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Modified Dai-Yuan Conjugate Gradient Method with Sufficient Descent Property for Nonlinear Equations

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Abstract The convex constraint nonlinear equation problem is to find a point q with the property that $q \in \mathcal{D}$ where $\mathcal{D}$ is a nonempty closed convex subset of Euclidean space $\mathbb{R}^n$. The convex constraint problem arises in many practical applications such as chemical equilibrium systems, economic equilibrium problems, and the power flow equations. In this paper, we extend the modified Dai-Yuan nonlinear conjugate gradient method with sufficiently descent property proposed for large-scale optimization problem to solve convex constraint nonlinear equation and establish the global convergence of the proposed algorithm under certain mild conditions. Our result is a significant improvement compared with related method for solving the convex constraint nonlinear equation.

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1. Introduction

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function, bounded from below. One of the fundamental problems in optimization is the minimization problem:

$$\min\{f(q) : q \in \mathbb{R}^n\}. \tag{1.1}$$

A well known class of iterative method for solving problem (1.1) is the conjugate gradient (CG) method [1, 2]. The CG methods are characterized by their simplicity, efficiency and low memory requirement. Furthermore, they are matrix free methods and do not require the computation of the Jacobian matrix at each iteration. The CG method have very efficient numerical performance with remarkable global convergence properties, they are widely used for solving unconstrained optimization problems by many researchers [3–8]. This method computes its sequence of iterates $\{q_t\}$ using the following updating rule

$$q_{t+1} := q_t + \varepsilon_t p_t, \ t = 0, 1, 2, \ldots. \tag{1.2}$$

The scalar $\varepsilon_t > 0$ denotes step–size that is obtained via a suitable line–search strategy and the vector p_t is a search direction given by

$$p_0 := -\nabla f(q_0), \ p_t := -\nabla f(q_t) + \beta_t p_{t-1}, \ \forall t > 0, \tag{1.3}$$

where β_t (called CG parameter) is a scalar that is updated in every iteration. Under differentiability assumption, a well–known necessary optimality condition for problem (1.1) is:

$$q^* \text{ is a minimizer of } f \implies \nabla f(q^*) = 0.$$

This fact is called the first–order necessary optimality condition for the minimizer q^* of problem (1.1). Therefore, we consider the following system of nonlinear equations

$$w(q) = 0. \tag{1.4}$$

where $w : \mathbb{R}^n \to \mathbb{R}^n$. In this context, the gradient of f in problem (1.1) is viewed as the function w in problem (1.4). Hence, algorithms for solving problem (1.1) can be extended to solve the nonlinear system of equations (1.4). In this paper, we consider the problem of finding a vector $q \in \mathcal{D} \subseteq \mathbb{R}^n$ for which (1.4) holds. The constrained set $\mathcal{D}$ is assumed to be nonempty, closed and convex. Thus, we refer problem (1.4) as nonlinear system of equations with convex constraints. Many mathematical problems arising from various applications such as fixed point problems, differential equations, variational inequality [9–11] problems and so on, can be reformulated into problem (1.4). Moreover, problem (1.4) also appears as a subproblem in generalized proximal algorithms with Bregman distance. Optimization problems containing least square errors and ℓ_1–norm can equally be translated into problem (1.4) (See, [12, 13]). This underlines the importance of system (1.4) and so also efficient algorithms with minimum computational cost for solving it.

Recently, the CG algorithms for solving unconstrained optimization problems has prompted several researchers to extend them to solve system of monotone nonlinear operator problems (1.4). For instance, Dai et al. [14] extended the modified Perry CG method in [15] to solve unconstrained version of problem (1.4) based on the hyperplane projection method proposed by Solodov and Svaiter in [16]. The numerical results presented show that their method works well. Yuan [17] extended the well-known CG_DESCENT method of Hager and Zhang to solve problem (1.4). Under some mild conditions, they proved the global convergence of their method. Furthermore, motivated by the idea of Yu in [18], Yuan [19] proposed a modified PRP formula for unconstrained optimization problems that possess a descent property without depending on any line search strategy. He established the global convergence of his method under weak Wolfe–Powell line search for nonconvex functions. Numerical results presented shows that his method works well. This was later modified by Liu and Feng in [20]. The algorithm proposed in [20] was shown to be efficient for solving convex constrained monotone operator equations of the form (1.4). Abubakar et al. [21] proposed a modified descent Dai–Yuan CG method for constrained nonlinear monotone operator equations which is a modification of the method proposed in [20]. Under some mild conditions, they established the global convergence of the method. Their numerical experiments performed better than the compared existing methods. They also applied their method in image de-blurring. For more on extension of the CG methods, author should refer to [21–44] and references therein.

Inspired by the above CG methods and specifically the modified Dai-Yuan nonlinear conjugate gradient method with sufficiently descent property proposed for large-scale optimization problem proposed by Yuan in [19], we propose, analyze and test a new Dai-Yuan-like derivative-free method for solving the system of nonlinear equations (1.4). The global convergence of the proposed method is established under some suitable assumptions. Numerical experiments indicate that the proposed method is competitive, practical and performs better than the compared CG methods.

The structure of this paper is as follows. In Section 2, we describe the new method and its global convergence is given in Section 3. In Section 4, we report numerical experiments to show the efficiency of our algorithm. Throughout this paper, $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$.

2. The Method

We begin this section with the definition of the projection map.

Definition 2.1. Let $\mathcal{D} \subseteq \mathbb{R}^n$ be a nonempty closed convex set. Then for any $y \in \mathbb{R}^n$, its projection onto $\mathcal{D}$, denoted by $P_{\mathcal{D}}[y]$, is defined by

$$P_{\mathcal{D}}[y] := \arg\min\{\|y - x\| \ : x \in \mathcal{D}\}.$$

The projection operator $P_{\mathcal{D}}$ has a well-known property, that is, for any $y, x \in \mathbb{R}^n$ the following nonexpansive property hold

$$\|P_{\mathcal{D}}[y] - P_{\mathcal{D}}[x]\| \leq \|y - x\|. \tag{2.1}$$

Using the projection technique in Solodov and Svaiter [16], Liu and Feng [20] extended the classical Dai-Yuan conjugate gradient method for unconstrained optimization problem to solve (1.4). Motivated by this, in this article, based on the conjugate gradient method introduced by Yuan [19], we propose a Dai-Yuan-like derivative-free method that generates iterates by the following relation:

$$z_t = q_t + \varepsilon_t p_t \tag{2.2}$$

where ε_t is the steplength and p_t is the search direction computed by

$$p_t := \begin{cases} -w(q_t) & \text{if } t = 0, \\ -w(q_t) + \beta_t^{EMDY} p_{t-1} & \text{if } t > 0, \end{cases} \tag{2.3}$$

where

$$\beta_t^{EMDY} := \frac{\|w(q_t)\|^2}{w_{t-1}^T p_{t-1}} + \min\left\{ -\frac{\|w(q_t)\|^2}{w_{t-1}^T p_{t-1}},\ \frac{\|w(q_t)\|^2}{(w_{t-1}^T p_{t-1})^2} w(q_t)^T p_{t-1} \right\}, \tag{2.4}$$

$$w_{t-1} := y_{t-1} + j_{t-1} p_{t-1},\ y_{t-1} := w(q_t) - w(q_{t-1}),\ j_{t-1} := 1 + \max\left\{0, -\frac{p_{t-1}^T y_{t-1}}{p_{t-1}^T p_{t-1}}\right\}. \tag{2.5}$$

The search direction defined by (2.3) satisfies the following Lemma.

Lemma 2.2. *Let p_t be the search direction generated by (2.3), then p_t is a sufficient descent direction. That is for all $t \geq 0$,*

$$w(q_t)^T p_t = -\|w(q_t)\|^2. \tag{2.6}$$

Proof. For $t = 0$, equation (2.6) obviously holds. For $t > 0$, we have

$$\begin{aligned} w(q_t)^T p_t &= -\|w(q_t)\|^2 + \beta_k^{EMDY} w(q_t)^T p_{t-1} \\ &= -\|w(q_t)\|^2 + \frac{\|w(q_t)\|^2}{w_{t-1}^T p_{t-1}} w(q_t)^T p_{t-1} \\ &\quad + \min\left\{ -\frac{\|w(q_t)\|^2}{w_{t-1}^T p_{t-1}},\ \frac{\|w(q_t)\|^2}{(w_{t-1}^T p_{t-1})^2} w(q_t)^T p_{t-1} \right\} w(q_t)^T p_{t-1} \\ &\leq -\|w(q_t)\|^2 + \left(\frac{\|w(q_t)\|^2}{w_{t-1}^T p_{t-1}} - \frac{\|w(q_t)\|^2}{w_{t-1}^T p_{t-1}} \right) w(q_t)^T p_{t-1} \\ &= -\|w(q_t)\|^2. \end{aligned}$$

■

In what follows, we state the iterative procedures/steps of our method.

Algorithm 1

Input. Set an initial point $q_0 \in \mathcal{D}$, the positive constants: $Tol > 0$, $r \in (0,1)$, $m \in (0,2)$, $a > 0$, $\mu > 0$. Set $t = 0$.

Step 1. Compute $w(q_t)$. If $\|w(q_t)\| \leq Tol$, stop. Otherwise, generate the search direction p_t using (2.3).

Step 2. Compute the trial point $z_t = q_t + \varepsilon_t p_t$, where the step-size $\varepsilon_t = ar^i$ and i is the least nonnegative integer satisfying

$$-w(q_t + \varepsilon_t p_t)^T p_t \geq \mu \varepsilon_t \|p_t\|^2. \tag{2.7}$$

Step 3. If $z_t \in \mathcal{D}$ and $\|w(z_t)\| \leq Tol$, stop. Otherwise, compute the next iterate by

$$q_{t+1} = P_{\mathcal{D}}\left[q_t - m \frac{w(z_t)^T (q_t - z_t)}{\|w(z_t)\|^2} w(z_t) \right]. \tag{2.8}$$

Step 4. Finally we set $t = t + 1$ and return to step 1.

3. Convergence Analysis

In this section, we obtain the global convergence property of Algorithm 1. We also make the following assumptions on the mapping w.

Assumption 1.

(1) The solution set of the constrained nonlinear (1.4), denoted by $\mathcal{D}^*$, is nonempty.

(2) The mapping w is Lipschitz continuous on $\mathbb{R}^n$. That is, there exists a constant $L > 0$ such that

$$\|w(\alpha) - w(\beta)\| \leq L\|\alpha - \beta\| \ \forall \alpha, \beta \in \mathbb{R}^n. \tag{3.1}$$

(3) For any $\beta \in \mathcal{D}^*$ and $\alpha \in \mathbb{R}^n$, it holds that

$$w(\alpha)^T(\alpha - \beta) \geq 0. \tag{3.2}$$

Lemma 3.1. *Let $\{p_t\}$ and $\{q_t\}$ be two sequences generated by Algorithm 1. Then, there exists a step size ε_t satisfying the line search (2.7) for all $t \geq 0$.*

Proof. Suppose (2.7) does not hold for the iterate t_0-th, then we have for all $i \geq 0$,

$$-w(q_{t_0} + ar^i p_{t_0})^T p_{t_0} < \mu a r^i \|p_{t_0}\|^2.$$

Thus, by the continuity of w and with $0 < r < 1$, it follows that by letting $i \to \infty$, we have

$$-w(q_{t_0})^T p_{t_0} \leq 0,$$

which contradicts (2.6).

∎

Lemma 3.2. *Let the sequences $\{q_t\}$ and $\{z_t\}$ be generated by the Algorithm 1 method under Assumption 1, then*

$$a \geq \varepsilon_t \geq \frac{r\|w(q_t)\|^2}{(L+\mu)\|p_t\|^2}. \tag{3.3}$$

Proof. From Step 1 of Algorithm 1 $\varepsilon_t = ar^i \leq a$. Also, $\hat{\varepsilon}_t = \varepsilon_t r^{-1}$ does not satisfy (2.7). That is,

$$-w(q_t + \hat{\varepsilon}_t p_t)^T p_t < \mu \hat{\varepsilon}_t \|p_t\|^2.$$

From (3.1) and (2.6), it can be obviously seen that

$$\begin{aligned}
\|w(q_t)\|^2 &= -w(q_t)^T p_t \\
&= (w(q_t + \hat{\varepsilon}_t p_t) - w(q_t))^T p_t - w(q_t + \hat{\varepsilon}_t p_t)^T p_t \\
&\leq L\hat{\varepsilon}_t \|p_t\|^2 + \mu \hat{\varepsilon}_t \|p_t\|^2 \\
&\leq \hat{\varepsilon}_t (L+\mu)\|p_t\|^2.
\end{aligned}$$

This gives the desired inequality (3.3).

∎

Lemma 3.3. *Suppose that Assumption 1 holds. Let $\{q_t\}$ and $\{z_t\}$ be sequences generated by the Algorithm 1, then for any solution q^* contained in the solution set $\mathcal{D}^*$ the inequality*

$$\|q_{t+1} - q^*\|^2 \leq \|q_t - q^*\|^2 - m(2-m)\frac{\mu^2\|q_t - z_t\|^4}{\|w(z_t)\|^2}. \tag{3.4}$$

holds. In addition, $\{q_t\}$ is bounded and

$$\sum_{t=0}^{\infty} \|q_t - z_t\|^4 < +\infty. \tag{3.5}$$

Proof. First, we begin by using the weakly monotonicity assumption (Assumption 1 (3)) on the mapping w. Thus, for any solution $q^* \in \mathcal{D}^*$,

$$w(z_t)^T(q_t - q^*) \geq w(z_t)^T(q_t - z_t).$$

By the definition of z_t and (2.7), we obtain

$$w(q_t + \varepsilon_t p_t)^T(q_t - z_t) \geq \mu\varepsilon_t^2\|p_t\|^2. \tag{3.6}$$

From (2.1) and (3.6), we have the following

$$\begin{aligned}
\|q_{t+1} - q^*\|^2 &= \left\| P_{\mathcal{D}}\left[q_t - m\frac{w(z_t)^T(q_t - z_t)}{\|w(z_t)\|^2}w(z_t)\right] - q^*\right\|^2 \\
&\leq \left\|\left[q_t - m\frac{w(z_t)^T(q_t - z_t)}{\|w(z_t)\|^2}w(z_t)\right] - q^*\right\|^2 \\
&= \|q_t - q^*\|^2 - 2m\left(\frac{w(z_t)^T(q_t - z_t)}{\|w(z_t)\|^2}\right)w(z_t)^T(q_t - q^*) + m^2\left(\frac{w(z_t)^T(q_t - z_t)}{\|w(z_t)\|}\right)^2 \\
&\leq \|q_t - q^*\|^2 - 2m\left(\frac{w(z_t)^T(q_t - z_t)}{\|w(z_t)\|^2}\right)w(z_t)^T(q_t - z_t) + m^2\left(\frac{w(z_t)^T(q_t - z_t)}{\|w(z_t)\|}\right)^2 \\
&= \|q_t - q^*\|^2 - m(2 - m)\left(\frac{w(z_t)^T(q_t - z_t)}{\|w(z_t)\|}\right)^2 \\
&\leq \|q_t - q^*\|^2 - m(2 - m)\frac{\mu^2\|q_t - z_t\|^4}{\|w(z_t)\|^2}.
\end{aligned}$$

Thus, the sequence $\{\|q_t - q^*\|\}$ has a nonincreasing and convergent property. Therefore, this makes $\{q_t\}$ to be bounded and from (3.4), it holds that

$$\mu^2\sum_{t=0}^{\infty}\|q_t - z_t\|^4 < \|q_0 - q^*\|^2 < +\infty.$$

■

Remark 3.4. Taking into account of the definition of z_t and also by (3.5), it can be deduced that

$$\lim_{t\to\infty} \varepsilon_t\|p_t\| = 0. \tag{3.7}$$

Theorem 3.5. *Suppose Assumption 1 holds. Let $\{q_t\}$ and $\{z_t\}$ be sequences generated by Algorithm 1, then*

$$\liminf_{t\to\infty} \|w(q_t)\| = 0. \tag{3.8}$$

Proof. Suppose (3.8) is not valid, that is, there exist a constant say $s > 0$ such that $s \leq \|w(q_t)\|, \forall t \geq 0$. From the sufficient descent condition (2.6) and Cauchy-Schwartz inequality, we get that $\|p_t\| \geq \|w(q_t)\|$. This implies that

$$\|p_t\| \geq s, \ \forall t \geq 0. \tag{3.9}$$

Obviously, it can be seen from the definition of w_{t-1} in (2.5), that

$$w_{t-1}^T p_{t-1} \geq y_{t-1}^T p_{t-1} + \|p_{t-1}\|^2 - y_{t-1}^T p_{t-1} = \|p_{t-1}\|^2. \tag{3.10}$$

Having in view that from Lemma 3.3 that the sequences $\{q_t\}$ is bounded by a positive constant say k_b. In addition with the continuity of w, it further implies that $\{\|w(q_t)\|\}$ is bounded by a constant say u. Thus, from (2.3) and (3.10), it follows that for all $t \geq 1$,

$$\begin{aligned}
&\|p_t\| = \left\|-w(q_t) + \beta_t^{EMDY} p_{t-1}\right\| \\
&= \left\|-w(q_t) + \left(\frac{\|w(q_t)\|^2}{w_{t-1}^T p_{t-1}} + \min\left\{-\frac{\|w(q_t)\|^2}{w_{t-1}^T p_{t-1}}, \frac{\|w(q_t)\|^2}{(w_{t-1}^T p_{t-1})^2} w(q_t)^T p_{t-1}\right\}\right) p_{t-1}\right\| \\
&\leq \|w(q_t)\| \\
&\leq u.
\end{aligned}$$

From (3.3), we have

$$\begin{aligned}
\varepsilon_t \|p_t\| &\geq \frac{r\|w(q_t)\|^2}{(L+\mu)\|p_t\|^2}\|p_t\| \\
&\geq \frac{rs^2}{(L+\mu)u} > 0,
\end{aligned}$$

which contradicts (3.7). Hence (3.8) is valid. ■

4. Numerical Experiments

This section focuses on the numerical experiments for testing the efficiency of the proposed method based on the Dolan and More performance profile [45]. Nine test problems were considered for the experiments with all codes written and implemented in MATLAB R2019b and performed on a PC Desktop Intel(R) Core(TM) i7-6600U CPU @ 2.60GHz 2.81 GHz, RAM 16.00 GB.

The proposed method is compared with methods which shares similar characteristics. For example, the derivative-free iterative method for nonlinear monotone equations with convex constraints (PDY) [20] and the conjugate gradient method to solve convex constrained monotone equations (CGD) [46]. All algorithms are terminated when

$$\|w(q_t)\| \leq 10^{-6}.$$

Our proposed algorithm (denoted as EMDY) was implemented using the following parameters: $a = 1,\ r = 0.8,\ m = 1.2,\ \mu = 10^{-4}$; while PDY and CGD were implemented using the parameters reported in their respective papers. More so, our experiment made use of various dimensions (for instance 1000, 5000, 10,000, 50,000, 100,000) and initial $\big(q_1 = (0.1, 0.1, \cdots, 0.1)^T,\ q_2 = (0.2, 0.2, \cdots, 0.2)^T,\ q_3 = (0.5, 0.5, \ldots, 0.5)^T,\ q_4 = (1.2, 1.2, \cdots, 1.2)^T,\ q_5 = (1.5, 1.5, \cdots, 1.5)^T,\ q_6 = (2, 2, \ldots, 2)^T,\ q_7 = \text{rand}(0, 1)\big)$.

The test problems with their mappings takes as $w = (w_1, w_2, \cdots, w_n)$ are given below:

Problem 1 [47] Exponential Function.

$$\begin{aligned}
&w_1(q) = e^{q_1} - 1, \\
&w_i(q) = e^{q_i} + q_i - 1, \text{ for } i = 2, 3, ..., n, \\
&\text{and } \mathcal{D} = \mathbb{R}_+^n.
\end{aligned}$$

Problem 2 [47] Modified Logarithmic Function.

$$w_i(q) = \ln(q_i+1) - \frac{q_i}{n}, \text{ for } i = 1,2,3,...,n,$$

$$\text{and } \mathcal{D} = \left\{ q \in \mathbb{R}^n : \sum_{i=1}^{n} q_i \le n, q_i > -1, i = 1,2,\cdots,n \right\}.$$

Problem 3 [48]

$$w_i(q) = \min\left(\min(|q_i|, q_i^2), \max(|q_i|, q_i^3)\right) \text{ for } i = 2,3,...,n,$$

$$\text{and} \mathcal{D} = \mathbb{R}^n_+.$$

Problem 4 [47] Strictly Convex Function I.

$$w_i(q) = e^{q_i} - 1, \text{ for } i = 1,2,...,n,$$

$$\text{and } \mathcal{D} = \mathbb{R}^n_+.$$

Problem 5 [47] Strictly Convex Function II.

$$w_i(q) = \frac{i}{n} e^{q_i} - 1, \text{ for } i = 1,2,...,n,$$

$$\text{and } \mathcal{D} = \mathbb{R}^n_+.$$

Problem 6 [49] Tridiagonal Exponential Function.

$$w_1(q) = q_1 - e^{\cos(h(q_1+q_2))},$$

$$w_i(q) = q_i - e^{\cos(h(q_{i-1}+q_i+q_{i+1}))}, \text{ for } i = 2,...,n-1,$$

$$w_n(q) = q_n - e^{\cos(h(q_{n-1}+q_n))},$$

$$h = \frac{1}{n+1}$$

Problem 7 [18] Nonsmooth Function.

$$w_i(q) = q_i - \sin|q_i - 1|, \ i = 1,2,3,...,n,$$

$$\text{and } \mathcal{D} = \left\{ q \in \mathbb{R}^n : \sum_{i=1}^{n} q_i \le n, q_i \ge -1, i = 1,2,\cdots,n \right\}.$$

Problem 8 [47] The Trig exp function

$$w_1(q) = 3q_1^3 + 2q_2 - 5 + \sin(q_1 - q_2)\sin(q_1 + q_2)$$

$$w_i(q) = 3q_i^3 + 2q_{i+1} - 5 + \sin(q_i - q_{i+1})\sin(q_i + q_{i+1}) + 4q_i - q_{i-1}e^{q_{i-1}-q_i} - 3$$

$$\text{for } i = 2,3,...,n-1$$

$$w_n(q) = q_{n-1}e^{q_{n-1}-q_n} - 4q_n - 3, \ \text{ where h} = \frac{1}{m+1} \text{ and } \mathcal{D} = \mathbb{R}^n_+..$$

Problem 9 [50]

$$t_i = \sum_{i=1}^{n} q_i^2, \ d = 10^{-5}$$

$$w_i(q) = 2d(q_i - 1) + 4(t_i - 0.25)q_i, \ i = 1,2,3,...,n.\text{and} \quad \mathcal{D} = \mathbb{R}^n_+.$$

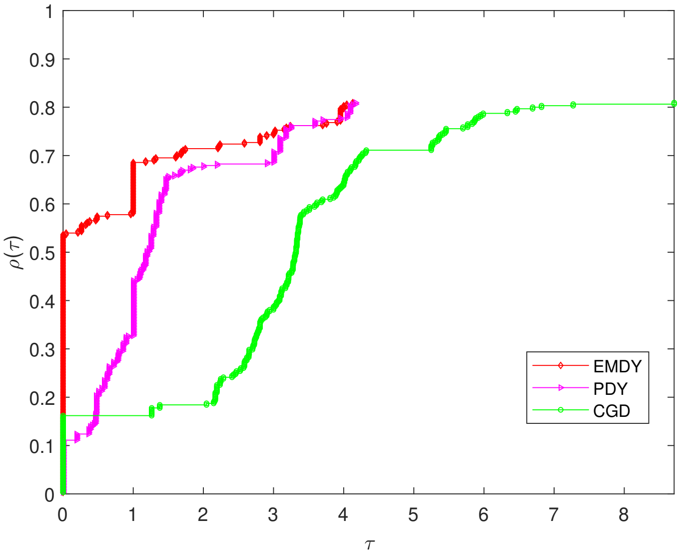

FIGURE 1. Performance profiles based on number of iterations.

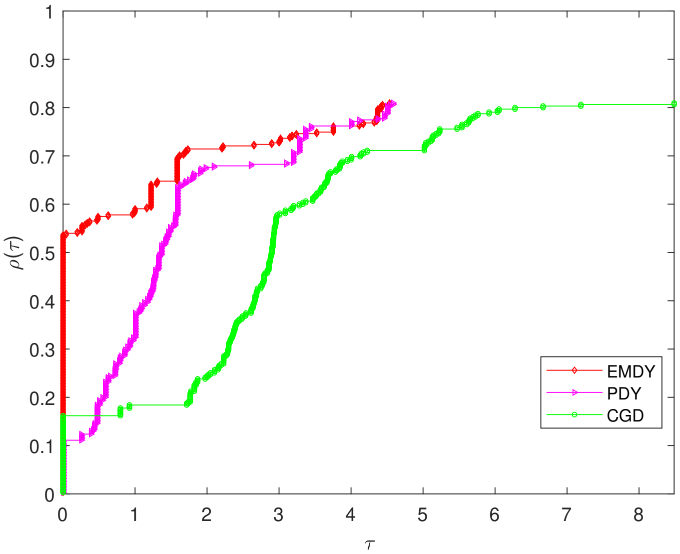

FIGURE 2. Performance profiles based on number of function evaluations.

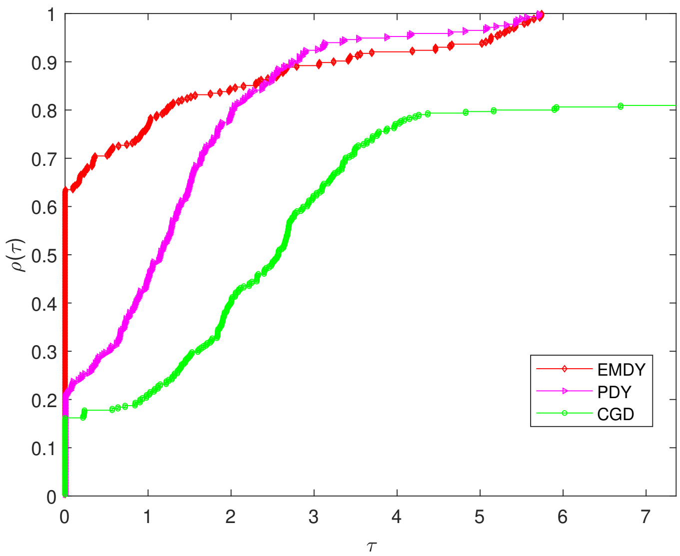

Figure 3. Performance profiles based on CPU time (in seconds).

In all the numerical results reported in Tables 1-9, it can be observed that our proposed EMDY method is competitive and promising. We note that, in the detailed numerical report presented in Table 1-9 of the Appendix section, "DM" denotes dimension, "IP" denotes initial point, "$\#IT$" denotes the number of iteration, "FV" denotes the number of function evaluation, "RT" denotes the CPU run time and "NM" is the norm. On the overall, to visualize the performance of EMDY verses the compared methods, we employ the well-known performance profiles of Dolan and Moré [45] defined as:

$$\rho(\tau) := \frac{1}{|T_P|}\left|\left\{t_p \in T_P : \log_2\left(\frac{t_{p,q}}{\min\{t_{p,q} : q \in Q\}}\right) \leq \tau\right\}\right|,$$

where T_P is the test set, $|T_P|$ is the number of problems in the test set T_P, Q is the set of optimization solvers, and $t_{p,q}$ is the NI (or the NF) for $t_p \in T_P$ and $q \in Q$. The following figures were obtained using the above performance profiles.

It can be observed from the above figures that with respect to number of iterations ($\#IT$), EMDY method was more successful as it has the least iteration number in over 50% of the problems as against PDY and CGD with less than 20% success. The interpretation is almost similar with respect to function evaluation number (FV). Finally, in terms of the CPU run time (RT), EMDY has the least in over 60% of the problems as against PDY and CGD with less than 25% success.

In summary, we can deduce that:

• The proposed method is efficient for solving the convex constraint nonlinear equation (1.4).

• Furthermore, the EMDY method outperforms the PDY and CGD method.

5. Conclusion

In this article, we proposed an extended modified Dai-Yuan conjugate gradient algorithm for solving constraint nonlinear equations. The search direction is sufficiently descent and bounded. The global convergence of the algorithm was established under some appropriate assumptions such as monotonicity and Lipschitz continuity on the operator. Numerical results were also provided to show the efficiency and competitiveness of the proposed algorithm compared with other existing algorithms. Based on the numerical results, we can conclude that the proposed algorithm is an alternative.

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Appendix

Table 1. Numerical result for problem 1

				EMDY				PDY				CGD	
DM	IP	#IT	FV	RT	NM	#IT	FV	RT	NM	#IT	FV	RT	NM
1000	q_1	2	7	0.005221	0	16	64	0.037824	3.45E-07	77	230	0.058808	8.66E-07
	q_2	2	7	0.004932	0	16	64	0.02392	7.03E-07	81	242	0.048604	9.44E-07
	q_3	2	7	0.005825	0	17	68	0.016334	6.22E-07	88	263	0.032011	9.54E-07
	q_4	2	7	0.004618	0	18	72	0.017108	4.54E-07	-	-	-	-
	q_5	2	7	0.004827	0	18	72	0.019795	3.65E-07	-	-	-	-
	q_6	2	7	0.004566	0	18	72	0.017994	3.80E-07	-	-	-	-
	q_7	7	28	0.008171	1.04E-07	17	68	0.01338	7.16E-07	-	-	-	-
5000	q_1	2	7	0.012558	0	16	64	0.066424	7.61E-07	76	227	0.099743	9.80E-07
	q_2	2	7	0.009678	0	17	68	0.057352	5.15E-07	82	245	0.09831	9.79E-07
	q_3	2	7	0.013249	0	18	72	0.054222	4.63E-07	88	263	0.094972	9.12E-07
	q_4	2	7	0.011553	0	19	76	0.065217	3.38E-07	-	-	-	-
	q_5	2	7	0.012447	0	18	72	0.049595	8.12E-07	-	-	-	-
	q_6	2	7	0.014483	0	18	72	0.049791	8.10E-07	-	-	-	-
	q_7	7	28	0.03735	2.27E-07	18	72	0.063816	5.34E-07	612	1835	0.66244	9.85E-07
10000	q_1	2	7	0.018914	0	17	68	0.087952	3.55E-07	76	227	0.1543	8.46E-07
	q_2	2	7	0.027386	0	17	68	0.079454	7.27E-07	83	248	0.1613	9.67E-07
	q_3	2	7	0.016454	0	18	72	0.0919	6.55E-07	87	260	0.17036	9.37E-07
	q_4	2	7	0.01758	0	19	76	0.15372	4.77E-07	-	-	-	-
	q_5	2	7	0.019436	0	20	80	0.11427	4.52E-07	-	-	-	-
	q_6	2	7	0.024777	0	19	76	0.099506	5.51E-07	-	-	-	-
	q_7	7	28	0.044221	3.16E-07	18	72	0.11131	7.55E-07	621	1862	1.2548	9.83E-07
50000	q_1	2	7	0.057066	0	17	68	0.37509	7.93E-07	76	227	0.63892	9.25E-07
	q_2	2	7	0.065781	0	18	72	0.37185	5.44E-07	82	245	0.6818	8.81E-07
	q_3	2	7	0.063818	0	19	76	0.3945	4.86E-07	85	254	0.71491	9.04E-07
	q_4	2	7	0.061612	0	20	80	0.41375	9.70E-07	-	-	-	-
	q_5	2	7	0.07211	0	22	88	0.53434	8.63E-07	-	-	-	-
	q_6	2	7	0.095546	0	23	92	0.59942	8.62E-07	-	-	-	-
	q_7	7	28	0.16406	7.22E-07	19	76	0.40117	5.61E-07	-	-	-	-
100000	q_1	2	7	0.12734	0	18	72	0.72616	3.76E-07	77	230	1.1887	9.97E-07
	q_2	2	7	0.10155	0	18	72	0.71564	7.69E-07	79	236	1.3242	8.85E-07
	q_3	2	7	0.12914	0	19	76	0.7505	6.88E-07	838	2513	13.3577	9.96E-07
	q_4	2	7	0.11379	0	23	92	0.98331	3.63E-07	-	-	-	-
	q_5	2	7	0.15285	0	23	92	1.0758	9.61E-07	-	-	-	-
	q_6	2	7	0.12571	0	26	104	1.2665	3.39E-07	-	-	-	-
	q_7	8	32	0.36467	1.78E-08	20	80	0.82349	7.79E-07	828	2483	13.0692	9.90E-07

Table 2. Numerical result for problem 2

		EMDY				PDY				CGD			
DM	IP	#IT	FV	RT	NM	#IT	FV	RT	NM	#IT	FV	RT	NM
1000	q_1	6	19	0.032101	3.20E-09	13	51	0.012439	7.68E-07	77	229	0.028973	9.99E-07
	q_2	6	19	0.020511	3.95E-09	15	59	0.042509	3.49E-07	927	2779	0.32168	9.97E-07
	q_3	6	19	0.009604	6.74E-07	16	63	0.020906	6.98E-07	-	-	-	-
	q_4	7	22	0.009733	3.73E-09	18	71	0.012086	3.52E-07	-	-	-	-
	q_5	6	19	0.017925	7.44E-07	18	71	0.015911	5.13E-07	116	343	0.056803	9.77E-07
	q_6	7	22	0.00891	3.50E-09	18	71	0.014349	8.59E-07	119	352	0.051033	8.86E-07
	q_7	10	32	0.010738	6.87E-07	17	67	0.014091	4.50E-07	111	331	0.051063	9.50E-07
5000	q_1	6	20	0.018643	5.43E-08	14	55	0.041353	5.44E-07	678	2032	1.1284	9.98E-07
	q_2	6	20	0.014783	6.64E-08	15	59	0.037994	7.63E-07	119	353	0.20315	9.01E-07
	q_3	6	19	0.050164	3.01E-07	17	67	0.05265	5.12E-07	112	332	0.17954	8.71E-07
	q_4	7	23	0.021263	6.33E-08	18	71	0.04944	7.73E-07	120	355	0.21445	9.97E-07
	q_5	6	19	0.02299	4.13E-07	19	75	0.078857	3.75E-07	120	355	0.22383	9.20E-07
	q_6	7	23	0.025022	5.80E-08	19	75	0.065929	6.27E-07	125	370	0.1982	9.33E-07
	q_7	11	35	0.037659	3.79E-07	17	67	0.074387	9.82E-07	118	352	0.2434	9.05E-07
10000	q_1	7	27	0.039059	1.22E-07	14	55	0.072486	7.66E-07	-	-	-	-
	q_2	7	26	0.034026	1.49E-07	16	63	0.081359	3.55E-07	126	374	0.37403	8.65E-07
	q_3	7	26	0.034292	6.48E-07	17	67	0.1068	7.23E-07	126	374	0.39658	9.26E-07
	q_4	8	30	0.040023	1.42E-07	19	75	0.096759	3.63E-07	119	352	0.36288	9.94E-07
	q_5	7	26	0.037387	8.95E-07	19	75	0.09909	5.29E-07	121	358	0.37717	9.20E-07
	q_6	8	30	0.043234	1.30E-07	19	76	0.10435	9.51E-07	116	345	0.3702	9.52E-07
	q_7	8	30	0.041806	4.39E-07	18	71	0.098908	4.63E-07	120	358	0.47533	8.98E-07
50000	q_1	7	27	0.12633	2.71E-07	15	59	0.32604	5.78E-07	127	377	1.57	9.75E-07
	q_2	7	27	0.12252	3.30E-07	16	63	0.30631	7.92E-07	130	386	1.6566	9.05E-07
	q_3	8	30	0.15369	5.74E-08	18	71	0.37626	5.36E-07	128	380	1.5649	9.12E-07
	q_4	8	30	0.15824	3.16E-07	21	84	0.46957	3.43E-07	129	382	1.9692	9.00E-07
	q_5	8	30	0.16261	8.07E-08	21	84	0.47051	4.72E-07	131	388	1.6508	9.64E-07
	q_6	8	30	0.15618	2.87E-07	21	84	0.46381	4.77E-07	130	385	1.8887	9.60E-07
	q_7	8	30	0.15421	9.92E-07	19	75	0.43375	3.46E-07	124	370	2.2707	9.58E-07
100000	q_1	7	27	0.25699	3.83E-07	15	59	0.59915	8.17E-07	132	392	3.4978	9.73E-07
	q_2	7	27	0.23116	4.67E-07	17	67	0.85174	3.76E-07	134	398	3.376	9.96E-07
	q_3	8	30	0.27668	8.10E-08	18	72	0.80688	9.65E-07	126	374	4.3133	9.02E-07
	q_4	8	30	0.27351	4.46E-07	22	88	0.97108	8.28E-07	130	385	5.6532	9.91E-07
	q_5	8	30	0.28719	1.14E-07	22	88	0.94559	8.18E-07	133	394	5.3457	9.78E-07
	q_6	8	30	0.30368	4.06E-07	22	88	1.0474	7.87E-07	129	384	3.9504	9.43E-07
	q_7	9	34	0.34015	5.55E-08	20	80	0.89246	5.46E-07	126	376	6.5207	9.70E-07

Table 3. Numerical result for problem 3

		EMDY				PDY				CGD			
DM	IP	#IT	FV	RT	NM	#IT	FV	RT	NM	#IT	FV	RT	NM
1000	q_1	2	6	0.030505	0	2	6	0.006131	0	1	2	0.004947	0
	q_2	2	6	0.002982	0	2	6	0.007275	0	1	2	0.002058	0
	q_3	2	6	0.00588	0	2	6	0.003828	0	1	2	0.002925	0
	q_4	2	7	0.007478	0	2	6	0.004078	0	1	3	0.002616	0
	q_5	2	7	0.004305	0	2	6	0.003418	0	1	3	0.004015	0
	q_6	2	7	0.004618	0	2	6	0.001919	0	1	3	0.002984	0
	q_7	6	23	0.009401	3.80E-07	2	6	0.002393	0	1	2	0.002801	0
5000	q_1	2	6	0.011101	0	2	6	0.008606	0	1	2	0.005551	0
	q_2	2	6	0.010828	0	2	6	0.011034	0	1	2	0.005309	0
	q_3	2	6	0.013438	0	2	6	0.009017	0	1	2	0.023812	0
	q_4	2	7	0.01146	0	2	6	0.009874	0	1	3	0.00584	0
	q_5	2	7	0.012405	0	2	6	0.009558	0	1	3	0.006556	0
	q_6	2	7	0.021066	0	2	6	0.006467	0	1	3	0.005306	0
	q_7	7	27	0.035447	5.55E-08	2	6	0.011348	0	1	2	0.007198	0
10000	q_1	2	6	0.016518	0	2	6	0.015772	0	1	2	0.008827	0
	q_2	2	6	0.02131	0	2	6	0.018236	0	1	2	0.008035	0
	q_3	2	6	0.017712	0	2	6	0.02033	0	1	2	0.020771	0
	q_4	2	7	0.023701	0	2	6	0.015231	0	1	3	0.01173	0
	q_5	2	7	0.018668	0	2	6	0.015895	0	1	3	0.008678	0
	q_6	2	7	0.019648	0	2	6	0.02343	0	1	3	0.009625	0
	q_7	7	27	0.054559	6.64E-08	2	6	0.019685	0	1	2	0.023142	0
50000	q_1	2	6	0.062189	0	2	6	0.06444	0	1	2	0.029067	0
	q_2	2	6	0.092421	0	2	6	0.061962	0	1	2	0.039209	0
	q_3	2	6	0.070691	0	2	6	0.071089	0	1	2	0.030268	0
	q_4	2	7	0.072241	0	2	6	0.048252	0	1	3	0.030737	0
	q_5	2	7	0.062769	0	2	6	0.056395	0	1	3	0.033978	0
	q_6	2	7	0.076984	0	2	6	0.045351	0	1	3	0.03421	0
	q_7	7	27	0.18641	1.65E-07	2	7	0.11221	0	1	2	0.031624	0
100000	q_1	2	6	0.11744	0	2	6	0.11724	0	1	2	0.05904	0
	q_2	2	6	0.12054	0	2	6	0.12336	0	1	2	0.061714	0
	q_3	2	6	0.1257	0	2	6	0.13151	0	1	2	0.086059	0
	q_4	2	7	0.12024	0	2	6	0.092938	0	1	3	0.062868	0
	q_5	2	7	0.13283	0	2	6	0.089919	0	1	3	0.079338	0
	q_6	2	7	0.12156	0	2	6	0.090501	0	1	3	0.064663	0
	q_7	7	27	0.40072	2.35E-07	2	7	0.1607	0	1	2	0.06281	0

Table 4. Numerical result for problem 4

		EMDY				PDY				CGD			
DM	IP	#IT	FV	RT	NM	#IT	FV	RT	NM	#IT	FV	RT	NM
	q_1	2	7	0.02032	0	15	60	0.011597	5.13E-07	104	311	0.030371	9.76E-07
	q_2	2	7	0.00331	0	16	64	0.017256	3.59E-07	109	326	0.031103	9.00E-07
	q_3	7	28	0.005993	7.05E-07	16	64	0.00812	9.42E-07	116	347	0.037892	9.53E-07
1000	q_4	2	7	0.003624	0	15	60	0.009109	6.44E-07	-	-	-	-
	q_5	2	7	0.006163	0	17	68	0.019514	3.91E-07	-	-	-	-
	q_6	2	7	0.004583	0	17	68	0.014318	7.89E-07	-	-	-	-
	q_7	13	52	0.009381	1.61E-07	17	68	0.011022	5.29E-07	113	338	0.030426	9.10E-07
	q_1	2	7	0.010876	0	16	64	0.029545	3.86E-07	109	326	0.12176	9.27E-07
	q_2	2	7	0.01285	0	16	64	0.032858	8.02E-07	116	347	0.10788	8.72E-07
	q_3	8	32	0.069383	3.15E-07	17	68	0.030346	7.00E-07	120	359	0.1134	9.91E-07
5000	q_4	2	7	0.012306	0	16	64	0.034047	4.74E-07	-	-	-	-
	q_5	2	7	0.010902	0	17	68	0.046379	8.74E-07	-	-	-	-
	q_6	2	7	0.013541	0	19	76	0.049272	5.11E-07	-	-	-	-
	q_7	10	40	0.029742	4.36E-07	18	72	0.033088	3.73E-07	122	365	0.12027	9.15E-07
	q_1	2	7	0.011806	0	16	64	0.065994	5.46E-07	114	341	0.19235	8.39E-07
	q_2	2	7	0.017303	0	17	68	0.049492	3.76E-07	118	353	0.22441	8.90E-07
	q_3	8	32	0.027319	4.46E-07	17	68	0.059866	9.90E-07	122	365	0.23321	9.11E-07
10000	q_4	2	7	0.018963	0	19	76	0.084262	3.70E-07	-	-	-	-
	q_5	2	7	0.022131	0	18	72	0.062557	4.15E-07	-	-	-	-
	q_6	2	7	0.021299	0	19	76	0.075499	7.22E-07	-	-	-	-
	q_7	10	40	0.055298	4.71E-07	18	72	0.056165	5.22E-07	123	368	0.21969	8.54E-07
	q_1	2	7	0.075152	0	17	68	0.21219	4.04E-07	117	350	0.78216	9.70E-07
	q_2	2	7	0.052256	0	17	68	0.2462	8.40E-07	124	371	0.86363	8.76E-07
	q_3	8	32	0.11188	9.97E-07	18	72	0.21904	7.39E-07	128	383	1.0627	9.89E-07
50000	q_4	2	7	0.058875	0	20	80	0.30397	6.25E-07	-	-	-	-
	q_5	2	7	0.063487	0	20	80	0.28388	8.13E-07	-	-	-	-
	q_6	2	7	0.066448	0	22	88	0.39048	9.65E-07	-	-	-	-
	q_7	10	40	0.14187	4.49E-07	19	76	0.24784	6.73E-07	130	389	0.91681	8.98E-07
	q_1	2	7	0.1051	0	17	68	0.41122	5.71E-07	121	362	1.6565	9.15E-07
	q_2	2	7	0.096758	0	18	72	0.49882	3.98E-07	127	380	1.6432	9.31E-07
	q_3	9	36	0.21759	2.82E-07	19	76	0.4804	9.57E-07	133	398	1.8604	9.16E-07
100000	q_4	2	7	0.091055	0	22	88	0.59813	3.99E-07	-	-	-	-
	q_5	2	7	0.097945	0	24	96	0.72938	3.66E-07	-	-	-	-
	q_6	2	7	0.11507	0	26	104	0.8351	3.55E-07	-	-	-	-
	q_7	11	44	0.26034	1.71E-07	19	76	0.48396	9.52E-07	132	395	1.6933	9.56E-07

Table 5. Numerical result for problem 5

		EMDY				PDY				CGD			
DM	IP	#IT	FV	RT	NM	#IT	FV	RT	NM	#IT	FV	RT	NM
1000	q_1	38	150	0.057238	8.47E-08	19	75	0.012902	6.70E-07	119	352	0.036197	9.35E-07
	q_2	22	86	0.019439	4.16E-07	19	75	0.020756	6.02E-07	119	353	0.034849	9.45E-07
	q_3	26	102	0.018633	2.36E-07	20	79	0.014886	8.17E-07	135	403	0.038107	8.58E-07
	q_4	24	96	0.057533	8.45E-07	20	80	0.011195	4.14E-07	137	410	0.044027	8.93E-07
	q_5	25	99	0.026711	5.17E-07	20	80	0.013151	3.51E-07	130	389	0.035436	9.32E-07
	q_6	15	60	0.020853	8.85E-07	21	84	0.014673	3.89E-07	-	-	-	-
	q_7	33	130	0.030795	3.77E-07	26	103	0.017087	5.20E-07	138	412	0.048861	9.72E-07
5000	q_1	53	208	0.23984	9.93E-07	20	79	0.044101	6.26E-07	-	-	-	-
	q_2	31	122	0.12894	5.58E-07	20	79	0.043518	5.64E-07	133	395	0.16445	9.99E-07
	q_3	29	115	0.1486	2.56E-07	21	83	0.060597	7.12E-07	147	439	0.23294	9.57E-07
	q_4	41	164	0.27392	7.97E-07	21	84	0.055785	3.38E-07	149	446	0.24793	9.59E-07
	q_5	97	388	0.74454	5.56E-08	21	84	0.069986	4.47E-07	142	425	0.19374	9.27E-07
	q_6	57	228	0.38549	4.04E-07	21	84	0.064768	6.59E-07	-	-	-	-
	q_7	90	359	0.63684	6.95E-08	28	111	0.076138	9.84E-07	151	451	0.18925	9.57E-07
10000	q_1	40	157	0.29205	9.17E-07	20	79	0.080163	9.79E-07	119	351	0.24959	9.30E-07
	q_2	40	158	0.43714	8.20E-07	20	79	0.074932	8.67E-07	154	460	0.37607	9.55E-07
	q_3	54	214	0.58976	2.19E-07	22	87	0.085226	4.07E-07	150	448	0.30268	9.36E-07
	q_4	70	279	1.037	2.10E-07	23	92	0.088099	4.76E-07	154	461	0.33386	8.50E-07
	q_5	98	392	1.2778	8.38E-07	21	84	0.098655	7.05E-07	147	440	0.33495	9.92E-07
	q_6	66	264	0.87059	4.47E-08	21	84	0.092469	5.31E-07	-	-	-	-
	q_7	77	307	0.9818	4.01E-07	25	100	0.091455	3.52E-07	151	451	0.34263	9.49E-07
50000	q_1	52	206	2.3011	4.89E-07	23	92	0.37009	4.69E-07	152	453	1.3242	8.54E-07
	q_2	77	306	4.2117	7.08E-07	23	92	0.36639	4.37E-07	162	484	1.471	9.01E-07
	q_3	72	287	3.666	8.86E-07	22	88	0.33903	8.93E-07	166	496	1.4469	8.91E-07
	q_4	192	768	13.4126	9.73E-07	24	96	0.39936	5.83E-07	166	497	1.4911	8.48E-07
	q_5	227	907	15.0917	1.57E-07	24	96	0.38467	5.87E-07	159	476	1.4245	8.71E-07
	q_6	149	596	9.6879	6.28E-07	23	92	0.43919	8.28E-07	-	-	-	-
	q_7	240	959	15.8727	2.10E-07	27	108	0.45222	4.05E-07	166	496	1.484	9.79E-07
100000	q_1	60	238	4.4751	5.61E-08	24	96	0.70772	8.11E-07	139	413	2.3777	9.00E-07
	q_2	77	306	6.3071	9.61E-07	24	96	0.75823	7.59E-07	168	502	2.9556	9.21E-07
	q_3	172	686	22.883	4.17E-08	23	92	0.65294	4.30E-07	172	514	2.7159	9.15E-07
	q_4	202	808	25.3557	2.13E-07	25	100	0.78073	3.79E-07	173	518	2.7994	8.68E-07
	q_5	227	907	26.8888	7.99E-08	25	100	0.77059	5.83E-07	163	488	2.6314	9.81E-07
	q_6	177	708	22.2072	5.34E-07	26	104	0.89351	3.96E-07	-	-	-	-
	q_7	195	779	26.3089	6.56E-07	24	96	0.71172	9.33E-07	173	517	2.7553	9.63E-07

Table 6. Numerical result for problem 6

		EMDY				PDY				CGD			
DM	IP	#IT	FV	RT	NM	#IT	FV	RT	NM	#IT	FV	RT	NM
1000	q_1	13	52	0.081128	3.38E-07	18	72	0.024111	4.82E-07	130	389	0.061581	9.85E-07
	q_2	13	52	0.011159	3.25E-07	18	72	0.022267	4.64E-07	130	389	0.059126	9.47E-07
	q_3	13	52	0.020428	2.86E-07	18	72	0.01605	4.08E-07	126	377	0.059245	8.84E-07
	q_4	12	48	0.010343	9.81E-07	17	68	0.016334	8.34E-07	124	371	0.067177	9.93E-07
	q_5	12	48	0.010191	7.87E-07	17	68	0.016901	6.69E-07	119	356	0.050634	9.34E-07
	q_6	12	48	0.013041	4.64E-07	17	68	0.019962	3.94E-07	120	359	0.049251	8.91E-07
	q_7	13	52	0.009868	2.89E-07	18	72	0.016165	4.12E-07	128	383	0.055122	9.12E-07
5000	q_1	13	52	0.043051	7.58E-07	19	76	0.079461	3.58E-07	132	395	0.25216	9.54E-07
	q_2	13	52	0.040608	7.29E-07	19	76	0.081992	3.44E-07	134	401	0.27542	9.28E-07
	q_3	13	52	0.045427	6.42E-07	18	72	0.082551	9.14E-07	131	392	0.26997	8.96E-07
	q_4	13	52	0.042083	4.40E-07	18	72	0.06617	6.26E-07	131	392	0.27896	8.66E-07
	q_5	13	52	0.03935	3.53E-07	18	72	0.067224	5.02E-07	126	377	0.24968	9.93E-07
	q_6	13	52	0.040382	2.08E-07	17	68	0.081465	8.83E-07	126	377	0.23986	9.16E-07
	q_7	13	52	0.043528	6.46E-07	18	72	0.076475	9.20E-07	133	398	0.27881	9.29E-07
10000	q_1	14	56	0.080183	2.14E-07	21	84	0.16321	4.00E-07	136	407	0.52609	8.81E-07
	q_2	14	56	0.10049	2.06E-07	21	84	0.16393	3.85E-07	136	407	0.52972	9.01E-07
	q_3	13	52	0.078795	9.09E-07	20	80	0.14211	5.83E-07	135	404	0.49628	8.67E-07
	q_4	13	52	0.068276	6.22E-07	18	72	0.15009	8.85E-07	133	398	0.614	8.99E-07
	q_5	13	52	0.069141	4.99E-07	18	72	0.1422	7.10E-07	131	392	0.63402	9.64E-07
	q_6	13	52	0.064335	2.94E-07	18	72	0.1296	4.19E-07	128	383	0.59342	9.99E-07
	q_7	13	52	0.079934	9.15E-07	20	80	0.14254	5.88E-07	135	404	0.68837	9.71E-07
50000	q_1	14	56	0.28325	4.80E-07	24	96	0.81018	7.08E-07	142	425	3.0408	9.29E-07
	q_2	14	56	0.29107	4.61E-07	24	96	0.80881	6.81E-07	142	425	2.1142	9.46E-07
	q_3	14	56	0.33891	4.06E-07	23	92	0.69993	7.26E-07	139	416	2.0689	9.98E-07
	q_4	14	56	0.27472	2.78E-07	21	84	0.61851	5.18E-07	140	419	2.0753	8.54E-07
	q_5	14	56	0.30714	2.23E-07	21	84	0.59983	4.16E-07	137	410	1.9041	9.70E-07
	q_6	13	52	0.26436	6.58E-07	18	72	0.51886	9.36E-07	133	398	1.8323	9.60E-07
	q_7	14	56	0.26555	4.10E-07	23	92	0.72386	7.32E-07	141	422	2.0335	9.41E-07
100000	q_1	14	56	0.59732	6.78E-07	29	116	2.19	5.93E-07	146	437	4.6878	8.55E-07
	q_2	14	56	0.79247	6.52E-07	28	112	2.1771	6.09E-07	147	440	4.4795	8.50E-07
	q_3	14	56	0.72287	5.75E-07	26	104	1.8398	6.39E-07	141	422	4.3297	9.12E-07
	q_4	14	56	0.63615	3.93E-07	23	92	1.5301	7.03E-07	142	425	4.3388	9.32E-07
	q_5	14	56	0.62979	3.16E-07	22	88	1.5168	3.66E-07	138	413	4.1973	9.04E-07
	q_6	13	52	0.65098	9.30E-07	20	80	1.2187	5.97E-07	135	404	4.19	9.69E-07
	q_7	14	56	0.66845	5.80E-07	26	104	1.8971	6.45E-07	143	428	4.2975	9.27E-07

Table 7. Numerical result for problem 7

		EMDY				PDY				CGD			
DM	IP	#IT	FV	RT	NM	#IT	FV	RT	NM	#IT	FV	RT	NM
1000	q_1	7	28	0.034773	3.18E-07	17	68	0.014092	6.92E-07	63	188	0.042695	8.68E-07
	q_2	7	28	0.005371	1.87E-07	17	68	0.012727	4.34E-07	61	182	0.019517	7.22E-07
	q_3	6	24	0.031885	2.92E-07	5	20	0.006327	4.50E-08	48	143	0.017961	7.05E-07
	q_4	7	28	0.008459	4.98E-07	18	72	0.010722	8.82E-07	63	188	0.020096	7.60E-07
	q_5	7	28	0.005978	8.54E-07	19	76	0.020154	8.09E-07	65	194	0.024797	8.17E-07
	q_6	8	31	0.008916	1.55E-07	18	71	0.011435	5.23E-07	64	190	0.027288	8.41E-07
	q_7	11	44	0.008701	5.33E-07	19	76	0.014425	4.01E-07	61	182	0.025103	9.64E-07
5000	q_1	7	28	0.020222	7.11E-07	18	72	0.052881	5.59E-07	67	200	0.10041	7.37E-07
	q_2	7	28	0.032154	4.19E-07	17	68	0.060047	9.70E-07	65	194	0.12889	9.46E-07
	q_3	6	24	0.016903	6.52E-07	5	20	0.015836	1.01E-07	52	155	0.073888	9.77E-07
	q_4	8	32	0.026291	4.61E-08	19	76	0.063913	7.14E-07	67	200	0.12227	7.98E-07
	q_5	8	32	0.021854	7.91E-08	20	80	0.067009	6.56E-07	68	203	0.12635	7.84E-07
	q_6	8	31	0.017087	3.46E-07	19	75	0.052514	4.22E-07	65	193	0.096178	9.55E-07
	q_7	14	56	0.036827	1.39E-07	19	76	0.060617	8.75E-07	64	191	0.13455	8.13E-07
10000	q_1	8	32	0.040875	4.16E-08	18	72	0.080435	7.90E-07	69	206	0.20249	8.48E-07
	q_2	7	28	0.031308	5.92E-07	18	72	0.086832	4.95E-07	65	194	0.19593	9.68E-07
	q_3	6	24	0.029814	9.22E-07	5	20	0.026455	1.42E-07	54	161	0.17907	8.85E-07
	q_4	8	32	0.032579	6.52E-08	20	80	0.095325	3.66E-07	68	203	0.19349	8.36E-07
	q_5	8	32	0.049485	1.12E-07	20	80	0.10367	9.28E-07	68	203	0.19112	8.04E-07
	q_6	8	31	0.028396	4.89E-07	21	84	0.11273	4.36E-07	66	196	0.18402	8.97E-07
	q_7	15	60	0.068762	7.66E-07	20	80	0.10925	4.83E-07	67	200	0.25543	8.23E-07
50000	q_1	8	32	0.12711	9.31E-08	19	76	0.3554	6.42E-07	69	206	1.0383	9.23E-07
	q_2	8	32	0.1292	5.48E-08	19	76	0.37087	4.02E-07	69	206	1.1211	8.24E-07
	q_3	7	28	0.102	8.53E-08	5	20	0.10411	3.18E-07	57	170	0.78127	8.91E-07
	q_4	8	32	0.1192	1.46E-07	21	84	0.42664	8.23E-07	-	-	-	-
	q_5	8	32	0.11524	2.50E-07	21	84	0.4116	7.14E-07	645	1934	6.8573	9.86E-07
	q_6	9	35	0.13619	4.53E-08	21	84	0.46219	9.75E-07	72	214	0.71757	8.55E-07
	q_7	12	48	0.26117	1.35E-07	21	84	0.44665	3.79E-07	69	206	1.0466	8.92E-07
100000	q_1	8	32	0.21963	1.32E-07	20	80	0.7103	7.45E-07	74	221	1.3959	8.25E-07
	q_2	8	32	0.2216	7.75E-08	19	76	0.69024	5.69E-07	73	218	1.3775	9.82E-07
	q_3	7	28	0.21668	1.21E-07	5	20	0.14594	4.50E-07	57	170	1.1078	8.45E-07
	q_4	8	32	0.21869	2.06E-07	22	88	0.87724	4.22E-07	74	221	1.4015	9.63E-07
	q_5	8	32	0.22502	3.54E-07	22	88	0.90385	7.50E-07	75	224	1.4108	8.29E-07
	q_6	9	35	0.28025	6.40E-08	22	88	0.89696	5.00E-07	74	220	1.3933	8.32E-07
	q_7	11	44	0.38459	8.43E-07	20	80	0.84964	6.68E-07	72	215	2.0602	8.38E-07

Table 8. Numerical result for problem 8

		EMDY				PDY				CGD			
DM	IP	#IT	FV	RT	NM	#IT	FV	RT	NM	#IT	FV	RT	NM
	q_1	31	124	0.22573	2.62E-07	36	144	0.18193	6.34E-07	2	6	0.012424	-
	q_2	30	120	0.21786	4.99E-07	35	140	0.16898	9.13E-07	2	6	0.005061	-
	q_3	28	112	0.1611	3.46E-07	35	140	0.17175	7.34E-07	-	-	-	-
1000	q_4	25	100	0.14315	5.48E-07	33	132	0.17954	2.30E-07	-	-	-	-
	q_5	25	100	0.16145	3.29E-07	31	124	0.17027	8.06E-07	-	-	-	-
	q_6	26	104	0.16143	7.00E-07	24	96	0.113	9.72E-07	2	6	0.006389	-
	q_7	29	116	0.18656	3.85E-07	34	136	0.14765	2.10E-07	-	-	-	-
	q_1	31	124	0.75135	4.06E-07	34	136	0.73104	8.36E-07	2	6	0.0184	-
	q_2	31	124	0.79394	4.27E-07	34	136	0.71489	7.93E-07	2	6	0.015833	-
	q_3	30	120	0.75333	3.34E-07	34	136	0.75698	6.18E-07	-	-	-	-
5000	q_4	31	124	0.82754	5.49E-07	31	124	0.70528	3.90E-07	-	-	-	-
	q_5	26	104	0.71815	4.09E-07	30	120	0.62952	8.11E-07	-	-	-	-
	q_6	31	123	0.80719	3.58E-07	24	96	0.53159	7.51E-07	2	6	0.021364	-
	q_7	32	128	0.93074	3.97E-07	24	96	0.5188	5.29E-07	-	-	-	-
	q_1	32	128	1.4992	4.61E-07	34	136	1.3758	6.78E-07	2	6	0.031455	-
	q_2	31	124	1.4344	3.25E-07	34	136	1.3977	6.42E-07	2	6	0.026895	-
	q_3	34	136	1.5395	4.11E-07	33	132	1.3753	7.57E-07	301	902	2.6217	9.82E-07
10000	q_4	32	128	1.5328	4.75E-07	30	120	1.217	3.94E-07	-	-	-	-
	q_5	25	100	1.3525	3.12E-07	30	120	1.2459	5.57E-07	-	-	-	-
	q_6	31	123	1.516	5.90E-07	24	96	1.006	7.21E-07	2	6	0.035741	-
	q_7	33	132	1.5945	8.93E-07	28	112	1.2535	2.91E-07	-	-	-	-
	q_1	31	124	5.9142	2.87E-07	34	136	5.8761	6.35E-07	2	6	0.11231	-
	q_2	32	128	6.3342	7.52E-07	33	132	5.681	6.12E-07	2	6	0.13194	-
	q_3	33	132	6.7763	2.60E-07	32	128	5.526	7.22E-07	7	21	0.30744	-
50000	q_4	34	136	6.4521	4.31E-07	24	96	4.1288	3.36E-07	-	-	-	-
	q_5	29	116	5.7229	3.63E-07	29	116	5.0629	5.83E-07	-	-	-	-
	q_6	31	123	5.929	3.60E-07	31	124	5.429	7.91E-07	2	6	0.14908	-
	q_7	20	77	4.5171	-	27	108	4.6515	4.16E-07	-	-	-	-
	q_1	27	108	11.0078	3.38E-07	33	132	12.0182	8.00E-07	2	6	0.25535	-
	q_2	35	139	14.973	3.12E-07	33	132	11.9965	7.49E-07	2	6	0.27981	-
	q_3	31	124	13.0731	3.65E-07	40	160	15.0111	9.75E-07	-	-	-	-
100000	q_4	33	132	13.6667	3.37E-07	30	120	11.5328	9.85E-07	-	-	-	-
	q_5	29	116	11.9657	4.22E-07	28	112	10.2117	9.46E-07	-	-	-	-
	q_6	33	130	13.5223	3.10E-07	26	104	9.8902	9.05E-07	2	6	0.29212	-
	q_7	31	121	16.4399	-	29	116	11.3768	2.77E-07	-	-	-	-

Table 9. Numerical result for problem 9

		EMDY				PDY				CGD			
DM	IP	#IT	FV	RT	NM	#IT	FV	RT	NM	#IT	FV	RT	NM
1000	q_1	8	28	0.030764	1.52E-07	11	42	0.007434	2.67E-07	39	110	0.011018	9.84E-07
	q_2	8	28	0.005022	1.52E-07	11	42	0.00735	2.67E-07	44	125	0.013022	9.55E-07
	q_3	8	28	0.004224	1.52E-07	11	42	0.010651	2.67E-07	53	153	0.030832	9.93E-07
	q_4	8	28	0.003639	1.52E-07	11	42	0.007526	2.67E-07	33	92	0.015758	9.83E-07
	q_5	8	28	0.004763	1.52E-07	11	42	0.008571	2.67E-07	59	171	0.017039	8.95E-07
	q_6	8	28	0.003651	1.52E-07	12	46	0.011781	2.67E-07	60	174	0.015494	8.67E-07
	q_7	8	28	0.008928	1.52E-07	11	42	0.008898	2.67E-07	56	162	0.016641	9.60E-07
5000	q_1	7	26	0.021359	4.20E-07	8	31	0.022631	1.59E-07	44	128	0.053302	9.03E-07
	q_2	7	26	0.022784	4.20E-07	8	31	0.023867	1.59E-07	44	128	0.050392	8.82E-07
	q_3	7	26	0.020035	4.20E-07	8	31	0.027595	1.59E-07	43	125	0.048859	9.14E-07
	q_4	7	26	0.019284	4.20E-07	9	35	0.04694	1.59E-07	44	128	0.052732	9.21E-07
	q_5	7	26	0.026307	4.20E-07	9	35	0.033777	1.59E-07	42	122	0.053127	9.29E-07
	q_6	7	26	0.015464	4.20E-07	9	35	0.035414	1.59E-07	42	122	0.049071	9.43E-07
	q_7	7	26	0.017105	4.20E-07	8	31	0.024653	1.59E-07	42	122	0.045245	9.35E-07
10000	q_1	7	26	0.035117	8.99E-07	11	43	0.067799	7.22E-07	32	92	0.076208	9.16E-07
	q_2	7	26	0.036305	8.99E-07	11	43	0.11292	7.22E-07	32	92	0.067837	8.10E-07
	q_3	7	26	0.03334	8.99E-07	11	43	0.075772	7.22E-07	32	92	0.13787	9.14E-07
	q_4	7	26	0.03278	8.99E-07	12	47	0.10103	7.22E-07	31	89	0.089473	8.53E-07
	q_5	7	26	0.032685	8.99E-07	13	51	0.14334	7.22E-07	33	95	0.080038	8.19E-07
	q_6	7	26	0.044189	8.99E-07	13	51	0.14162	7.22E-07	33	95	0.093456	8.52E-07
	q_7	7	26	0.039107	8.99E-07	11	43	0.069372	7.22E-07	32	92	0.077062	9.18E-07
50000	q_1	4	15	0.085176	9.63E-07	10	40	0.34801	7.59E-07	18	51	0.16825	8.12E-07
	q_2	4	15	0.088199	9.63E-07	10	40	0.34841	7.59E-07	18	51	0.18522	8.76E-07
	q_3	4	15	0.089543	9.63E-07	11	44	0.4185	7.59E-07	19	54	0.20809	7.87E-07
	q_4	4	15	0.081827	9.63E-07	13	52	0.67152	7.59E-07	18	51	0.16654	8.06E-07
	q_5	4	15	0.089199	9.63E-07	14	56	0.76269	7.59E-07	18	51	0.16034	8.15E-07
	q_6	4	15	0.11738	9.63E-07	16	64	1.02	7.59E-07	18	51	0.19372	7.48E-07
	q_7	4	15	0.082849	9.63E-07	11	44	0.44169	7.59E-07	19	54	0.18019	7.59E-07
100000	q_1	5	19	0.257	8.39E-07	9	36	0.61378	2.19E-07	12	33	0.2135	7.61E-07
	q_2	5	19	0.28101	8.39E-07	9	36	0.64652	2.19E-07	12	33	0.21885	7.09E-07
	q_3	5	19	0.245	8.39E-07	11	44	1.0263	2.19E-07	12	33	0.28442	5.40E-07
	q_4	5	19	0.27126	8.39E-07	14	56	1.7274	2.19E-07	13	36	0.24529	8.69E-07
	q_5	5	19	0.24215	8.39E-07	16	64	2.1972	2.19E-07	12	33	0.21371	8.65E-07
	q_6	5	19	0.28187	8.39E-07	18	72	2.7076	2.19E-07	13	36	0.23409	3.10E-07
	q_7	5	19	0.23839	8.39E-07	11	44	1.0178	2.19E-07	12	33	0.2229	9.47E-07

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