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Approximation of Common Fixed Points of Suzuki-Square-α-Nonexpansive Mappings in CAT(0) Spaces

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Abstract In this paper, we will study research by the following process. First, we introduce Suzuki-square-α-nonexpansive mappings in CAT(0) spaces by using the concept of a Suzuki condition and α-nonexpansive mappings. Second, we create results with respect to approximation of common fixed points of Suzuki-square-α-nonexpansive mappings in CAT(0) spaces by the concept of iterative process of Muangchoo-in et al. Finally, We obtain the approximation of common fixed point of Suzuki-square-α-nonexpansive mappings in CAT(0) spaces and prove that results.

MSC: 47H09; 47H10.
Keywords: Suzuki-square-α-nonexpansive mapping; CAT(0) spaces; fixed point; Δ-converge.

1. Introduction and Preliminaries

In 2008, Suzuki [1] introduced the condition C as follows. S is said to satisfy condition C if

$$\frac{1}{2}d(x,Sx) \leq d(x,y) \Rightarrow d(Sx,Sy) \leq d(x,y),$$

for all x, y in metric spaces X.

In 2008, Kikkawa and Suzuki [2] generalized the Kannan mapping resulting in the following condition. Let S be a mapping on complete metric space (X,d) and let φ be a non-increasing function from $[0,1)$ into $(\frac{1}{2},1]$ defined by

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$$\varphi(r) = \begin{cases} 1 & \text{if } 0 \leq r < \frac{1}{\sqrt{2}}, \\ \frac{1}{1+r} & \text{if } \frac{1}{\sqrt{2}} \leq r < \frac{1}{2}. \end{cases}$$

Let $\alpha \in [0, \frac{1}{2})$ and put $r = \frac{\alpha}{1-\alpha} \in [0, 1)$. Suppose that

$$\varphi(r)d(x, Sx) \leq d(x, y) \Rightarrow d(Sx, Sy) \leq \alpha d(x, Sx) + \alpha d(y, Sy),$$

for all $x, y \in X$.

In 2011, Karapinar and Tas [3] stated some new conditions which are modifications of Suzuki's condition C, as follows. S is said to satisfy condition SCC if

$$\tfrac{1}{2}d(x, Sx) \leq d(x, y) \Rightarrow d(Sx, Sy) \leq M(x, y),$$

where $M(x, y) = \{d(x, y), d(x, Sx), d(y, Sy), d(x, Sy), d(y, Sx)\}$, for all x, y in metric spaces X.

In the same way, in 2021, Aoyama and Kohsaka [4] introduced the class of α-nonexpansive mapping in Banach spaces.

Let E be a Banach space, let C be a nonempty subset of E, and let α be a real number such that $\alpha < 1$. A mapping $S : C \to E$ is said to be α-nonexpansive if

$$||Sx - Sy||^2 \leq \alpha||Sx - y||^2 + \alpha||x - Sy||^2 + (1 - 2\alpha)||x - y||^2,$$

for all $x, y \in C$. Now, we give an example for a square α-nonexpansive mapping as follows :

Example 1.1. Let M be a nonempty closed and convex subset of a complete CAT(0) space X, and let $S_1, S_2 : M \to M$ be firmly nonexpansive mappings such that $S_1(M)$ and $S_2(M)$ are contained by rB_M for some positive real number r. Let α and δ be real numbers such that $0 < \alpha \leq 1$ and $\delta \geq (1 + 2/\sqrt{\alpha})r$. Then the mapping $U : M \to M$ is defined by

$$Ux = \begin{cases} S_1x & (x \in \delta B_M); \\ S_2x & (\text{otherwise}), \end{cases} \tag{1.1}$$

U is a square α-nonexpansive (See [4]).

Next, we extend definitions of α-nonexpansive mappings in Banach spaces to α-nonexpansive mappings in metric spaces.

Let (X, d) be a metric space and C be a nonempty subset. Then $S : C \to C$ said to be a square α-nonexpansive mapping (or α-noexpansive mapping) [5–7], if $\alpha < 1$ such that

$$d^2(Sx, Sy) \leq \alpha d^2(Sx, y) + \alpha d^2(x, Sy) + (1 - 2\alpha)d^2(x, y)$$

for all $x, y \in C$.

On the other hand, we recall definitions of CAT(0) spaces, let (X, d) be a metric space and $x, y \in X$ with $l = d(x, y)$. A geodesic path from x to y is an isometry $\zeta : [0, l] \to X$ such that $\zeta(0) = x$, $\zeta(l) = y$, and $d(\zeta(s_1), \zeta(s_2)) = |s_1 - s_2|$ for any $s_1, s_2 \in [0, l]$. We will say that (X, d) is a (uniquely) geodesic metric space if any two points are connected by a (unique) geodesic. In this case, we denote such geodesic by $[x, y]$. In general, geodesic is not uniquely determined by its endpoints. For a point $z \in [x, y]$, we will use the notation $z = (1-s)x \oplus sy$, where $s = \frac{d(x,z)}{d(x,y)}, 1-s = \frac{d(y,z)}{d(x,y)}$ assuming $x \neq y$. Let (X, d) be a geodesic metric space. A geodesic triangle consists of three points $p, q, r \in X$ and three geodesics $[p, q], [q, r], [r, p]$ denoted $\Delta([p, q], [q, r], [r, p])$. For such a triangle, there is a comparison triangle $\overline{\Delta}(\overline{p}, \overline{q}, \overline{r}) \to \mathbb{E}^2 :$ $d(p, q) = d(\overline{p}, \overline{q})$, $d(q, r) = d(\overline{q}, \overline{r})$, $d(r, p) = d(\overline{r}, \overline{p})$.

Definition 1.2. A geodesic space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom.
Cat(0): Let $\Delta = (x_1, x_2, x_3)$ be a geodesic triangle in b-metric space X and let $\bar{\Delta} \in \mathbb{E}^2$ be a comparison triangle for Δ. Then Δ is said to satisfy the CAT(0) inequality if for all $x, y \in \Delta$ and all comparison points $\overline{x}, \overline{y} \in \overline{\Delta} := (\overline{x_1}, \overline{x_2}, \overline{x_3})$ such that $d(x, y) \leq d^2_{\mathbb{E}}(\overline{x}, \overline{y})$.

It is easy to see that a CAT(0) space is uniquely geodesic.

It is well known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space. Other examples include inner product spaces, R-trees (see, for example, [8]), Euclidean building (see, for example, [9]), and the complex Hilbert ball with a hyperbolic metric (see, for example, [10]). For a thorough discussion on other spaces in geometry, see, for example, [8]-[19]. We collect some properties of CAT(0) spaces. For more details, we refer the readers to [11]-[21].

Lemma 1.3. [11] *Let (X, d) be a CAT(0) space. Then the following assertions hold.*
(i) For x, y in X and s in [0, 1], there exists a unique point $z \in [x, y]$ such that

$$d(x, z) = sd(x, y) \quad \text{and} \quad d(y, z) = (1 - s)d(x, y). \tag{1.2}$$

We use the notation $(1 - s)x \oplus sy$ for the unique point z satisfying (1.2)
(ii) For x, y in X and s in $[0, 1]$, we have

$$d((1 - s)x \oplus sy, z) \leq (1 - s)d(x, z) + sd(y, z). \tag{1.3}$$

Example 1.4. (I). Let $X := l_p(\mathbb{R})$ where $l_p(\mathbb{R}) := \{\{x_n\} \subset \mathbb{R} : \sum_{i=1}^{\infty} |x_i| < \infty\}$. Define $d : X \times X \to [0, \infty)$ as:

$$d(x, y) = (\sum_{i=1}^{\infty} |x_i - y_i|)$$

where $x = \{x_n\}, y = \{y_n\}$. Then d is a metric space, see([22] -[24]). And, define a continuous mapping $\zeta : [0, d(x, y)] \to X$ by $\zeta(z) = (1 - s)x + sy$ for all $s \in [0, d(x, y)]$ and all $z \in X$. Then (X, d) is a CAT(0) space.

(II). Let $X := L_p[0, 1]$ be the space of all real functions $x(s)$, $s \in [0, 1]$ such that $\int_0^1 |x(s)|ds < \infty$. Define $d : X \times X \to [0, \infty)$ as:

$$||x|| = (\int_0^1 |x(s)|ds)$$

where $x = x(s)$. Then d is a metric space, see([22] -[24]). And, define a continuous mapping $\zeta : [0, d(x, y)] \to X$ by $\zeta(z) = (1 - s)x + sy$ for all $s \in [0, d(x, y)]$ and all $z \in X$. Then (X, d) is a CAT(0) space.

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\},$$

and the asymptotic center A $(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}$$

A sequence $\{x_n\}$ in X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta - \lim_n x_n = x$ and call x the Δ-limit of $\{x_n\}$, see [25].

Lemma 1.5. [26] *Every bounded sequence in a complete CAT(0) space X has a Δ-convergent subsequence.*

Lemma 1.6. [27] *Let C be a closed and convex subset of a complete CAT(0) space X. If $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C.*

Lemma 1.7. [28] *Let X be a complete CAT(0) space and let $x \in X$. Suppose that $0 < b \leq s_n \leq c < 0$ and $x_n, y_n \in X$ for $n = 1, 2, \ldots$. If for some $r \geq 0$ we have*

$$\limsup_{n\to\infty} d(x_n, x) \leq r, \quad \limsup_{n\to\infty} d(y_n, x) \leq r,$$

and $\lim_{n\to\infty} d(s_n x_n \oplus (1 - s_n)y_n, x) = r$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Lemma 1.8. [29] *Let C be a nonempty closed and convex subset of a complete CAT(0) space X and let $S : C \to C$ be an α-nonexpansive mapping for some $\alpha < 1$. If $\{x_n\}$ is a sequence in C such that $d(Sx_n, x_n) \to 0$ and $\Delta - \lim_{n\to\infty} x_n = z$ for some $z \in X$, then $z \in C$ and $Sz = z$.*

Definition 1.9. [30] Let (X, d) be a metric space and C be nonempty subset. Then $S : C \to C$ said to be a quasi-nonexpansive if $F(S) \neq \emptyset$; and $d(Sx, p) \leq d(x, p)$ for all $p \in F(S); = \{x \in X | x = Sx\}$, and $x \in C$.

Lemma 1.10. [30] *Let C be a nonempty subset of a $CAT(0)$ space X. Let $S : C \to C$ be a square α-nonexpansive mapping and $F(S) \neq \emptyset$, then S is quasi-nonexpansive.*

In this paper, we will study research by the following process. First, we introduce Suzuki-square-α-nonexpansive mappings in CAT(0) spaces by using the concept of a Suzuki condition and α-nonexpansive mappings. Second, we create results with respect to approximation of common fixed points of Suzuki-square-α-nonexpansive mappings in CAT(0) spaces by the concept of iterative process of Muangchoo-in et al. [31]. Finally, We obtain the approximation of common fixed point of Suzuki-square-α-nonexpansive mappings in CAT(0) spaces and prove that results.

2. Main Results

In this section, we state some useful lemmas as follows. Next, we introduce definitions of Suzuki-square-α-nonexpansive mappings in metric spaces.

Definition 2.1. Let (X, d) be a metric space and C be a nonempty subset. Then $S : C \to C$ said to be a Suzuki-square-α-nonexpansive mapping (or Suzuki-α-noexpansive mapping), if $\alpha < 1$ such that

$$\tfrac{1}{2} d(x, Sx) \leq d(x, y) \Rightarrow d^2(Sx, Sy) \leq \alpha d^2(Sx, y) + \alpha d^2(x, Sy) + (1 - 2\alpha) d^2(x, y)$$

for all $x, y \in C$.

Remark 2.2. A Suzuki condition and square-α-nonexpansive mapping are a Suzuki-square-α-nonexpansive mappings.

Example 2.3. By conditions of an example 1.1, the mapping $U : M \to M$ is defined by

$$Ux = \begin{cases} S_1x & (x \in \delta B_M); \\ S_2x & (\text{otherwise}), \end{cases} \tag{2.1}$$

where $\frac{1}{2}d(x, S_ix) \leq d(x, y)$, for $x, y \in M$, $\quad i = 1, 2$. Then, we see that U is a Suzuki-square-α-nonexpansive mapping.

Lemma 2.4. *Let C be a nonempty subset of a $CAT(0)$ space X. Let $S : C \to C$ be a Suzuki-square-α-nonexpansive mapping and $F(S) \neq \emptyset$, then S is quasi-nonexpansive.*

Proof. Since $F(S) \neq \emptyset$, we get that $\frac{1}{2}d(x, Sp) \leq d(x, p)$. By using lemma 1.10, we can prove S is quasi-nonexpansive. ■

Next, we recall that iterations in CAT(0) spaces. We begin the Ishikawa iteration in CAT(0) spaces is described as follows: For any initial point $x \in C$, we define the iterates $\{x_n\}$ by

$$\begin{cases} x_{n+1} = \gamma_n y_n \oplus (1 - \gamma_n)x_n \\ y_n = \beta_n Sx_n \oplus (1 - \beta_n)x_n \ n \in \mathbb{N}, \end{cases} \tag{2.2}$$

where $\{\beta_n\}$ and $\{\gamma_n\}$ are in $(0, 1)$, see [32]. In 2018, Muangchoo-in et al. [31] introduced and approximated common fixed points of two alpha-nonexpansive mappings through weak and strong convergence of an iterative sequence in a uniformly convex Babach space. For any initial point $x \in C$, we define the iterates $\{x_n\}$ by

$$\begin{cases} x_{n+1} = \gamma_n Sy_n \oplus (1 - \gamma_n)x_n \\ y_n = \beta_n Sx_n \oplus (1 - \beta_n)x_n \ n \in \mathbb{N}, \end{cases} \tag{2.3}$$

where $\{\beta_n\}$ and $\{\gamma_n\}$ are in $(0, 1)$.

Next, we would like to introduce lemma for approximation of sequence by using the concept of Suzuki-square-α-nonexpansive mappings.

Lemma 2.5. *Let C be a nonempty closed convex subset of a complete CAT(0) space (X, d). Suppose that $S_1, S_2 : C \to C$ are Suzuki-square-α-nonexpansive mappings and $F(S_1) \cap F(S_2)$ be a set of all common fixed points of two nonexpansive mappings S_1 and S_2 of C. Assume there exists $p \in F(S_1) \cap F(S_2)$. Suppose that $\{x_n\}$ is defined by iteration, for any initial point $x \in C$, we define the iterates $\{x_n\}$ by*

$$\begin{cases} x_{n+1} = \gamma_n S_1 y_n \oplus (1 - \gamma_n)x_n \\ y_n = \beta_n S_2 x_n \oplus (1 - \beta_n)x_n \ n \in \mathbb{N}, \end{cases} \tag{2.4}$$

where $\{\beta_n\}$ and $\{\gamma_n\}$ are in $(0, 1)$ and $d(x_n, S_1y_n) \leq 2d(x_n, y_n)$ for all $n \in \mathbb{N}$. Then

$$\lim_{n\to\infty} d(S_1x_n, x_n) = \lim_{n\to\infty} d(S_2x_n, x_n) = 0.$$

Proof. Let $p \in F(S_1) \cap F(S_2)$. Then, we see that $\frac{1}{2}d(x, S_i p) \leq d(x,p)$, for $i = 1, 2$. By lemma 2.4 we get

$$\begin{aligned}
d(x_{n+1}, p) &= d((1-\gamma_n)x_n \oplus \gamma_n S_1 y_n, p)\\
&\leq (1-\gamma_n)d(x_n,p) + \gamma_n d(S_1 y_n, p)\\
&\leq (1-\gamma_n)d(x_n,p) + \gamma_n d(y_n, p)\\
&= (1-\gamma_n)d(x_n,p) + \gamma_n d((1-\beta_n)x_n \oplus \beta_n S_2 x_n, p)\\
&\leq (1-\gamma_n)d(x_n,p) + \gamma_n(1-\beta_n)d(x_n,p) + \gamma_n\beta_n d(S_2x_n, p)\\
&\leq (1-\gamma_n)d(x_n,p) + \gamma_n(1-\beta_n)d(x_n,p) + \gamma_n\beta_n d(x_n, p)\\
&\leq (1-\gamma_n)d(x_n,p) + \gamma_n(1-\beta_n)d(x_n,p) + \gamma_n\beta_n d(S_2x_n, p)\\
&= d(S_2x_n, p)
\end{aligned}$$

Hence $\lim_{n\to\infty} d(x_n,p)$ exists. Let $\lim_{n\to\infty} d(x_n,p) = r$ where r is a real number. By S_2 is quasi-nonexpansive mapping then we have $d(S_2x_n,p) \leq d(x_n,p)$ for all $n = 1, 2, \ldots$. So $\limsup_{n\to\infty} d(S_2x_n,p) = \limsup_{n\to\infty} d(x_n,p) = r$. Also,

$$\begin{aligned}
d(y_n, p) &= d((1-\beta_n)x_n \oplus \beta_n S_2x_n, p)\\
&\leq (1-\beta_n)d(x_n,p) + \beta_n d(S_2x_n,p)\\
&\leq (1-\beta_n)d(x_n,p) + \beta_n d(x_n,p)\\
&= d(x_n,p),
\end{aligned} \tag{2.5}$$

and by S_1 is a quasi-nonexpansive mapping then we obtain that

$$\limsup_{n\to\infty} d(S_1y_n,p) \leq \limsup_{n\to\infty} d(y_n,p) \leq r. \tag{2.6}$$

Moreover, $\lim_{n\to\infty} d(x_{n+1},p) = r$ means that

$$\lim_{n\to\infty} d(\gamma_n S_1 y_n \oplus (1-\gamma_n)x_n, p) = r. \tag{2.7}$$

By Lemma 1.7 , we get that

$$\lim_{n\to\infty} d(S_1y_n, x_n) = 0. \tag{2.8}$$

Since $d(x_n,p) \leq d(x_n, S_1y_n) + d(S_1y_n,p) \leq d(x_n, S_1yn) + d(y_n,p)$, then we obtain that

$$r \leq \liminf_{n\to\infty} d(y_n,p) \tag{2.9}$$

By inequality (2.6) and (2.9), we obtain that

$$\lim_{n\to\infty} d(\beta_n S_2x_n \oplus (1-\beta_n)x_n, p) = \lim_{n\to\infty} d(y_n,p) = 0 \tag{2.10}$$

By Lemma 1.7 , we get that

$$\lim_{n\to\infty} d(S_2x_n, x_n) = 0. \tag{2.11}$$

Noe, we consider

$$\begin{aligned}
d(S_2x_n, y_n) &= d(S_2x_n, \beta_n S_2x_n \oplus (1-\beta_n)x_n)\\
&\leq (1-\beta_n)d(S_2x_n, S_2x_n) + \beta_n d(S_2x_n, x_n)\\
&= \beta_n d(S_2x_n, x_n),
\end{aligned} \tag{2.12}$$

then by inequality (2.11), we have

$$\lim_{n\to\infty} d(S_2x_n, y_n) = 0. \tag{2.13}$$

By definition 2.1 and the hypothesis $\frac{1}{2}d(x_n, S_1y_n) \le d(x_n, y_n)$, we get that

$$\begin{aligned} d(S_1x_n, x_n)^2 \le& (d(S_1x_n, S_1y_n) + d(S_1y_n, x_n))^2 \\ =& d(S_1x_n, S_1y_n)^2 + 2d(S_1x_n, S_1y_n)d(S_1y_n, x_n)) + d(S_1y_n, x_n))^2 \\ \le& \alpha d(S_1x_n, y_n)^2 + \alpha d(x_n, S_1y_n)^2 + (1-2\alpha)d(x_n, y_n)^2 \\ &+ 2d(S_1x_n, S_1y_n)d(S_1y_n, x_n)) + d(S_1y_n, x_n))^2 \\ \le& \alpha(d(S_1x_n, x_n) + d(x_n, y_n))^2 + (1-2\alpha)d(x_n, y_n)^2 \\ &+ 2d(S_1x_n, S_1y_n)d(S_1y_n, x_n)) + (1+\alpha)d(S_1y_n, x_n))^2 \\ \le& \alpha d(S_1x_n, x_n)^2 + \alpha 2d(S_1x_n, x_n)d(x_n, y_n) + \alpha d(x_n, y_n)^2 \\ &+ (1-2\alpha)d(x_n, y_n)^2 + 2d(S_1x_n, S_1y_n)d(S_1y_n, x_n)) \\ &+ (1+\alpha)d(S_1y_n, x_n))^2, \end{aligned} \tag{2.14}$$

so

$$\begin{aligned} (1-\alpha)d(S_1x_n, x_n)^2 \le& (1-\alpha)d(x_n, y_n)^2 + \alpha 2d(S_1x_n, x_n)d(x_n, y_n) \\ &+ 2d(S_1x_n, S_1y_n)d(S_1y_n, x_n)) + (1+\alpha)d(S_1y_n, x_n))^2 \\ \le& (1-\alpha)(d(x_n, S_2x_n) + d(S_2x_n, y_n))^2 \\ &+ 2\alpha d(S_1x_n, x_n)(d(x_n, S_2x_n) + d(S_2x_n, y_n)) \\ &+ 2d(S_1x_n, S_1y_n)d(S_1y_n, x_n)) \\ &+ (1+\alpha)d(S_1y_n, x_n))^2 \end{aligned} \tag{2.15}$$

By inequality (2.8), (2.11) and (2.13), we conclude that

$$\lim_{n\to\infty} d(S_1x_n, x_n) = \lim_{n\to\infty} d(S_2x_n, x_n) = 0.$$

■

Next, we would like to introduce a main theorem using the concept of Suzuki-square-α-nonexpansive mappings and the iterates 2.4.

Theorem 2.6. *Let C be a nonempty closed convex subset of a complete CAT(0) space (X, d). Suppose that $S_1, S_2 : C \to C$ are Suzuki-square-α-nonexpansive mappings. Assume C satisfies Opial's condition and the sequence defined be the iteration, for any initial point $x \in C$, we define the iterates 2.4 If $F(S_1) \cap F(S_2) \neq \emptyset$ then then $\{x_n\}$ Δ-converges to a unique common fixed point of S_1 and S_2.*

Proof. Begin proof by let p be a common fixed point of S_1 and S_2 and $\lim_{n\to\infty} d(x_n, p)$ exists. Thus, we have $\{x_n\}$ is bounded.

Thus, $\{x_n\}$ has a Δ-convergent subsequence and the asymptotic center of $\{x_n\}$ is in C, by using lemmas 1.5 and 1.6. We next prove that every Δ-convergent subsequence of $\{x_n\}$ has a unique Δ-limit in $F(S_1) \cap F(S_2)$. Supose that u and v be two Δ-limits of the subsequences $\{a_n\}$ and $\{b_n\}$ of $\{x_n\}$, respectively. By definition $A(\{a_n\}) = \{a\}$ and $A(\{b_n\}) = \{b\}$. By lemma 2.5, $\lim_{n\to\infty} d(S_1a_n, a_n) = 0 = \lim_{n\to\infty} d(S_2a_n, a_n)$. Now using the Δ-convergence of $\{u_n\}$ to u and the Suzuki-square-α-nonexpansive mappings of T and S, we obtain $a \in F(S_1) \cap F(S_2)$ by a repeated application of lemma 1.8 on S_1 and S_2. Again in the same fashion, we can prove that $b \in F(S_1) \cap F(S_2)$. Next, we prove the uniqueness. To this end, if a and b are distinct then by the uniqueness of asymptotic centers,

$$\begin{aligned}
\lim_{n\to\infty} d(x_n, a) &= \limsup_{n\to\infty} d(a_n, a) \\
&< \limsup_{n\to\infty} d(a_n, b) \\
&= \limsup_{n\to\infty} d(x_n, b) \\
&= \limsup_{n\to\infty} d(b_n, b) \\
&< \limsup_{n\to\infty} d(b_n, a) \\
&= \limsup_{n\to\infty} d(x_n, a) \\
&= \lim_{n\to\infty} d(x_n, a). \qquad (2.16)
\end{aligned}$$

This is a contradiction, so $a = b$. ∎

Next, we would like to introduce a theorem using the concept of Opial's condition.

Theorem 2.7. *Let C be a nonempty closed convex subset of a complete CAT(0) space (X, d). Suppose that $S_1, S_2 : C \to C$ be Suzuki-square-α-nonexpansive mappings. Assume C satisfies Opial's condition and the sequence defined be the iteration 2.4 . If $F(S_1) \cap F(S_2) \neq \emptyset$ then $\{x_n\}$converges strongly to a common fixed point of S_1 and S_2 if and only if* $\liminf_{n\to\infty} d(x_n, F(S_1) \cap F(S_2)) = 0$, *where* $d(x, F(S_1) \cap F(S_2)) := \inf\{d(x, p) | p \in F(S_1) \cap F(S_2)\}$.

Proof. Necessity is obvious. Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F(S_1) \cap F(S_2)) = 0$. As proved in lemma 2.5, we have

$$d(x_{n+1}, p) \leq d(x_n, p), \text{ for all } p \in F(S_1) \cap F(S_2).$$

This implies that $d(x_{n+1}, F(S_1) \cap F(S_2)) \leq d(x_n, F(S_1) \cap F(S_2))$, so that $d(x_n, F(S_1) \cap F(S_2))$ exists. Thus by hypothesis $\lim_{n\to\infty} d(x_n, F(S_1) \cap F(S_2)) = 0$. Next, we show that $\{x_n\}$ is a Cauchy sequence in C. Let $\epsilon > 0$ be arbitrarily chosen. Since $\lim_{n\to\infty} d(x_n, F(S_1) \cap F(S_2)) = 0$, there exists a positive integer n_0 such that $d(x_n, F(S_1) \cap F(S_2)) < \frac{\epsilon}{4}, \forall n \geq n_0$. In particular, $\inf\{d(x_{n_0}, p) | p \in F(S_1) \cap F(S_2)\} < \frac{\epsilon}{4}$. Thus there must exist $p^* \in F(S_1) \cap F(S_2)$ such that $d(x_{n_0}, p^*) < \frac{\epsilon}{2}$. Now, for all $m, n \geq n_0$, we have

$$d(x_{n+m}, x_n) \leq d(x_{n+m}, p^*) + d(p^*, x_n) \leq 2d(x_{n_0}, p^*) < \epsilon.$$

Hence $\{x_n\}$ is a Cauchy sequence in a closed subset C of a complete CAT (0) space, and so it must converge to a point p in C. Now, $\lim_{n\to\infty} d(x_n, F(S_1) \cap F(S_2)) = 0$ gives that $d(p, F(S_1) \cap F(S_2)) = 0$. Since F is closed, so we have $p \in F(S_1) \cap F(S_2)$. ∎

3. Conclusion

The purpose of this paper is to create results with respect to approximation of common fixed points of Suzuki-square-α-nonexpansive mappings in $CAT(0)$ spaces. Results: We prove convergence theorems fixed points of Suzuki-square-α-nonexpansive mappings in $CAT(0)$ spaces.

(1) Let (X, d) be a metric space and C be nonempty subset. Then $S : C \to C$ said to be a Suzuki-square-α-nonexpansive mapping (or Suzuki-α-noexpansive mapping), if $\alpha < 1$ such that

$$\frac{1}{2}d(x, Sx) \leq d(x, y) \Rightarrow d^2(Sx, Sy) \leq \alpha d^2(Sx, y) + \alpha d^2(x, Sy) + (1 - 2\alpha)d^2(x, y)$$

for all $x, y \in C$

(2) Let C be a nonempty closed convex subset of a complete CAT(0) space (X, d). Suppose that $S_1, S_2 : C \to C$ are Suzuki-square-α-nonexpansive mappings and $F(S_1) \cap F(S_2)$ be a the set of all common fixed points of two nonexpansive mappings S_1 and S_2 of C. Assume there exists $p \in F(S_1) \cap F(S_2)$. Suppose that $\{x_n\}$ is defined by iteration, for any initial point $x \in C$, we define the iterates $\{x_n\}$ by

$$\begin{cases} x_{n+1} = \gamma_n S_1 y_n \oplus (1 - \gamma_n)x_n \\ y_n = \beta_n S_2 x_n \oplus (1 - \beta_n)x_n \ n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $\{\beta_n\}$ and $\{\gamma_n\}$ are in $(0, 1)$ and $d(x_n, S_1 y_n) \leq 2d(x_n, y_n)$ for all $n \in \mathbb{N}$. Then

$$\lim_{n\to\infty} d(S_1 x_n, x_n) = \lim_{n\to\infty} d(S_2 x_n, x_n) = 0.$$

(3) Let C be a nonempty closed convex subset of a complete CAT(0) space (X, d). Suppose that $S_1, S_2 : C \to C$ are Suzuki-square-α-nonexpansive mappings. Assume C satisfies Opial's condition and the sequence defined be the iteration, for any initial point $x \in C$, we define the iterates 2.4. If $F(S_1) \cap F(S_2) \neq \emptyset$ then then $\{x_n\}$ Δ-converges to a unique common fixed point of S_1 and S_2.

(4) Let C be a nonempty closed convex subset of a complete CAT(0) space (X, d). Suppose that $S_1, S_2 : C \to C$ be Suzuki-square-α-nonexpansive mappings. Assume C satisfies Opial's condition and the sequence defined be the iteration 2.4 . If $F(S_1) \cap F(S_2) \neq \emptyset$ then $\{x_n\}$ converges strongly to a common fixed point of S_1 and S_2 if and only if $\liminf_{n\to\infty} d(x_n, F(S_1) \cap F(S_2)) = 0$, where $d(x, F(S_1) \cap F(S_2)) := \inf\{d(x, p)|p \in F(S_1) \cap F(S_2)\}$.

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