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On the Ulam Stability of Two Fuzzy Number-valued Functional Equations Via the Metric Defined on a Fuzzy Number Space

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Abstract In this research, two novel fuzzy number-valued functional equations are studied. The first fuzzy number-valued functional equation is motivated from the fuzzy number-valued additive functional equation, and it is of the form

$$rf\left(\frac{x+y}{2}\right)+sf\left(\frac{x-y}{2}\right)=\frac{r+s}{2}f\left(x\right),$$

where f is a fuzzy number-valued function and $r, s \in \mathbb{R}$ with $r+s \neq 0$. Meanwhile, the second fuzzy number-valued functional equation is

$$rf\left(\frac{x+y}{s}\right)=\left(\frac{r}{s}+\frac{s}{r}\right)f\left(x\right)+\left(\frac{r}{s}-\frac{s}{r}\right)f\left(y\right),$$

where f is a fuzzy number-valued mapping and $r, s \in \mathbb{R}\setminus\{0\}$ and $0 < s < 2$. The stability results of two proposed functional equations are proved by using the metric related to fuzzy numbers and the Hausdorff metric.

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1. Introduction

The stability is one of the most important topics in the functional equation theory. The first investigation of the stability for functional equations starts with a group homomorphisms question of Ulam [1] in the mid of the 19th century. In 1941, this question was studied by Hyers [2] who gave the first significant answer for the functional equation in Banach spaces. The question of Ulam [1] and the answer of Hyers [2] can be considered a great thing to open avenues for further developments of the stability for functional

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equations. Afterward, many researchers studied stability results in a variety of ways of functional equations.

About fifty years ago, there has been a large amount of research in many branches of mathematics studying the fuzzy theory, which was introduced by Zadeh [3]. In 2019, Wu and Jin [4] gave the first exploration of the stability problem for fuzzy number-valued functional equations. They proved two stability results using some properties of the metric related to the Hausdorff metric and fuzzy numbers. One of these results deals with the following fuzzy number-valued functional equation:

$$f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)=f\left(x\right), \tag{1.1}$$

where f is a fuzzy number-valued mapping on a subspace of a Banach space.

It points out that the above functional equation can be generalized as follows:

$$rf\left(\frac{x+y}{2}\right)+sf\left(\frac{x-y}{2}\right)=\frac{r+s}{2}f\left(x\right), \tag{1.2}$$

where f is a fuzzy number-valued mapping on a subspace of a Banach space and $r, s \in \mathbb{R}$ with $r+s \neq 0$. In this research, we establish the Ulam stability result for (1.2). Moreover, the Ulam stability result for the following functional equation is proved:

$$rf\left(\frac{x+y}{s}\right)=\left(\frac{r}{s}+\frac{s}{r}\right)f\left(x\right)+\left(\frac{r}{s}-\frac{s}{r}\right)f\left(y\right), \tag{1.3}$$

where f is a fuzzy number-valued mapping and $r, s \in \mathbb{R}\setminus\{0\}$ and $0 < s < 2$.

2. Preliminaries

Throughout this paper,

- $\mathbb{N}$ denotes the set of all positive integers;
- $\mathbb{R}^+$ denotes the set of all positive real numbers;
- $\mathbb{R}$ denotes the set of all real numbers;
- X and Y stand for Banach spaces unless otherwise specified;
- $P_{kc}\left(X\right) := \{\emptyset \neq A \subseteq X : A \text{ is a compact convex set}\}$;
- $\overline{A}$ denotes the closure of $A \subseteq X$;
- B stands for a subspace of Y.

Now, we give some necessary definitions and give fundamental properties needed to prove main theorems in this paper.

Definition 2.1. The Hausdorff metric of two non-empty subsets A and B of a metric space (X, d) is denoted by $d_H\left(A, B\right)$, and it is given by

$$d_H\left(A,B\right)=\max\left\{\sup_{x\in A}\inf_{y\in B}d\left(x,y\right), \sup_{y\in B}\inf_{x\in A}d\left(x,y\right)\right\}.$$

Remark 2.2. If d_H is a metric on $CB\left(X\right)$ which is defined by the set of all nonempty compact subsets of a metric space (X, d). Moreover, $CB\left(X\right)$ is complete whenever X is complete.

Definition 2.3 ([4])**.** A function u from X into a closed interval $[0, 1]$ is said to be a fuzzy number on X if two following conditions hold:

(i) for each $\alpha \in (0, 1]$, we have $\left[u\right]^{\alpha} := \{x \in X : u\left(x\right) \geq \alpha\} \in P_{kc}\left(X\right)$;

(ii) $\left[u\right]^{0} := \overline{\{x \in X : u\left(x\right) > 0\}}$ is a compact set.

In this paper, the notion X_F represents the set of all fuzzy numbers on X.

Example 2.4. Let $X = \mathbb{R}$ be a Euclidean Banach space. Define a function $u : X \to [0, 1]$ by

$$u(x) = \begin{cases} x; & x \in [0, 0.2], \\ 0.2; & x \in (0.2, 0.3], \\ 8x - 2.2; & x \in (0.3, 0.4], \\ 0; & \text{otherwise.} \end{cases}$$

It is easy to see that

$$[u]^\alpha = \begin{cases} [\alpha, 0.4]; & 0 \le \alpha \le 0.2, \\ \left[\dfrac{\alpha + 2.2}{8}, 0.4\right]; & \text{otherwise.} \end{cases}$$

For each $\alpha \in (0, 1]$, we obtain $[\alpha, 0.4]$ and $\left[\dfrac{\alpha + 2.2}{8}, 0.4\right]$ are closed and bounded. By Heine-Borel theorem, we have $[\alpha, 0.4]$ and $\left[\dfrac{\alpha + 2.2}{8}, 0.4\right]$ are compact in X for each $\alpha \in (0, 1]$. This yields that $[u]^\alpha \in P_{kc}(X)$ for all $\alpha \in (0, 1]$. Moreover, we obtain

$$[u]^0 = \overline{[0, 0.4]} = [0, 0.4],$$

which is a compact set. Therefore, $u \in X_F$.

By using the Zadeh extension principle, we can prove the following assertions involving two operations on a set of all fuzzy numbers X_F:

(i) $[a + b]^\alpha = [a]^\alpha + [b]^\alpha$;
(ii) $[k \cdot a]^\alpha = k [a]^\alpha$

for all $a, b \in X_F$ and for all $k \in \mathbb{R}$.

Proposition 2.5 ([5]). *Let $D : X_F \times X_F \to \mathbb{R}^+ \cup \{0\}$ be defined by*

$$D(a, b) = \sup_{\alpha \in [0,1]} d_H\left([b]^\alpha, [b]^\alpha\right)$$

for all $a, b \in X_F$, where d_H is the Hausdorff metric. Then the following assertions hold:

(P1) $D(\lambda a, \lambda b) = |\lambda| \cdot D(a, b)$ *for all* $\lambda \in \mathbb{R}$ *and* $a, b \in X_F$*;*
(P2) $D(a + c, b + c) = D(a, b)$ *for all* $a, b, c \in X_F$*;*
(P3) $D(a + b, c + d) \le D(a, c) + D(b, d)$ *for all* $a, b, c, d \in X_F$*.*

Moreover, D is a metric on X_F and X_F is complete.

3. Main theoretical results

This section aims to investigate the stability theorems of two fuzzy number-valued functional equations (1.2) and (1.3), which are the main results of this research.

Lemma 3.1. *Let V be a vector space and $f : V \to X_F$ be a fuzzy number-valued mapping.*

(1) *If (1.2) holds, then for given $n \in \mathbb{N}$, we have*

$$2^n f(x) = f(2^n x)$$

for all $x \in V$.

(2) *If* (1.3) *holds, then for given* $n \in \mathbb{N}$, *we have*

$$f\left(\left(\frac{2}{s}\right)^n x\right) = \left(\frac{2}{s}\right)^n f(x)$$

for all $x \in V$.

Proof. (1) Replacing y by 0 in (1.2), we obtain $f\left(\frac{x}{2}\right) = \frac{1}{2}f(x)$ for all $x \in V$, that is, $2f(x) = f(2x)$ for all $x \in V$. By repeating this process, for each $n \in \mathbb{N}$, we get the conclusion in this claim.

(2) Replacing y by x in (1.3) we obtain $f\left(\frac{2x}{s}\right) = \frac{2}{s}f(x)$ for all $x \in V$. By repeating this process, for each $n \in \mathbb{N}$, we get the conclusion in this claim. ∎

Theorem 3.2. *Suppose that* $\varepsilon > 0$ *and* $f : B \to X_F$ *satisfies*

$$D\left(rf\left(\frac{x+y}{2}\right) + sf\left(\frac{x-y}{2}\right), \frac{r+s}{2}f(x)\right) < \varepsilon \tag{3.1}$$

for all $x, y \in B$, *where* $r, s \in \mathbb{R}$ *with* $r + s \neq 0$. *Then there is a unique mapping* $T : B \to X_F$ *satisfying* (1.2) *and*

$$D(f(x), T(x)) \leq \frac{2\varepsilon}{|r+s|}$$

for all $x \in B$.

Proof. From (3.1) with $y = 0$, we have

$$D\left(rf\left(\frac{x}{2}\right) + sf\left(\frac{x}{2}\right), \frac{r+s}{2}f(x)\right) < \varepsilon$$

and thus

$$D\left(f\left(\frac{x}{2}\right), \frac{1}{2}f(x)\right) < \frac{\varepsilon}{|r+s|} \tag{3.2}$$

for all $x \in B$. For each $n \in \mathbb{N}$, replacing x by $2^{n+1}x$ in (3.2), we get

$$D\left(f(2^n x), \frac{1}{2}f\left(2^{n+1}x\right)\right) < \frac{\varepsilon}{|r+s|},$$

that is,

$$D\left(\frac{f(2^n x)}{2^n}, \frac{f\left(2^{n+1}x\right)}{2^{n+1}}\right) < \frac{\varepsilon}{2^n|r+s|} \tag{3.3}$$

for all $x \in B$. For each $x \in B$ and $n \in \mathbb{N}$, we define $f_n(x) := \frac{f(2^n x)}{2^n}$.

Now, we will show that for $x \in B$, $\{f_n(x)\}$ is a Cauchy sequence in X_F. Let $m, n \in \mathbb{N}$. Without loss of generality, we may assume that with $m < n$. Then

$$\begin{aligned} D(f_m(x), f_n(x)) &\leq D(f_m(x), f_{m+1}(x)) + D(f_{m+1}(x), f_{m+2}(x)) \\ &\quad + \cdots + D(f_{n-1}(x), f_n(x)) \\ &< \frac{\varepsilon}{2^m |r+s|} + \frac{\varepsilon}{2^{m+1}|r+s|} + \ldots + \frac{\varepsilon}{2^{n-1}|r+s|} \\ &< \frac{\varepsilon}{2^m |r+s|} + \frac{\varepsilon}{2^{m+1}|r+s|} + \ldots \\ &= \frac{\varepsilon}{2^{m-1}|r+s|} \end{aligned} \tag{3.4}$$

for all $x \in B$. Letting $m, n \to \infty$ in (3.4), we obtian $\{f_n(x)\}$ is a Cauchy sequence in X_F. Since X_F is complete, we can construct a mapping $T : B \to X_F$ as

$$T(x) = \lim_{n\to\infty} f_n(x).$$

for all $x \in B$. From (3.3), for each $n \in \mathbb{N}$, we obtain

$$D(f_n(x), f(x)) \leq \sum_{n=1}^{\infty} D(f_n(x), f_{n-1}(x)) \leq \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n-1}|r+s|} = \frac{2\varepsilon}{|r+s|} \tag{3.5}$$

for all $x \in B$. The following inequality is obtained by taking the limit as $n \to \infty$ in (3.5):

$$D(T(x), f(x)) \leq \frac{2\varepsilon}{|r+s|}$$

for all $x \in B$.

Next, we prove that T satisfies (1.2). For each $n \in \mathbb{N}$, we obtain

$$\begin{aligned} & D\left(rf_n\left(\frac{x+y}{2}\right) + sf_n\left(\frac{x-y}{2}\right), \frac{r+s}{2} f_n(x)\right) \\ = \; & D\left(r\frac{f\left(\frac{2^n x + 2^n y}{2}\right)}{2^n} + s\frac{f\left(\frac{2^n x - 2^n y}{2}\right)}{2^n}, \frac{r+s}{2}\frac{f(2^n x)}{2^n}\right) \\ = \; & \frac{1}{2^n} D\left(rf\left(\frac{2^n x + 2^n y}{2}\right) + sf\left(\frac{2^n x - 2^n y}{2}\right), \frac{r+s}{2} f(2^n x)\right) \\ \leq \; & \frac{\varepsilon}{2^n} \end{aligned} \tag{3.6}$$

for all $x, y \in B$. This implies that

$$\lim_{n\to\infty} D\left(rf_n\left(\frac{x+y}{2}\right) + sf_n\left(\frac{x-y}{2}\right), \frac{r+s}{2} f_n(x)\right) = 0$$

for all $x, y \in B$. Since the metric D is continuous, we have

$$D\left(rT\left(\frac{x+y}{2}\right) + sT\left(\frac{x-y}{2}\right), \frac{r+s}{2} T(x)\right) = 0,$$

that is,

$$rT\left(\frac{x+y}{2}\right) + sT\left(\frac{x-y}{2}\right) = \frac{r+s}{2} T(x) \tag{3.7}$$

for all $x, y \in B$.

To show that T is unique, let $T_1, T_2 : B \to X_F$ be mappings satisfying (1.2) and

$$D\left(T_i\left(x\right), f\left(x\right)\right) \leq \frac{2\varepsilon}{|r+s|}$$

for all $x \in B$ and for all $i = 1, 2$. By Lamma 3.1, we get $2^n T_i\left(x\right) = T_i\left(2^n x\right)$ for all $x \in B$ and for all $i = 1, 2$. Then

$$\begin{aligned} D\left(T_1\left(x\right), T_2\left(x\right)\right) &= D\left(\frac{T_1\left(2^n x\right)}{2^n}, \frac{T_2\left(2^n x\right)}{2^n}\right) \\ &= D\left(\frac{T_1\left(2^n x\right) + f\left(2^n x\right)}{2^n}, \frac{T_2\left(2^n x\right) + f\left(2^n x\right)}{2^n}\right) \\ &= \frac{1}{2^n} D\left(T_1\left(2^n x\right) + f\left(2^n x\right), T_2\left(2^n x\right) + f\left(2^n x\right)\right) \\ &\leq \frac{1}{2^n} D\left(T_1\left(2^n x\right), f\left(2^n x\right)\right) + \frac{1}{2^n} D\left(T_2\left(2^n x\right), f\left(2^n x\right)\right) \\ &\leq \frac{4\varepsilon}{2^n |r+s|} \qquad (3.8) \end{aligned}$$

for all $x \in B$ and $n \in \mathbb{N}$. As $n \to \infty$ in in (3.8), we get $T_1\left(x\right) = T_2\left(x\right)$ for all x as desired. Hence, T is unique. ∎

In [4], Wu and Jin investigated the stability result of the functional equation as follows:

$$f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) = f\left(x\right), \qquad (3.9)$$

which is the special case of the functional equation (1.2) whenever $r = s \neq 0$. Therefore, the following result can be obtained from Theorem 3.2.

Corollary 3.3 ([4])**.** *Suppose that $\varepsilon > 0$ and $f : B \to X_F$ satisfies*

$$D\left(f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right), f\left(x\right)\right) < \varepsilon$$

for all $x, y \in B$. Then there is a unique mapping $T : B \to X_F$ satisfying (3.9) and

$$D\left(f\left(x\right), T\left(x\right)\right) \leq \varepsilon$$

for all $x \in B$.

In the last part of this section, we will present the stability of (1.3).

Theorem 3.4. *Suppose that $\varepsilon > 0$ and $f : B \to X_F$ satisfies*

$$D\left(rf\left(\frac{x+y}{s}\right), \left(\frac{r}{s} + \frac{s}{r}\right) f\left(x\right) + \left(\frac{r}{s} - \frac{s}{r}\right) f\left(y\right)\right) < \varepsilon \qquad (3.10)$$

for all $x, y \in B$, where $r, s \in \mathbb{R} \setminus \{0\}$ with $0 < s < 2$. Then there is a unique mapping $T : B \to X_F$ satisfying (1.3) and

$$D\left(f\left(x\right), T\left(x\right)\right) \leq \frac{s\varepsilon}{(2-s)|r|}$$

for all $x \in B$.

Proof. From (3.10), we replace y by x and so

$$D\left(f\left(\frac{2x}{s}\right), \frac{2}{s}f(x)\right) < \frac{\varepsilon}{|r|} \tag{3.11}$$

for all $x \in B$. For each $n \in \mathbb{N}$, we replace x by $\left(\frac{2}{s}\right) x$ in (3.11), we get

$$D\left(f\left(\left(\frac{2}{s}\right)^{n+1} x\right), \frac{2}{s}f\left(\left(\frac{2}{s}\right)^{n} x\right)\right) < \frac{\varepsilon}{|r|}$$

for all $x \in B$. On dividing the above inequality by $\left(\frac{2}{s}\right)^{n+1}$, we get

$$D\left(\frac{f\left(\left(\frac{2}{s}\right)^{n+1} x\right)}{\left(\frac{2}{s}\right)^{n+1}}, \frac{f\left(\left(\frac{2}{s}\right)^{n} x\right)}{\left(\frac{2}{s}\right)^{n}}\right) < \frac{\frac{\varepsilon}{|r|}}{\left(\frac{2}{s}\right)^{n+1}}. \tag{3.12}$$

For each $x \in B$, we define $f_n(x) = \dfrac{f\left(\left(\frac{2}{s}\right)^{n} x\right)}{\left(\frac{2}{s}\right)^{n}}$ for all $n \in \mathbb{N}$.

Now, we will claim that $\{f_n(x)\}$ is a Cauchy sequence in X_F for all $x \in B$. Let $m, n \in \mathbb{N}$. Without loss of generality, we may assume that with $m < n$. Then

$$\begin{aligned} D(f_m(x), f_n(x)) &\leq D(f_m(x), f_{m+1}(x)) + D(f_{m+1}(x), f_{m+2}(x)) \\ &\quad + \cdots + D(f_{n-1}(x), f_n(x)) \\ &< \frac{\varepsilon}{|r|}\left(\frac{s}{2}\right)^{m+1} + \frac{\varepsilon}{|r|}\left(\frac{s}{2}\right)^{m+2} + \ldots + \frac{\varepsilon}{|r|}\left(\frac{s}{2}\right)^{n} \\ &< \frac{\varepsilon}{|r|}\left(\frac{s}{2}\right)^{m+1} + \frac{\varepsilon}{|r|}\left(\frac{s}{2}\right)^{m+2} + \ldots \\ &= \frac{2\varepsilon}{(2-s)|r|}\left(\frac{s}{2}\right)^{m+1} \end{aligned} \tag{3.13}$$

for all $x \in B$. This yields that $\{f_n(x)\}$ is a Cauchy sequence by letting $m, n \to \infty$ in (3.13). By using the completeness of X_F, a mapping $T : B \to X_F$ can be defined by

$$T(x) = \lim_{n\to\infty} f_n(x).$$

for all $x \in B$. From (3.12), for each $n \in \mathbb{N}$, we obtain

$$\begin{aligned} D(f_n(x), f(x)) &\leq \sum_{n=1}^{\infty} D(f_n(x), f_{n-1}(x)) \\ &\leq \sum_{n=1}^{\infty} \frac{\frac{\varepsilon}{|r|}}{\left(\frac{2}{s}\right)^{n}} \\ &= \frac{s\varepsilon}{(2-s)|r|} \end{aligned}$$

for all $x \in B$. The following inequality is obtained by taking the limit as $n \to \infty$ in the above inequality:

$$D\left(T\left(x\right), f\left(x\right)\right) \leq \frac{s\varepsilon}{\left(2-s\right)\left|r\right|}$$

for all $x \in B$.

Next, we prove that T satisfies (1.3). For each $n \in \mathbb{N}$, we obtain

$$\begin{aligned}
& D\left(rf_n\left(\frac{x+y}{s}\right), \left(\frac{r}{s}+\frac{s}{r}\right)f_n\left(x\right)+\left(\frac{r}{s}-\frac{s}{r}\right)f_n\left(y\right)\right) \\
= \quad & D\left(r\frac{f\left(\left(\frac{2}{s}\right)^n\left(\frac{x+y}{s}\right)\right)}{\left(\frac{2}{s}\right)^n}, \left(\frac{r}{s}+\frac{s}{r}\right)\frac{f\left(\left(\frac{2}{s}\right)^n x\right)}{\left(\frac{2}{s}\right)^n}+\left(\frac{r}{s}-\frac{s}{r}\right)\frac{f\left(\left(\frac{2}{s}\right)^n y\right)}{\left(\frac{2}{s}\right)^n}\right) \\
= \quad & \left(\frac{s}{2}\right)^n D\left(rf\left(\left(\frac{2}{s}\right)^n\left(\frac{x+y}{s}\right)\right), \left(\frac{r}{s}+\frac{s}{r}\right)f\left(\left(\frac{2}{s}\right)^n x\right)+\left(\frac{r}{s}-\frac{s}{r}\right)f\left(\left(\frac{2}{s}\right)^n y\right)\right) \\
\leq \quad & \left(\frac{s}{2}\right)^n \varepsilon
\end{aligned} \tag{3.14}$$

for all $x, y \in B$. This yields the following inequality by taking the limit as $n \to \infty$ of (3.14):

$$\lim_{n\to\infty} D\left(rf_n\left(\frac{x+y}{s}\right), \left(\frac{r}{s}+\frac{s}{r}\right)f_n\left(x\right)+\left(\frac{r}{s}-\frac{s}{r}\right)f_n\left(y\right)\right) = 0$$

for all $x, y \in B$. Since D is continuous, we obtain

$$D\left(rT\left(\frac{x+y}{s}\right), \left(\frac{r}{s}+\frac{s}{r}\right)T\left(x\right)+\left(\frac{r}{s}-\frac{s}{r}\right)T\left(y\right)\right) = 0,$$

that is,

$$rT\left(\frac{x+y}{s}\right) = \left(\frac{r}{s}+\frac{s}{r}\right)T\left(x\right)+\left(\frac{r}{s}-\frac{s}{r}\right)T\left(y\right) \tag{3.15}$$

for all $x, y \in B$.

Finally, we will show that T is unique. Assume that $T_1, T_2 : B \to X_F$ are mappings satisfying (1.3) and $D\left(T_i\left(x\right), f\left(x\right)\right) \leq \frac{s\varepsilon}{\left(2-s\right)\left|r\right|}$ for all $x \in B$ and for all $i = 1, 2$. By

Lamma 3.1, we get $T_i\left(\left(\frac{2}{s}\right)^n x\right) = \left(\frac{2}{s}\right)^n T_i(x)$ for all $x \in B$ and for all $i = 1, 2$. Then,

$$\begin{aligned}
D\left(T_1(x), T_2(x)\right) &= D\left(\frac{T_1\left(\left(\frac{2}{s}\right)^n x\right)}{\left(\frac{2}{s}\right)^n}, \frac{T_2\left(\left(\frac{2}{s}\right)^n x\right)}{\left(\frac{2}{s}\right)^n}\right) \\
&= \left(\frac{s}{2}\right)^n D\left(T_1\left(\left(\frac{2}{s}\right)^n x\right), T_2\left(\left(\frac{2}{s}\right)^n x\right)\right) \\
&= \left(\frac{s}{2}\right)^n D\left(T_1\left(\left(\frac{2}{s}\right)^n x\right) + f\left(\left(\frac{2}{s}\right)^n x\right), T_2\left(\left(\frac{2}{s}\right)^n x\right) + f\left(\left(\frac{2}{s}\right)^n x\right)\right) \\
&\leq \left(\frac{s}{2}\right)^n D\left(T_1\left(\left(\frac{2}{s}\right)^n x\right), f\left(\left(\frac{2}{s}\right)^n x\right)\right) \\
&\quad + \left(\frac{s}{2}\right)^n D\left(T_2\left(\left(\frac{2}{s}\right)^n x\right), f\left(\left(\frac{2}{s}\right)^n x\right)\right) \\
&\leq \left(\frac{s}{2}\right)^n \frac{2s\varepsilon}{(2-s)|r|} \qquad (3.16)
\end{aligned}$$

for all $x \in B$ and $n \in \mathbb{N}$. As $n \to \infty$ in (3.16), we get $T_1(x) = T_2(x)$ for all x. Hence, T is unique. ∎

4. Conclusion

We established Ulam stability results of following functional equations via the metric defined on a space of fuzzy numbers under some reasonable conditions:

$$rf\left(\frac{x+y}{2}\right) + sf\left(\frac{x-y}{2}\right) = \frac{r+s}{2} f(x) \quad (r, s \in \mathbb{R} \text{ with } r+s \neq 0)$$

and

$$rf\left(\frac{x+y}{s}\right) = \left(\frac{r}{s} + \frac{s}{r}\right) f(x) + \left(\frac{r}{s} - \frac{s}{r}\right) f(y) \quad (r \in \mathbb{R} \setminus \{0\}, 0 < s < 2),$$

where f is an unknown fuzzy number-valued mapping. One of the main results is the extension of some parts of Wu and Jin [4] which is the first stability result for fuzzy number-valued mappings. Meanwhile, the second main theorem in this paper is the new result which was never mentioned in the literature. However, there are several functional equations that are still open to the reader to investigate using a similar technique with this paper.

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References

[1] S. M. Ulam, Problems in Modern Mathematics, Wiley, New York, 1940.

[2] D.H. Hyers, On the stability of the linear functional equation, Proc. Nat. Acad. Sci. USA. 27 (1941) 222-224.

[3] L.A. Zadeh, Fuzzy sets, Information and control 8 (1965) 338-353.

[4] J.R. Wu, Z.Y. Jin, A note on Ulam stability of some fuzzy number-valued functional equations, Fuzzy Sets Syst. 375 (2019) 191-195.

[5] J. Ban, Ergodic theorems for random compact sets and fuzzy variables in Banach spaces, Fuzzy Sets Syst. 44 (1991) 71-82.