Thai Journal of **Math**ematics
Special Issue (2022): Annual Meeting in Mathematics 2021
Pages 124 – 129
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

A Luxurious Proof of the Generalized Caristi-Kirk Fixed Point Result Using Theory on Ball Spaces

Woraphak Nithiarayaphaks, Wutiphol Sintunavarat*

Department of Mathematics and Statistics Faculty of Science and Technology Thammasat University Rangsit Center Pathum Thani 12120 Thailand
e-mail: woraphak.n@gmail.com (W. Nithiarayaphaks)
e-mail: wutiphol@mathstat.sci.tu.ac.th (W. Sintunavarat)

Abstract In this work, we study the famous fixed point result named Caristi-Kirk's fixed point theorem (CK-FPT, for short) in partial metric spaces and merge such a theorem with the theory of ball spaces. We begin our work with new results in the framework of partial metric spaces concerning Caristi-Kirk ball spaces. At the final section, our results are applied to provide a short and luxurious proof for CK-FPT on partial metric spaces.

MSC: 54H25; 47H09, 47H10.
Keywords: Ball space; Caristi-Kirk ball space; partial metric space

1. Introduction and Fundamental Results

Based on the fact that a fixed point theorem is a good choice for solving many mathematics problems, fixed point results have many applications in real-world phenomena. For example, some fixed point theorem is used to confirm the existence and uniqueness of solutions for several equations such as differential equations, integral equations, stochastic equations, etc. Next, we quote one of the famous fixed point theorems called Caristi-Kirk's fixed point theorem (C-K FPT, for short) as follows:

Theorem 1.1 ([1]). *Let φ be a lower semicontinuous function (LSF, for short) from a complete metric space (X, d) into $[0, \infty)$ and T be a self mapping on X. If the following condition holds:*

$$d(x, Tx) \leq \varphi(x) - \varphi(Tx) \tag{1.1}$$

for all $x \in X$, then the fixed point of T exists.

This theorem was first introduced by Caristi [1] in 1976, and it is applied to nonconvex minimization problems three years later (see mention 6). Furthermore, Theorem 1.1 can also be used for other mathematical problems and various problems in several branches of

*Corresponding author.

science. Based on various applications of Theorem 1.1, there are a lot of generalizations of this result in several directions. One of theses directions is to investigate this result in new spaces.

Next, we give a definition of one of the extensions of metric spaces, which is the motivation of the focussed results in this paper. This space was presented by Matthews [2] in 1994, and its definition is shown below.

Definition 1.2 ([2]). The pair (X, p) is called a partial metric space (PMS, for short) if $X \neq \emptyset$ and p is a partial metric (PM, for short), that is, p is a nonnegative valued-real function from $X \times X$ satisfyingthe the following conditions for all $x, y, z \in X$:

(PM1) $p(x, y) = p(y, x)$;
(PM2) $p(x, x) = p(x, y) = p(y, y)$ if and only if $x = y$;
(PM3) $p(x, x) \leq p(x, y)$;
(PM4) $p(x, z) + p(y, y) \leq p(x, y) + p(y, z)$.

For the conciseness of this paper, we omit the other definitions in PMS such as a convergence sequence, a Cauchy sequence and a completeness in a PMS. The reader can see more details in [2]

In [3], Theorem 1.1 is extended from the framework of matric spaces to partial matric spaces as follows:

Theorem 1.3 ([3]). *Let ψ be a LSF from a complete PMS (X, p) into $[0, \infty)$ and T be a self mapping on X. If the following condition holds:*

(CC) $p(a, Ta) \leq \psi(a) - \psi(Ta)$ *for all* $a \in X$,

then the fixed point of T exists.

In 2015, Kuhlmann [4] firstly introduced a concept of a ball space and many ideas related to a ball space as follows:

Definition 1.4 ([4]). Let X be a nonempty set.

(1) The pair $(X, \mathcal{B})$ is called a ball space if $\mathcal{B}$ is a nonempty set of subsets of X.
(2) A set in a ball space is called a ball.
(3) A nonempty set of balls in a ball space is called a nest if it is totally orderd by inclusion.

Definition 1.5 ([4]). The ball space $(X, \mathcal{B})$ is said to be spherical complete if and only if for every nest $\mathcal{N}$ of balls in $\mathcal{B}$, we have $\bigcap \mathcal{N} \neq \emptyset$.

Definition 1.6 ([4]). A function f is called a self-contractive function on a ball space $(X, \mathcal{B})$ if there is a function $B : X \to \mathcal{B}$ satisfying the following conditions for all $x \in X$:

(S1) $x \in B_x$;
(S2) $B_{f(x)} \subseteq B_x$ and if x is not a fixed point of f, then $B_{f(x)} \subsetneq B_x$;
(S3) if $\mathcal{N}$ is a nest of balls such that every ball $B_x \in \mathcal{N}$ contains $B_{f(x)}$ and $z \in \bigcap \mathcal{N}$, then $B_z \subseteq \bigcap \mathcal{N}$,

where B_a represents $B(a)$ for all $a \in X$.

Based on Zorn's lemma, the next fixed point result in ball spaces is proved in [4].

Theorem 1.7 ([4]). *Every self-contractive function on a spherically complete ball space admits a fixed point.*

In [5], Kuhlmann et al. also gave a new short proof of Theorem 1.1 using the above result in ball spaces.

Inspired by the above literature, this paper aims to investigate the connection between Theorem 1.7 and ball spaces in the framework of PMSs. Our main results provide a short and luxurious proof of Caristi Kirk's fixed point theorem on PMSs in the last section.

2. Main Results on Partial Caristi-Kirk Ball Spaces

In this section, for a PMS (X, p), a given point $x \in X$, a given LSF $\psi : X \to [0, \infty)$, the symbol B_x^ψ is defined by the following set:

$$B_x^\psi := \{a \in X : p(x, a) \leq \psi(x) - \psi(a) + \max\{p(x, x), p(a, a)\}\}. \tag{2.1}$$

The collection $\mathcal{B}_\psi := \{B_x^\psi : x \in X\}$ is called a partial Caristi-Kirk ball space. If p is a metric, then a partial Caristi-Kirk ball space reduces to the idea of a classical Caristi-Kirk ball space.

Next, we give main theoretical results related on partial Caristi-Kirk ball spaces.

Theorem 2.1. *Let ψ be a LSF from a complete PMS (X, p) into $[0, \infty)$. Suppose that there is $M \subseteq X$ such that p is continuous and*

$$\lim_{n\to\infty} p(x_n, x_n) = 0 \tag{2.2}$$

for $\{x_n\} \subseteq M$ with $\lim_{n\to\infty} \psi(x_n) = \inf_{x\in M} \psi(x)$. Then a partial Caristi-Kirk ball space $(X, \mathcal{B}_\psi)$ is spherically complete.

Proof. Take a nest $\mathcal{N} = \{B_x^\psi : x \in M\}$ in $\mathcal{B}_\psi$. For each $x \in X$, we have

$$p(x, x) \leq \psi(x) - \psi(x) + p(x, x), \tag{2.3}$$

that is, $x \in B_x^\psi$. This implies that for every $x, y \in M$, we get $x \in B_y^\psi$ or $y \in B_x^\psi$ since $\mathcal{N}$ is a nest. It yields that

$$p(x, y) \leq |\psi(x) - \psi(y)| + \max\{p(x, x), p(y, y)\} \tag{2.4}$$

for all $x, y \in M$. Consider $\{x_n\}$ in M such that

$$\lim_{n\to\infty} \psi(x_n) = r := \inf_{x\in M} \psi(x).$$

Since $\{\psi(x_n)\}$ is a Cauchy sequence in $\mathbb{R}$, from (2.4) together with (PM1), (PM2) and (2.2), we obtain $\{x_n\}$ is a Cauchy sequence in (X, p). It follows from the completeness of (X, p) that $\{x_n\}$ converges to some $z \in X$.

Finally, for each $x \in M$, we have

$$\begin{aligned} p(x, z) &= \lim_{n\to\infty} p(x, x_n) \\ &\leq \lim_{n\to\infty} [|\psi(x) - \psi(x_n)| + \max\{p(x, x), p(x_n, x_n)\}] \\ &\leq \psi(x) - \psi(z) + \max\{p(x, x), p(z, z)\}. \end{aligned}$$

Hence, $z \in B_x^\psi$ for every $x \in M$. It yields that $\bigcap \mathcal{N} \neq \emptyset$. ∎

Theorem 2.2. *Let ψ be a LSF from a complete PMS (X, p) into $[0, \infty)$ and T be a self mapping on X. Suppose that*

$$p(a, Ta) \leq \psi(a) - \psi(Ta) \tag{2.5}$$

for all $a \in X$. Then T is a self-contractive on $(X, \mathcal{B}_\psi)$.

Proof. The proof of Theorem 2.1 shows that $x \in B_x^\psi$ for all $x \in X$. It implies that (S1) holds.

To prove (S2), we will show that $B_{Tx}^\psi \subseteq B_x^\psi$ for all $x \in X$. For each $y \in B_{Tx}^\psi$, we get

$$\begin{aligned} p(x,y) &\leq p(x,Tx) + p(Tx,y) - p(Tx,Tx) \\ &\leq \psi(x) - \psi(Tx) + \psi(Tx) - \psi(y) + \max\{p(Tx,Tx), p(y,y)\} - p(Tx,Tx) \\ &\leq \psi(x) - \psi(Tx) + \psi(Tx) - \psi(y) + p(y,y) \\ &\leq \psi(x) - \psi(y) + \max\{p(x,x), p(y,y)\}. \end{aligned}$$

Therefore, $y \in B_x^\psi$ and so $B_{Tx}^\psi \subseteq B_x^\psi$. Next, we will show that $B_{Tx}^\psi \subsetneq B_x^\psi$ whenever $Tx \neq x$. It is easy to see that if $Tx \in B_x^\psi$ and $x \in B_{Tx}^\psi$, then $\psi(x) = \psi(Tx)$. From (2.5), we get $p(x,Tx) = 0$. This implies that Tx and x are identical. Therefore, if $x \neq Tx$, then $Tx \notin B_x^\psi$ or $x \notin B_{Tx}^\psi$. So $B_{Tx}^\psi \subsetneq B_x^\psi$.

Finally, for (S3), we consider a nest $\mathcal{N}$ such that every ball $B_x^\psi \in \mathcal{N}$ contains B_{Tx}^ψ and $z \in \bigcap \mathcal{N}$. This implies $z \in B_{Tx}^\psi$ for every $B_{Tx}^\psi \in \mathcal{N}$. For each $y \in B_z^\psi$ we have $y \in B_x^\psi$. Hence, $B_z^\psi \subseteq B_x^\psi$ for all $B_x^\psi \in \mathcal{N}$. Therefore, $B_z^\psi \in \bigcap \mathcal{N}$. ∎

Theorem 2.3. *If (X,p) is a PMS and $(X, \mathcal{B}_\vartheta := \{B_x^\vartheta : x \in X\})$ is a spherical complete partial Caristi-Kirk ball space for all continuous functions $\vartheta : X \to [0,\infty)$ and*

$$\lim_{n\to\infty} p(x_n, x_n) = 0, \tag{2.6}$$

for every Cauchy sequence $\{x_n\} \subseteq X$, then X is complete.

Proof. We consider a Cauchy sequence $\{x_n\} \subseteq X$, and construct a function $\vartheta : X \to [0,\infty)$ by

$$\vartheta(x) = \lim_{n\to\infty} d_p(x, x_n)$$

for all $x \in X$, where d_p is defined as

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

for all $x, y \in X$. It is well-known that (X, d_p) is a metric space and ϑ is continuous. Now, we consider a subsequence $\{y_k\}$ of $\{x_n\}$, where $y_k = x_{n_k}$. Then there exists some $m > n_k$ satisfying $\vartheta(y_k) \geq \vartheta(x_m)$, which yields that

$$\frac{1}{2} d_p(y_k, x_m) \leq \vartheta(y_k) - \vartheta(x_m). \tag{2.7}$$

Here, we can pick up one of such m and let $n_{k-1} := m$. Next, we use the spherical completeness of the partial Caristi-Kirk ball space. By construction and the inequality (2.7), it yields that

$$\begin{aligned} 2p(y_k, y_{k+1}) &\leq 2\vartheta(y_k) - 2\vartheta(y_{k+1}) + p(y_k, y_k) + p(y_{k+1}, y_{k+1}) \\ p(y_k, y_{k+1}) &\leq \vartheta(y_k) - \vartheta(y_{k+1}) + \max\{p(y_k, y_k), p(y_{k+1}, y_{k+1})\} \end{aligned}$$

that impiles $y_{k+1} \in B_{y_k}^\vartheta$. Consider the set $\mathcal{N} := \{B_{y_k}^\vartheta : k \in \mathbb{N}\}$. By the proof of Theorem 2.2, it shows that $B_{y_{k+1}}^\vartheta \subseteq B_{y_k}^\vartheta$ and hence $\mathcal{N}$ is a nest. Since $(X, \mathcal{B}_\vartheta)$ is a spherical complete partial Caristi-Kirk ball space, there is $z \in \bigcap \mathcal{N}$. It follows that

$$d_p(y_k, z) \leq \vartheta(y_k) - \vartheta(y_{k+1}).$$

This shows that a Cauchy sequence $\{x_n\}$ converges to $z \in X$. Hence, X is complete. ∎

Corollary 2.4 ([5]). *Let ψ be a LSF from a complete metric space (X, d) into $[0, \infty)$. Then a Caristi-Kirk ball space $(X, \mathcal{B}_\psi)$ is spherically complete.*

Corollary 2.5 ([5]). *Let ψ be a LSF from a complete metric space (X, d) into $[0, \infty)$ and T be a self mapping on X. Suppose that*

$$p(x, Tx) \leq \psi(x) - \psi(Tx) \tag{2.8}$$

for all $x \in X$. Then T is a self-contractive on a ball space $(X, \mathcal{B}_\psi)$.

Corollary 2.6 ([5]). *If (X, d) is a metric space and $(X, \mathcal{B}_\vartheta)$ is a spherical complete Caristi-Kirk ball spaces for all continuous functions $\vartheta : X \to [0, \infty)$, then (X, d) is complete.*

3. Application on Partial Caristi-Kirk Fixed Point Results

This section presents a short proof of Theorem 1.3 whenever p is continuous and it satisfies (2.2) by using results in the previous section.

Theorem 3.1. *Theorem 1.3 holds provided that p is continuous and satisfies (2.2).*

Proof. By Theorem 2.1, we conclude that a partial Caristi-Kirk ball space $(X, \mathcal{B}_\psi)$ is spherically complete. Hence, the conclusion of Theorem 2.2 shows that T is a self-contractive on a ball space $(X, \mathcal{B}_\psi)$. Therefore, Theorem 1.7 yields that T has a fixed point. ∎

Theorem 3.2. *Let (X, p) be a PMS such that p is continuous and every Cauchy sequence satisfies (2.6). Suppose that $(X, \mathcal{B}_\vartheta)$ is a spherical complete partial Caristi-Kirk ball space for every continuous function $\vartheta : X \to [0, \infty)$ and $T : X \to X$ be a mapping. If there is $k \in [0, 1)$ and the following conditions hold for all $a, b \in X$:*

(T1) $p(Ta, Tb) \leq p(a, b)$;
(T2) $p(Ta, T^2a) \leq kp(a, Ta)$,

then the fixed point of T exists.

Proof. Define a function $\vartheta : X \to [0, \infty)$ by

$$\vartheta(a) := \frac{p(a, Ta)}{1 - k}$$

for all $a \in X$. We obtain

$$\begin{aligned} \vartheta(Ta) &= \frac{p(Ta, T^2a)}{1 - k} \\ &\leq \frac{kp(a, Ta)}{1 - k}, \end{aligned}$$

for all $a \in X$. Hence,

$$\vartheta(a) - \vartheta(Ta) \geq p(a, Ta)$$

for all $a \in X$. This implies that T satisfies (CC).

To show the continuity of ϑ, we consider $a, b \in X$ and assume without loss of generality that $\vartheta(a) - \vartheta(b) \geq 0$. Then

$$\begin{aligned} \vartheta(a) - \vartheta(y) &= \frac{1}{1-k}(p(a,Ta) - p(b,Tb)) \\ &\leq \frac{1}{1-k}(p(a,b) + p(b,Ta) - p(b,b) - p(b,Tb)) \\ &\leq \frac{1}{1-k}(p(a,b) + p(Ta,Tb) - 2p(b,b)) \\ &\leq \frac{2}{1-k}p(a,b). \end{aligned}$$

This implies ϑ is continuous. Theorem 2.3 tells us X is complete. Finally, Theorem 2.2 and Theorem 1.7 make the conclusion that T has a fixed point. ■

Acknowledgement

This work was supported by Thammasat University Research Unit in Fixed Points and Optimization.

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