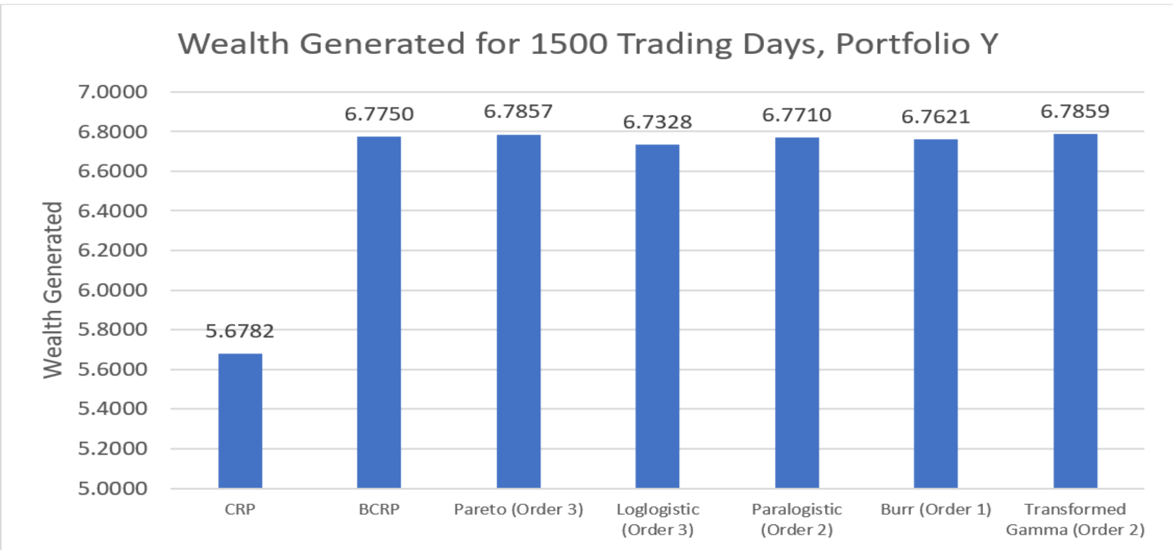
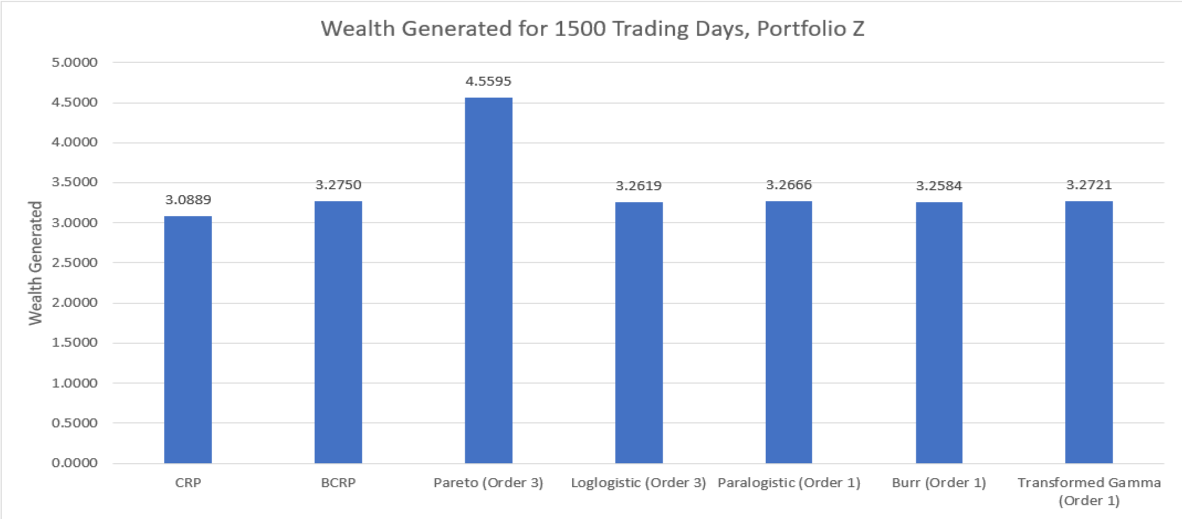


Figure 8: Comparison of 1500 trading days universal portfolio's wealth generated by selected distributions with selected parameter values with BCRP and CRP for portfolio Z.



Next, to study the performance of the long and short run of the universal portfolio with selected distributions, additional opening and closing prices of the selected companies were collected from Yahoo Finance from 1^{st} December 2008 to 27^{th} January 2021. This paper aimed to study the performance of low order universal portfolio generated by selected distributions during the economic downturn; therefore, the selected trading period includes the period during the Covid-19 Pandemic. A different range of trading periods was studied in this section. The results were shown in Tables 8, 9, and 10 for portfolios X, Y, and Z, respectively. It can be noticed that as the range of trading days increase, the wealth of the universal portfolio generated with selected distributions also increases. Therefore, Pareto distribution can generate wealth that can outperform wealth generated by the other distributions most of the time, whether in the short or long term. For example, although it did not perform well when generating a universal portfolio for 3000 trading days with portfolio Z, it can generate comparable wealth and even outperform BCRP. Therefore, Pareto distribution is preferable among the other selected distributions in generating a universal portfolio.

Table 8: The wealth achieved of the universal portfolios generated by selected distributions for portfolio X in the short and long term.

Distribution	**Trading Days**					
	500	**1000**	**1500**	**2000**	**2500**	**3000**
Pareto (Order 3)	0.8827	1.4352	2.7708	2.6894	7.4344	16.6243
Loglogistic (Order 3)	0.8823	1.4329	2.7544	2.6759	7.3384	16.3575
Paralogistic (Order 2)	0.8827	1.4351	2.7702	2.6888	7.4306	16.6118
Burr (Order 2)	0.8832	1.4319	2.7475	2.6702	7.2980	16.2454
Transformed Gamma (Order 2)	0.8827	1.4352	2.7708	2.6891	7.4343	16.6240
CRP	0.8684	1.3810	2.4399	2.3970	5.5591	11.6285
BCRP	0.8800	1.4370	2.7740	2.6810	7.5070	16.6810

Table 9: The wealth achieved of the universal portfolios generated by selected distributions for portfolio Y in the short and long term.

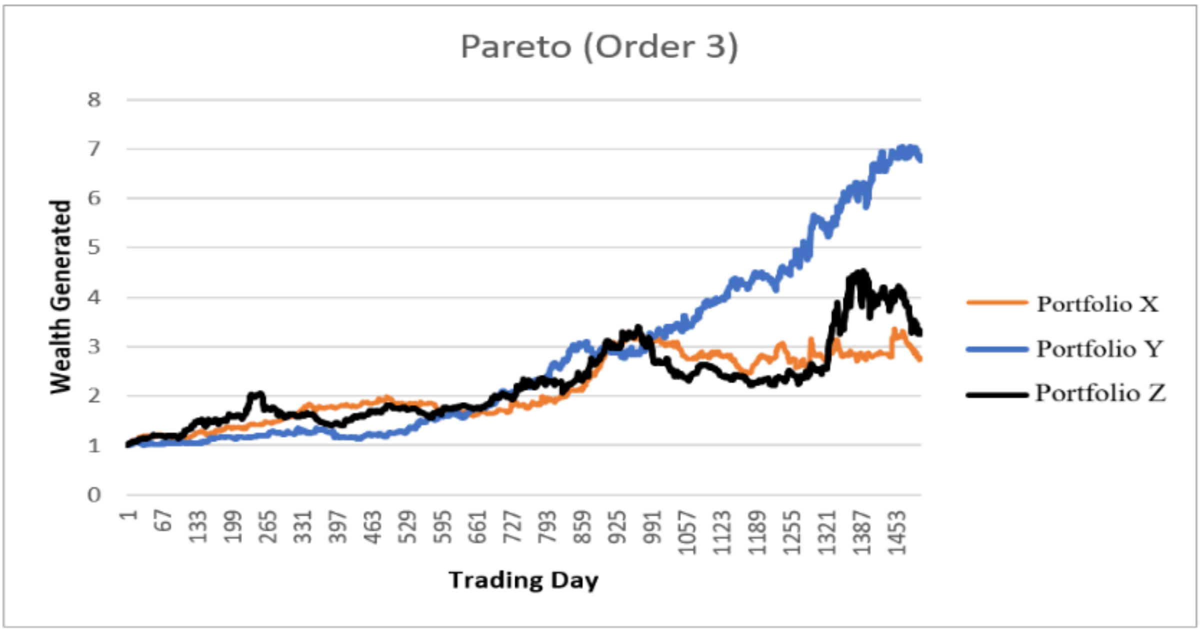
Distribution	**Trading Days**					
	500	**1000**	**1500**	**2000**	**2500**	**3000**
Pareto (Order 3)	2.0964	5.4572	6.7857	9.6267	8.8638	16.8175
Loglogistic (Order 3)	2.0858	5.4021	6.7328	9.5183	8.7862	16.6569
Paralogistic (Order 2)	2.0934	5.4421	6.7710	9.5967	8.8421	16.7725
Burr (Order 1)	2.0916	5.4329	6.7621	9.5785	8.8290	16.4958
Transformed Gamma (Order 2)	2.0964	5.4575	6.7859	9.6271	8.8641	16.5648
CRP	1.8835	4.3670	5.6782	7.5641	7.2975	13.5262
BCRP	2.1120	5.4190	6.7750	9.7270	8.9110	16.8850

Table 10: The wealth achieved of the universal portfolios generated by selected distributions for portfolio Z in the short and long term.

Distribution	**Trading Days**					
	500	**1000**	**1500**	**2000**	**2500**	**3000**
Pareto (Order 3)	1.3727	2.2478	4.5595	4.8771	5.1274	9.9323
Loglogistic (Order 3)	1.1907	1.7781	3.2619	3.1383	5.5908	11.1068
Paralogistic (Order 2)	1.1858	1.7763	3.2598	3.1370	5.6517	11.2571
Burr (Order 1)	1.1580	1.7562	3.2584	3.1471	5.9767	12.1133
Transformed Gamma (Order 1)	1.2029	1.7886	3.2721	3.1452	5.5092	10.8861
CRP	1.2697	1.8185	3.0889	2.9746	6.2685	13.3944
BCRP	1.3300	1.8450	3.2750	3.1480	7.4140	16.6310

Lastly, the result generated by Pareto distribution for order 3 universal portfolio with selected parameter value in 1500 trading days was used to compare the performance of universal portfolio generated by selected distributions with diversified and non-diversified portfolio during the Covid-19 pandemic. In this paper, the selected portfolios are formed according to the industry of the companies. Three companies were chosen from the financing services sector to form a non-diversified portfolio X, and these companies have a strong track record of consistent earning, and higher growth potential is limited. While for portfolio Y, three of the companies are from different industry which forms a diversified portfolio. These companies are from financing services, health care, and consumer products and services sector. Portfolio Y consists of stable earning stocks and stocks that will not affect the economy's growth. While portfolio Z consists of two companies from the financing service sector and another company from the health care sector, making it a weakly diversified portfolio. A closer study on Figure 9, portfolio Y, shows a steadily higher performance than the other two portfolios. It even has better performance during the Covid-19 pandemic. It can be observed that the wealth generated by portfolios X and Z was showing a decreasing trend in the last few trading days, which is during the Covid-19 pandemic.

Figure 9: Comparison of the order 3 universal portfolio wealth generated by Pareto distribution with diversified and non-diversified portfolios for 1500 trading days.



4. Conclusion

In conclusion, the selected distributions cannot achieve the highest wealth of low-order universal portfolios with the randomly generated parameter value. However, with the parameter sensitivity test, the optimum parameter value for each distribution can be determined. With the selected parameter value, the selected distributions can generate wealth comparable to BCRP and outperform CRP. Furthermore, by increasing the range of trading days, the performance of universal portfolios was improved. Moreover, Pareto distribution shows an outstanding performance in generating order 3 universal portfolio. It can consistently generate a high-performance universal portfolio that outperforms the other selected distributions and CRP most of the time. It was also able to generate wealth that is outperforming BCRP. Finally, the universal portfolio had better performance than diversified portfolios even during the Covid-19 pandemic, where the economy was downturned.

However, with the selected parameter value, some combination of parameter values for the selected distributions may be excluded, which causes the wealth generated for the universal portfolio not to be the maximum wealth. As there is a limitation in VBA that cannot loop an extensive range of parameter values, Software R or python would be suggested to run a more extensive sensitivity testing on the distributions' parameter values. In future research studies, inverse transformed gamma, beta or mixture distributions (e.g., Pareto mixed with transformed gamma distribution) can be used to generate a low-order universal portfolio. Besides that, it can extend the study to the mid-order universal portfolio, order 4 to order 6 universal portfolios, or even high-order universal portfolio. Furthermore, a well-diversified portfolio should be determined by identifying the correlation of stocks. A portfolio structured with highly negatively correlated stocks should generate high performance of the universal portfolio.

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