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On Generalized Distinguished Prime Submodules

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Abstract : Distinguished prime submodules of a module over a commutative ring with non-zero identity have been investigated in [1]. Here we study generalized distinguished prime submodules of modules and we will give a condition which allow us to determine whether the radical of submodules of a module are a generalized distinguished prime submodule.

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1 Introduction

In this paper all rings are commutative rings with non-zero identity and all modules are unital. Assume that N is a proper submodule of any R-module M and let P be a prime ideal of a ring R. We recall from [7, 1] the subset $M(P)$ of M defined by $\{m \in M : cm \in PM \text{ for some } c \in R - P\}$. Then it is clear that $M(P)$ is a submodule of M containing PM, $P \subseteq (PM : M) \subseteq (M(P) : M)$ and $M(P)$ is then called distinguished submodule. Now we shall denote the subset $M(N, P)$ of M by $\{m \in M : cm \in PM + N \text{ for some } c \in R - P\}$. It is clear that $M(N, P)$ is a submodule of M and $N + PM \subseteq M(N, P)$. In this case we also say that $M(N, P)$ is a generalized distinguished submodule of M. Recently extensive research has been done on prime submodules (see, for example [7,8,1]). Here we study some properties of generalized distinguished submodules of a module. In general, the radical of a primary submodule is not prime and the radical does not split intersections of submodules, as is valid in the ideal case. We study conditions for which these properties hold in the some module setting. Although a characterrization of the elements of the radical of a submodule of a free R-module is given in [9], no method for finding a generating set of the radical of a submodule appears in the literature. The aim of this paperis to present a method for finding the internal structure of the radical of submodules of a module. A number of results concerning of these concepts are given (see sections 2, 3 and 4).

For the sake of completeness, we state some definitions and notations used throughout. A proper submodule N of a module M over a ring R is said to be prime (respect. primary) if for any $r \in R$ and $m \in M$ such that $rm \in N$, either

$m \in N$ or $r \in (N : M) = \{a \in R : aM \subseteq N\}$ (resp. $r^n M \subseteq N$ for some positive integer n). It is easy to show that if N is a prime (resp. primary) submodule of M then the annihilator P of the module M/N (resp. the radical annihilator P' of the module M/N) is a prime ideal of R, and N is said to be P-prime submodule (resp. P'-primary submodule) of M. The radical of N in M, denoted by $\mathrm{rad}_M(N)$, is defined to be the intersection of all prime submodules of M containing N. Should there be no prime submodule of M containing N, then we put $\mathrm{rad}_M(N) = M$ [7, 8].

An R-module M is said to be secondary (resp. co-primary) if $M \neq 0$ and for each $r \in R$ the R-endomorphism of M produced by multiplication by r is either surjective (resp. injective) or nilpotent, so $\mathrm{nilrad}(M) = P$ is a prime ideal of R, and M is said to be P-secondary (resp. P-co-primary). A module M is said to be representable if it can be written as a sum $M_1 + ..., M_k$ of secondary modules; such a sum is called a secondary representation of M. if this representation is irredundant, we say that the attached primes of M are $\mathrm{Att}_R(M) = \{P_1, ..., P_k\}$, where $\mathrm{nilrad}(M_i) = P_i$ [6]. Let N be an R-submodule of M. Then N is pure in M if $rN = N \cap rM$ for every element $r \in R$.

Given a maximal ideal P of R. An R-module M will be called P-special if for each $a \in P$, $m \in M$, there exists a positive integer n and an element $c \in R - P$ such that $ca^n m = 0$. Moreover, the R-module M is called special if M is P-special for every maximal ideal P of R [8]. Recall that an R-module M is called a multiplication module if for each submodule N of M, $N = IM$ for some ideal I of R [3].

2 The radical of submodules

In this section we list some basic properties concerning generalized distinguished submodules.

Remark 2.1. *Let P, Q be prime ideals of a ring R and let N_1, N_2 be proper submodules of an R-module M. Then:*

(i) If $N_1 \subseteq PM$, then $M(N_1, P) = M(0, P) = M(P)$. Moreover, if $N_1, N_2 \subseteq PM$, then $M(N_1, P) = M(N_2, P)$.

(ii) If $M(N_1, P) \subseteq M(N_1, Q) \neq M$, then $P \subseteq Q$.

(ii) If R is a domain and M is an R-torsion module, then $0M = 0$ and $M(N, 0) = M = T(M)$, where $T(M)$ is the torsion submodule of M. Moreover, $T(M/N) = \{m + N \in M/N : rm \in N \text{ for some } 0 \neq r \in R\} = M(N, 0)/N$. In particular, if M is a torsion module, then M/N is torsion.

The following Proposition is used widely in the sequel.

Proposition 2.2. *Let N be a proper submodule of an R-module M and P a prime ideal of R such that $M \neq M(N, P)$. Then $M(N, P)$ is a P-prime submodule of M and $M(N, P)$ is the intersection of all P-prime submodules of M containing N.*

Proof. Let $rx \in M(N,P)$, for $r \notin (M(N,P):M)$ and $x \in M$. Then $r \notin P$ and $crx \in PM+N$ for some $c \in R-P$. Since $cr \notin P$, we must have $x \in M(N,P)$. This shows that $M(N,P)$ is a prime submodule of M. Clearly, $P \subseteq (M(N,P):M)$. Now suppose that $s \in (M(N,P):M)$ such that $s \notin P$. Then $sM \subseteq M(N,P)$. Consequently, for each $m \in M$, we get $sam \in M(N,P)$ for some $a \in R-P$. As $as \notin P$, we must have $m \in M(N,P)$; hence $M = M(N,P)$ which is a contradiction. So we have $(M(N,P):M) = P$ and $M(N,P)$ is a P-prime submodule of M.

Next, let K be a P-prime submodule of M containing N and suppose that $z \in M(N,P)$. Then $tz \in PM+N$ for some $t \notin P$. Since $(K:M) = P$, we must have $tz \in N+K = K$ and $t \notin (K:M)$. Then K prime gives $z \in K$. Since $M(N,P)$ is a P-prime submodules of M containing N, we have $M(N,P) = \bigcap\{K : K$ is a prime submodule of M containing $N\}$. □

Theorem 2.3. *Let N be a proper submodule of an R-module M. Then*

$$\mathrm{rad}_M(N) = \bigcap\{M(N,P) : P \textit{ is a prime ideal of } R\}.$$

Proof. Assume that $m \in \bigcap\{M(N,P) : P$ is a pime ideal of $R\} = T$ and let K be a prime submodule of M containing N. Since $(K:M)$ is a prime ideal of R, we must have $m \in M(N,(K:M))$, so $rm \in (K:M)M+N$ for some $r \notin (K:M)$; hence $rm \in K$. Then K prime gives $m \in K$. Thus $T \subseteq \mathrm{rad}_M(N)$. Now let $x \in \mathrm{rad}_M(N)$, and let $M(N,P) \in T$. By Proposition 2.2, we must have either $M = M(N,P)$ or $M(N,P)$ is a P-prime submodule of M containing N. In any case $m \in M(N,P)$. Thus $\mathrm{rad}_M(N) \subseteq T$, so we have equality. □

Theorem 2.4. *Let N be a P-primary submodule of a module M over a zero-dimential ring R. Then $\mathrm{rad}_M(N) = M(N,P)$.*

Proof. The inclusion $\mathrm{rad}_M(N) \subseteq M(N,P)$ follows from Proposition 2.2. For the other containment, let $m \in M(N,P)$. Then $rm \in PM+N$ for some $r \notin P$. Let K be a prime submodule of M containing N. By assumption, it is easy to see that $P = (K:M)$. So $rm \in K$. Then K prime gives $m \in K$; hence $M(N,P) \subseteq K$, and the proof is complete. □

Lemma 2.5. *Let M be an R-module, N a proper R-submodule of M and P a maximal ideal of R such that $M \neq PM+N$. Then $PM+N$ is a P-prime submodule of M.*

Proof. First, we show that $(PM+N:M) = P$. If $r \in P$, then $rM \subseteq PM+N$, so $P \subseteq (PM+N:M)$; hence we have equality since P is maximal. Next, since $M/(PM+N)$ is a torsion-free R/P-module, we must have $PM+N$ is prime. □

Theorem 2.6. *Let R be a one-dimensional domain, M an R-module and N a proper submodule of M. Then $\mathrm{rad}_M(N) = M(N,0) \cap (\bigcap\{PM+N : P$ is a maximal ideal of $R\})$.*

Proof. By Lemma 2.5, it is clear that $\mathrm{rad}_M(N) \subseteq M(N,0) \cap (\bigcap\{PM + N : P \text{ is a maximal ideal of } R\}) = H$. Assume that L is a prime submodule of M containing N and let $(L : M) = P$. By hypothesis, if $P \neq 0$, then P is a maximal ideal of R and $PM + N \subseteq L$. If $P = 0$, then $M(N,0) \subseteq L$. In any case, we must have $H \subseteq \mathrm{rad}_M(N)$, as required. □

Proposition 2.7. *Let M be an R-module, N a proper R-submodule of M and P a prime ideal of R. Then N is a P-prime submodule of M if and only if $N = M(N,P)$.*

Proof. Suppose that N is a P-prime submodule of M. It suffices to show that $M(N,P) \subseteq N$. Let $x \in M(N,P)$. Then $cx \in PM + N = N$ for some $c \notin P$; hence $x \in N$, so we have equality. The other implication follows from Proposition 2.2. □

Let M be an R-module. $m \in M$ is said to be torsion-free element if $rm = 0$ implies that $r = 0$ for every $r \in R$.

Theorem 2.8. *Let R be a domain, M a co-primary R-module such that M has a torsion-free element t and N a proper pure R-submodule of M. Then $\mathrm{rad}_M(N) = M(N,0) = N$.*

Proof. By Proposition 2.7, it suffices to show that $N = M(N,0)$ is a 0-prime submodule of M. First, we show that $N = M(N,0)$. Since the inclusion $N \subseteq M(N,0)$ is trivial, we will prove the reverse inclusion. Let $m \in M(N,0)$. Then there is an element $0 \neq c \in R$ such that $cm \in N \cap cM = cN$, so $cm = cn$ for som $n \in N$; hence M co-primary gives either $n = m \in N$ or $c^k M = 0$ for some positive integer k. Therefore, $c^k t = 0$ which is a contradiction, so we have equality, as needed. □

Theorem 2.9. *Let N be a proper submodule of a especial module M over a local ring (R,P). Then $\mathrm{rad}_M(N) = M(N,P)$.*

Proof. First, we show that if L is a prime submodule of M, then $(L : M) = P$. Suppose not. So there is an element $a \in P$ such that $a \notin (L : M)$. Let $m \in M$. Then M especial gives there exists a positive integer s and $c \notin P$ such that $ca^s m = 0$. As c is a unit element of R, we must have $a^s m = 0 \in L$, so $m \in L$ which is a contradiction. Thus $(L : M) = P$.

Next, by Proposition 2.2, it suffices to show that $M(N,P) \subseteq \mathrm{rad}_M(N)$. Let $x \in M(N,P)$. Then $cx \in PM + N$ for some $c \notin P$. Let L be a prime submodule of M containing N. Since $(L : M) = P$, we must have $cx \in L$, so $x \in L$; hence $M(N,P) \subseteq L$. This proves that $M(N,P) \subseteq \mathrm{rad}_M(N)$, as required. □

Theorem 2.10. *Let N be a proper submodule of a Noetherian module M over a ring R. Then There exist prime ideals $P_1, ..., P_n$ of R such that $\mathrm{rad}_M(N) = \bigcap_{i=1}^n M(N,P_i)$.*

Proof. By [7, Theorem 4.2], there are only finite number of minimal prime submodules $L_1, ..., L_n$ of M containing N, so $\mathrm{rad}_M(N) = \bigcap_{i=1}^n L_i$. For each i, $i = 1, ..., n$, let $(L_i : M) = P_i$. Since the inclusion $\mathrm{rad}_M(N) \subseteq \bigcap_{i=1}^n M(N, P_i)$ is trivial, we will prove the reverse inclusion. Let $x \in \bigcap_{i=1}^n M(N, P_i)$. Then there exists $c_i \notin P_i$ such that $c_i x \in P_i M + N$ for every i, so $c_i x \in L_i$; hence $x \in L_i$ for every i. Therefore, $x \in \mathrm{rad}_M(N)$, as needed. □

3 Representable modules

. Let N be an R-submodule of M. In this section, we study relation between Att$(M/N$ and the prime submodules of M containing N.

Lemma 3.1. *Let P be an ideal of R, M an R-module and N a proper submodule of M. Then there exists a proper submodule K of M containing N such that $P = (K : M)$ if and only if $PM + N \neq M$ and $P = (PM + N : M)$.*

Proof. Let K be a proper submodule of M containing N such that $P = (K : M)$. Then $PM + N \subseteq K \neq M$. Since $P \subseteq (PM + N : M)$, it suffices to show that $(PM + N : M) \subseteq P$. Let $r \in (PM + N : M)$. Then $rM \subseteq PM + N \subseteq K$; hence $r \in P$. The other implication is clear. □

Theorem 3.2. *Let N be a proper submodule of an Artinian module M over a ring R. Also suppose that M/PM is a finitely generated for some $P \in Att_R(M/N)$. Then $M(N, P)$ is a prime submodule of M with $P = (M(N, P) : M)$.*

Proof. By [12, Corollary 2.6], there exists a proper submodule K of M such that $P = (K : M)$. Then Lemma 3.1 gives $PM + N \neq M$ and $P = (PM + N : M)$. By assumption, let $M/PM = R(x_1+N)+...+R(x_n+N)$, so $M = Rx_1+...+Rx_n+PM$. Now we claim that $M \neq M(N, P)$. Otherwise, for each i, there is an element $c_i \in R - P$ such that $c_i x_i \in PM + N$. If we put $c = c_1 c_2 ... c_n$, then $cM \subseteq PM + N$ which is a contradiction. Now the assertion follows from Proposition 2.2. □

Theorem 3.3. *Let N be a proper submodule of an Artinian module M over a ring R and $M = \sum_{i=1}^n M_i$ a minimal secondary representation of M with* $\mathrm{Att}(M) = \{P_1, ..., P_n\}$. *Then*

$$\mathrm{rad}_M(N) = \bigcap \{M(N, P) : P \in \mathrm{Att}(M) \textit{ and } (N : M) \subseteq P\}.$$

Proof. It is enough to show that

$$H = \bigcap \{M(N, P) : P \in \mathrm{Att}(M) \text{ and } (N : M) \subseteq P\} \subseteq \mathrm{rad}_M(N).$$

Let $x \in H$. Then $x \in M(N, P)$ for every $P \in \mathrm{Att}(M)$ with $(N : M) \subseteq P$. Suppose that L is a prime submodule of M containing N. Then by [12, Lemma 2.6], there exists $Q \in \mathrm{Att}(M/N) \subseteq \mathrm{Att}(M)$ such that $Q = (L : M)$. As $(N : M) \subseteq Q$, we must have $x \in M(N, Q) \subseteq L$, as required. □

Lemma 3.4. *Let N be a proper submodule of an Artinian module M over a ring R and P a prime ideal of R. Then $PM + N \neq M$ and $P = (PM + N : M)$ if and only if $P \in \mathrm{Att}(M)$.*

Proof. This follows from Lemma 3.1 and [12, Corollary 2.6]. □

Theorem 3.5. *If N is a proper submodule of an Artinian module M over a one-dimensional domain R, then $\mathrm{rad}_M(N) = \bigcap\{PM + N : P \in \mathrm{Att}(M/N)\}$.*

Proof. Let $P \in \mathrm{Att}(M/N)\}$. Then Lemma 3.4 and Lemma 2.5 gives $PM + N$ is a P-prime submodule of M containing N, so $PM+N = M(N, P$; hence $\mathrm{rad}_M(N) \subseteq \bigcap\{PM + N : P \in \mathrm{Att}(M/N)\} = H$. For the other containment, assume thar $m \in H$. Then $m \in PM + N$ for every $P \in \mathrm{Att}(M/N)$. Let K be a prime submodule of M containing N and $Q = (N : M)$. Then [12, Lemma 2.6] gives $Q \in \mathrm{Att}(M/N)$, so $QM + N \subseteq K$; hence $m \in K$, as needed. □

Theorem 3.6. *If N is a P-primary submodule of an Artinian module M over a one-dimensional domain R, then $\mathrm{rad}_M(N) = PM + N$.*

Proof. This follows from Theorem 3.6 and [5, Theorem 2.2]. □

Theorem 3.7. *Let R be a one-dimensional domain, M an R-module and N a proper submodule of M with $(\mathrm{rad}_M(N) :_R M) = P$. Then the following hold:*

(i) If M is secondary, then $\mathrm{rad}_M(N) = M(N, P)$.

(ii) If M is a torsion module such that 0 is a prime submodule, then $\mathrm{rad}_M(N) = M(N, P)$.

Proof. In any case, $\mathrm{rad}_M(N)$ is a prime submodule of M by [4, Theorem 3.5 and Corollary 3.7]. Now the assertion follows from Proposition 2.7. □

Theorem 3.8. *Let R be a Noetherian domain, M a secondary R-module and N a proper submodule of M with $(\mathrm{rad}_M(N) :_R M) = P$. If every prime submodule of M contains only finitely many prime submodules, then $\mathrm{rad}_M(N) = M(N, P)$.*

Proof. Since $\mathrm{rad}_M(N)$ is prime by [4, Corollary 3.6], the result follows from Proposition 2.7. □

4 Submodules of a finitely generated module

. Recall that the set of supported prime ideals of a given R-module M is defined as: $\mathrm{Supp}_R(M) = \{P \in \mathrm{Spec}(R) : M_P \neq 0\}$. It is well-known that if M is a finitely generated, then $\mathrm{Supp}_R(M)$ is the set of prime ideals of R which contain $(0 : M)$ (see [11, Lemma 9.20].

Lemma 4.1. *Let N be a proper submodule of an R-module M and P a prime ideal of R such that $M \neq M(N, P)$. Then $P \in \mathrm{Supp}_R(M/N)$.*

Proof. Suppose not. So $(M/N)_P = 0$. Let $m \in M$. Then $(m+N)/1 = 0$, so there is an element $t \notin P$ such that $tm \in N \subseteq N + PM$; hence $m \in M(N, P)$ which is a contradiction. Thus $P \in \mathrm{Supp}_R(M/N)$. $\square$

Proposition 4.2. *Let N be a proper submodule of a finitely generated R-module M and P a prime ideal of R such that $P \in \mathrm{Supp}_R(M/N)$. Then $M \neq M(N, P)$.*

Proof. Suppose not. First, we show that $M_P = (PR_P)M_P + N_P$. It suffices to show that $M_P \subseteq (PR_P)M_P + N_P$. Let $z = m/s \in M_P$ for some $m \in M = M(N, P)$ and $s \in R - P$, so there is an element $c \notin P$ such that $cm \in PM + N$; hence $z = (cm)/(cs) \in (PM + N)_P = (PR_P)M_P + N_P$. By hypothesis, Nakayama's lemma gives $M_P = N_P$. Since $P \in \mathrm{Supp}(M/N)$, we must have $0 \neq u = (a + N)/t \in (M/N)_P$ for some $x \in M$ and $t \notin P$. As $a/t \in N_P$, there exist $b \in N$ and $w \notin P$ such that $a/t = b/w$. Then $aws' = ts'b \in N$ for some $s' \notin P$; thus $u = (aws' + N)/(s'wt) = 0$ which is a contradiction. Therefore $M(N, P) \neq M$. $\square$

Corollary 4.3. *Let N be a proper submodule of a multiplication R-module M and P a prime ideal of R such that $P \in \mathrm{Supp}_R(M/N)$. Then $M \neq M(N, P)$.*

Proof. By [2, Proposition 1], the proof is similar to the Proposition 4.2. $\square$

Theorem 4.4. *Let M be a finitely generated module (resp. multiplication module) over a ring R. Then every proper submodule of M is contained in a prime submodule of M.*

Proof. Let N be a proper submodule of M. Then there is a maximal ideal P of R such that $(N : M) \subseteq P$, so $P \in \mathrm{Supp}_R(M/N)$; hence $M \neq M(N, P)$ by Proposition 3.2. Now the assertion follows from Proposition 2.2. $\square$

Corollary 4.5. *Let R be a commutative ring, M a finitely generated P-secondary R-module and N a proper submodule of M. Then $\mathrm{rad}_M(N) = M(N, P) = N$.*

Proof. By Theorem 4.4, $\mathrm{rad}_M(N) \neq M$. Therefore, $\mathrm{rad}_M(N) = M(N, P)$ by [10, Proposition 3] and Proposition 2.7. $\square$

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