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Unshackle Game on 2D Grid and Shadow Strategy

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Abstract Unshackle game is a 2-person combinatorial game starting with prisoners and shackles on the board such that each shackle has two ends, each end is shackled to one prisoner and the number of shackles that are shackled to each prisoner is at least one. Players alternately play a turn by destroying a shackle on the board until all shackles are destroyed. A prisoner is *free* if all shackles that are shackled to him are destroyed, and the player who makes the most prisoners free *wins* and the other *loses*. Both players *draw* if no one wins. The prisoners and the shackles on the board can be considered as vertices and edges of a graph in the plane. This article constructs shadow strategies of playing the game that starts on a $2D$ grid which is a symmetric plane graph. Possible outcomes of the game when the players play using the proposed shadow strategies are provided.

MSC: 05C57; 91A05; 91A43; 91A46

Keywords: Unshackle game; combinatorial game; impartial game; 2-person game; game on graph; symmetric plane graph; shadow strategy; winning strategy

1. Introduction

Let us introduce a new game called *Unshackle game* which is an impartial combinatorial game, the definition of the impartial combinatorial game was introduced in [1, 2]. There are two players in the game called *Player I* and *Player II*, and they are the *opponents* of each other. The game starts with a finite set of *prisoners* and a finite set of *shackles* on the board such that each shackle has exactly two *ends*, each end is shackled to one prisoner and the number of shackles that are shackled to each prisoner is at least one. Players alternately play a *turn* by destroying a shackle on the board. A prisoner is *free* if all shackles that are shackled to him are destroyed. The game ends when all prisoners on the board are free, i.e., all shackles are destroyed, and the player who makes the most prisoners free *wins* and the other *loses*. However, both players *draw* if no one wins.

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The prisoners and the shackles on the board can be considered as vertices (or points) and edges (or arcs) of a finite graph in the Euclidean plane. The starting graph is called the *initial graph*. Then, the initial graph contains no isolated vertices. However, the initial graph is not necessary simple, planar or connected, i.e., it may contain loops, multiple edges, crossing of two edges or more than one component. Each player's turn can be defined as a removal of an edge and the number of isolated vertices (free prisoners) that each player makes can be counted as *score points* of that player such that both players have 0 score points when the game starts. When all edges are removed, the player who has the most score points wins.

The Unshackle game is similar to Strings-and-Coins game introduced in [3], but their rules are a bit different. For example, in the Strings-and-Coins game, a player who gets score points can remove more than one edge in one turn. For those who interested in the related results about the Strings-and-Coins game can see in [4] and references therein. This article is interested in the Unshackle game that starts on a 2-dimensional ($2D$) grid which is defined in Section 2, a strategy and all outcomes when the players play using the strategy are given in Section 3, and more strategies are given and some outcomes are improved in Section 4. Finally, conclusion and discussion are provided in the last section.

2. Preliminaries

The definitions of graphs are various in several sources. For convenience, definitions of all terminologies involving graphs are given as follows.

Definition 2.1. A *graph* G consists of a *vertex set*, denoted by $V(G)$, and an *edge set*, denoted by $E(G)$, such that $V(G)$ is a non-empty countable set of points and $E(G)$ is a countable set of arcs joining two vertices which are not necessary distinct called *endpoints*. The *degree* of a vertex v of a graph G, denoted by $deg_G(v)$, is the number of times that v occurs in endpoints of all edges of G. A vertex v of a graph G is *isolated* if $deg_G(v) = 0$. Let I be the set of isolated vertices of a graph G, $G - I$ denote a graph such that $V(G - I) = V(G) \setminus I$ and $E(G - I) = E(G)$.

Definition 2.2. A *plane graph* is a graph in the Euclidean plane $\mathbb{R}^2$ such that every two edges are not crossing, i.e., every two edges contain no common points which are not their endpoints.

Definition 2.3. A $2D$ *grid* or an $m \times n$ *grid* is a finite plane graph such that the vertex set is $\{1, 2, 3, \dots, m\} \times \{1, 2, 3, \dots, n\}$ where m and n are positive integers and the edge set is the set of line segments joining two vertices with the Euclidean distance 1.

Note that the Euclidean distance between two points $p(x_1, y_1)$ and $q(x_2, y_2)$ in $\mathbb{R}^2$, denoted by $d_u(p, q)$, is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

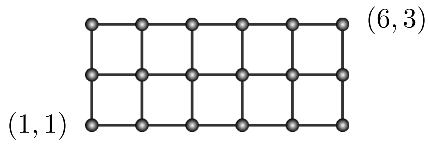


Figure 1. A $2D$ grid or a 6×3 grid

Figure 1 shows an example of $2D$ grid. Clearly, for each $m \times n$ grid G, $|V(G)| = mn$ and $|E(G)| = 2mn - m - n$.

3. Strategy and Outcomes

A *strategy* of playing a game is a plan constructed for turns of a player. An *outcome* of a game is a result, which is winning, drawing or losing, when a player plays using the strategy.

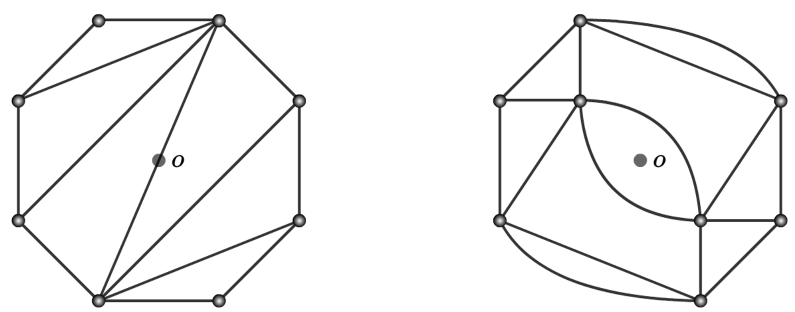
In this section, we define a point symmetric plane graph and construct a point reflection shadow strategy which is a strategy of playing an Unshackle game that starts on a point symmetric plane graph. Although we are interested in a game that starts on a $2D$ grid, we first analyze the strategy of playing a game that starts on a point symmetric plane graph because it is easier to use some terminologies which are defined on the point symmetric plane graph. Next, we find outcomes of all games such that each game starts on a point symmetric plane graph. After that, we apply our obtained outcomes to the game that starts on an $m \times n$ grid for all positive integers m and n.

Definition 3.1. A plane graph G is *point symmetric* if there is a point o in $\mathbb{R}^2$ called the *central point of symmetry*, satisfying the following conditions.

(1) For each $v \in V(G)$, there is a vertex $v' \in V(G)$ such that $d_u(v,o) = d_u(v',o) := d$ and $d_u(v',v) = 2d$ for some $d \geq 0$, the vertex v' is called the *shadow* of v under the reflection through the central point of symmetry o, denoted by $\partial_o v$.

(2) For each $e \in E(G)$, there is an edge $e' \in E(G)$ such that, for each point $p \in \mathbb{R}^2$ lying on e, there is a point $p' \in \mathbb{R}^2$ lying on e' such that $d_u(p,o) = d_u(p',o) := k$ and $d_u(p',p) = 2k$ for some $k \geq 0$, the edge e' is called the *shadow* of e under the reflection through the central point of symmetry o, denoted by $\partial_o e$.

A point symmetric plane graph is invariant under the reflection through the central point of symmetry. The central point of symmetry of a point symmetric plane graph is unique and it is not necessary a vertex of the graph. Clearly, each vertex or edge of a point symmetric plane graph has a unique shadow under the reflection through the central point of symmetry. Figure 2 shows examples of point symmetric plane graphs with their central points of symmetry.



(A) (B)

Figure 2. Point symmetric plane graphs with the central points of symmetry

For each Unshackle game, let G_0 denote the initial graph, and let G_i denote a graph changed by a turn from a graph G_{i-1} for all $i \in \{1, 2, 3, \ldots, t\}$ where t is the number of all turns in the game. By the rule of the game introduced in the first section, G_0 contains no isolated vertices and $t = |E(G_0)|$.

Strategy 1. (Point Reflection Shadow Strategy) Assume that G_0 is a point symmetric plane graph. Then, a player plays each turn according to the following plan.

(1) If G_0 contains an edge e_0 through the central point of symmetry which is not its endpoint, then the player has to remove e_0 from G_0.

(2) For all $i \in \{1, 2, 3, \ldots, t-1\}$, if the opponent removes some edge e from G_{i-1}, then the player has to remove the shadow of e from G_i.

A strategy of playing a game is called a *winning strategy* for a player who plays according to the plan of the strategy if that player wins no matter how the opponent plays, and the strategy is called a *drawing strategy* for a player who plays according to the plan of the strategy if that player draws in at least one case and wins in the other cases.

Theorem 3.2. *For playing an Unshackle game that starts on a point symmetric plane graph containing no edges through the central point of symmetry o,*

(1) if $o \notin V(G_0)$, then Strategy 1 is a drawing strategy for Player II;

(2) if $o \in V(G_0)$, then Strategy 1 is a winning strategy for Player II.

Proof. Assume that G_0 is a point symmetric plane graph containing no edges through the central point of symmetry o. Clearly, each edge of G_0 is distinct from its shadow. Suppose that Player II plays according to Strategy 1. Then, for each edge $e \in E(G_0)$, Player II removes $\partial_o e$ from G_i when Player I removes e from G_{i-1}; $i \in \{1, 2, 3, \ldots, t-1\}$.

Case 1. G_0 contains no o. Clearly, each vertex of G_0 is distinct from its shadow. By Definition 3.1, v is an endpoint of e if and only if $\partial_o v$ is an endpoint of $\partial_o e$. Then, for each vertex $v \in V(G_0)$, the degree of $\partial_o v$ is decreased by 1 in a Player II's turn from G_i when the degree of v is decreased by 1 in a Player I's turn from G_{i-1}. Then, Player II gets a score point from $\partial_o v$ in the turn from G_i when Player I gets a score point from v in the turn from G_{i-1}. Hence, the number of score points of both players are equal, i.e., both players draw.

Case 2. G_0 contains o. Then, $V(G_0) \setminus \{o\}$ can be considered similarly to Case 1. It is enough to consider only $o \in V(G_0)$. By Definition 3.1, o is an endpoint of e if and only if $o = \partial_o o$ is an endpoint of $\partial_o e$, i.e., $deg_{G_0}(o)$ is even. Then, the degree of o is decreased by 1 in a Player II's turn from G_i when the degree of o is decreased by 1 in a Player I's turn from G_{i-1}. Then, Player II gets a score point from o. Hence, the number of score points of Player II is greater than Player I by 1, i.e., Player II wins. ∎

Theorem 3.3. *For playing an Unshackle game that starts on a point symmetric plane graph containing an edge e_0 through the central point of symmetry o,*

(1) if the degree of the endpoints of e_0 of G_0 are both at least 2, then Strategy 1 is a drawing strategy for Player I;

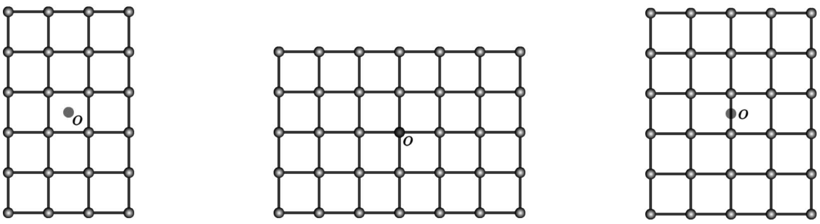
(2) if the degree of the endpoints of e_0 of G_0 are both 1, then Strategy 1 is a winning strategy for Player I.

Proof. Assume that G_0 is a point symmetric plane graph containing an edge e_0 through the central point of symmetry o. Suppose that Player I plays according to Strategy 1. Then, Player I removes e_0 from G_0.

Case 1. The degree of the endpoints of e_0 of G_0 are both at least 2. Then, G_1 contains no isolated vertices. Clearly, G_1 is a point symmetric plane graph containing no edges through o and $o \notin V(G_1)$. Then, G_1 can be considered as the initial graph of a new game such that the first turn is Player II's. By Theorem 3.2 (1), both players draw in the game that starts on G_1. Consequently, both players draw.

Case 2. The degree of the endpoints v_1 and v_2 of e_0 of G_0 are both 1. Then, Player I gets 2 score points from v_1 and v_2 in the turn from G_0. Clearly, $G_1 - \{v_1, v_2\}$ is a point symmetric plane graph containing no edges through o and $o \notin V(G_1 - \{v_1, v_2\})$. Similar to Case 1, both players draw in the game that starts on $G_1 - \{v_1, v_2\}$. Consequently, Player I wins. ∎

Strategy 1 is a strategy of playing an Unshackle game that starts on a point symmetric plane graph, and Theorem 3.2 and Theorem 3.3 show outcomes of all games such that each game starts on a point symmetric plane graph. Clearly, an $m \times n$ grid is a point symmetric plane graph with the central point of symmetry $(\frac{m+1}{2}, \frac{n+1}{2})$. Then, terminologies in Definition 3.1 can be used with the $m \times n$ grid, and Theorem 3.2 and Theorem 3.3 can be applied to the game that starts on an $m \times n$ grid for all positive integers m and n. Figure 3 shows examples of $m \times n$ grids with their central points of symmetry.



(A) A 4×6 grid (B) A 7×5 grid (C) A 5×6 grid

FIGURE 3. $m \times n$ grids with the central points of symmetry

Theorem 3.4. *For playing an Unshackle game that starts on an $m \times n$ grid which is neither a 1×2 grid nor a 2×1 grid,*

(1) if m and n are both even, Strategy 1 is a drawing strategy for Player II;

(2) if m and n are both odd, Strategy 1 is a winning strategy for Player II;

(3) if m and n have different parities, Strategy 1 is a drawing strategy for Player I.

Proof. Assume that G_0 is an $m \times n$ grid which is neither a 1×2 grid nor a 2×1 grid. Then, G_0 is a point symmetric plane graph with the central point of symmetry o.

Case 1. m and n are both even. Then, $|V(G_0)|$ and $|E(G_0)|$ are both even. Clearly, $o \notin V(G_0)$ and G_0 contains no edges through o. By Theorem 3.2 (1), Strategy 1 is a drawing strategy for Player II.

Case 2. m and n are both odd. Then, $|V(G_0)|$ is odd and $|E(G_0)|$ is even. Clearly, $o \in V(G_0)$ and G_0 contains no edges through o. By Theorem 3.2 (2), Strategy 1 is a winning strategy for Player II.

Case 3. m and n have different parities. Then, $|V(G_0)|$ is even and $|E(G_0)|$ is odd. Clearly, G_0 contains an edge e_0 through o and the degree of the endpoints of e_0 of G_0 are both at least 2 since G_0 is neither a 1×2 grid nor a 2×1 grid. By Theorem 3.3 (1), Strategy 1 is a drawing strategy for Player I. ∎

Note that an Unshackle game that starts on either a 1×2 grid or a 2×1 grid is *trivial* because it has only one turn which is Player I's and the outcome is Player I wins. However, the initial graph and the outcome of the game that starts on either a 1×2 grid or a 2×1 grid satisfies Theorem 3.3 (2).

Table 1 shows the outcomes of the games such that each game starts on an $m \times n$ grid, when players play by using Strategy 1, that are proved in Theorem 3.4.

TABLE 1. Outcomes of Unshackle games on an $m \times n$ grid with Strategy 1

$m \backslash n$	1	2	3	4	5	6	7	8	9	10	$\cdots$
1		I_1	II_1	I_0	II_1	I_0	II_1	I_0	II_1	I_0	$\cdots$
2	I_1	II_0	I_0	II_0	I_0	II_0	I_0	II_0	I_0	II_0	$\cdots$
3	II_1	I_0	II_1	I_0	II_1	I_0	II_1	I_0	II_1	I_0	$\cdots$
4	I_0	II_0	I_0	II_0	I_0	II_0	I_0	II_0	I_0	II_0	$\cdots$
5	II_1	I_0	II_1	I_0	II_1	I_0	II_1	I_0	II_1	I_0	$\cdots$
6	I_0	II_0	I_0	II_0	I_0	II_0	I_0	II_0	I_0	II_0	$\cdots$
7	II_1	I_0	II_1	I_0	II_1	I_0	II_1	I_0	II_1	I_0	$\cdots$
8	I_0	II_0	I_0	II_0	I_0	II_0	I_0	II_0	I_0	II_0	$\cdots$
9	II_1	I_0	II_1	I_0	II_1	I_0	II_1	I_0	II_1	I_0	$\cdots$
10	I_0	II_0	I_0	II_0	I_0	II_0	I_0	II_0	I_0	II_0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

I_1 : Strategy 1 is a winning strategy for Player I.
I_0 : Strategy 1 is a drawing strategy for Player I.
II_1 : Strategy 1 is a winning strategy for Player II.
II_0 : Strategy 1 is a drawing strategy for Player II.

4. Improving Outcomes

For playing a non-trivial Unshackle game that starts on an $m \times n$ grid, Strategy 1 is a winning strategy for Player II in the case that m and n are both odd, by Theorem 3.4. Although Strategy 1 is not a winning strategy for some player in the case that m or n is even, there may be winning strategies in these cases.

First, we consider a non-trivial Unshackle game that starts on either a $1 \times n$ grid or an $n \times 1$ grid. Strategy 1 is a winning strategy for Player II in the case that n is odd, and it is a drawing strategy for Player I in the case that $n \geq 4$ is even. We improve the outcomes by showing that there is a winning strategy for Player I in the case that $n \geq 4$ is even.

Theorem 4.1. *For playing an Unshackle game that starts on either a $1 \times n$ grid or an $n \times 1$ grid where $n \geq 4$ is even, there is a winning strategy for Player I.*

Proof. It is enough to consider only the game that starts on a $1 \times n$ grid where $n \geq 4$ is even. Assume that G_0 is a $1 \times n$ grid where $n \geq 4$ is even. Suppose that Player I removes an edge joining $(1, n-1)$ and $(1, n)$ from G_0. Then, Player I gets a score point from $(1, n)$. Clearly, $G_1 - \{(1, n)\}$ is a $1 \times (n-1)$ grid where $n-1$ is odd. Then, $G_1 - \{(1, n)\}$ can be considered as the initial graph of a new game such that the first turn is Player II's. Suppose that Player I plays according to Strategy 1 in the game that starts on $G_1 - \{(1, n)\}$. By Theorem 3.4 (2), Player I wins in the game that starts on $G_1 - \{(1, n)\}$. Consequently, Player I wins. ∎

Next, to improve outcomes of some other cases, we need to introduce a linear symmetric plane graph and construct a linear reflection shadow strategy which is a strategy of playing an Unshackle game that starts on a linear symmetric plane graph. Then, we find outcomes of some games such that each game starts on a linear symmetric plane graph. After that, we apply our obtained outcomes to the game that starts on an $m \times n$ grid for some positive integers m and n.

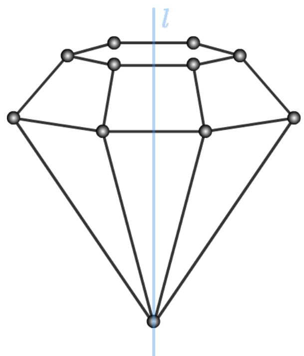
Definition 4.2. A plane graph G is *linear symmetric* if there is a straight line l in $\mathbb{R}^2$ called a *line of symmetry*, satisfying the following conditions.

(1) For each $v \in V(G)$, there is a vertex $v' \in V(G)$ such that $d_u(v,l) = d_u(v',l) := d$ and $d_u(v',v) = 2d$ for some $d \geq 0$, the vertex v' is called the *shadow* of v under the reflection across the line of symmetry l, denoted by $\partial_l v$.

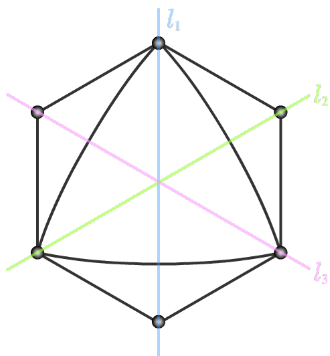
(2) For each $e \in E(G)$, there is an edge $e' \in E(G)$ such that, for each point $p \in \mathbb{R}^2$ lying on e, there is a point $p' \in \mathbb{R}^2$ lying on e' such that $d_u(p,l) = d_u(p',l) := k$ and $d_u(p',p) = 2k$ for some $k \geq 0$, the edge e' is called the *shadow* of e under the reflection across the line of symmetry l, denoted by $\partial_l e$.

Note that the Euclidean distance between a point $p(x,y)$ and a straight line l in $\mathbb{R}^2$, denoted by $d_u(p,l)$, is $min\{d_u(p,q)|q$ lies on $l\}$.

A linear symmetric plane graph is invariant under the reflection across a line of symmetry. It is the same meaning as a symmetric plane graph introduced in [5]. The number of lines of symmetry of a linear symmetric plane graph can be more than one. Clearly, each vertex or edge of a linear symmetric plane graph has a unique shadow under the reflection across a line of symmetry. Figure 4 shows examples of linear symmetric plane graphs with their lines of symmetry.



(A)



(B)

FIGURE 4. Linear symmetric plane graphs with the lines of symmetry

Strategy 2. (Linear Reflection Shadow Strategy) Assume that G_0 is a linear symmetric plane graph and l is a line of symmetry such that G_0 contains at most one edge passing across or lying on l. Then, a player plays each turn according to the following plan.

(1) If G_0 contains an edge e_0 passing across or lying on l, then the player has to remove e_0 from G_0.

(2) For all $i \in \{1,2,3,\ldots,t-1\}$, if the opponent removes some edge e from G_{i-1}, then the player has to remove the shadow of e from G_i.

Theorem 4.3. *For playing an Unshackle game that starts on a linear symmetric plane graph containing no edges passing across or lying on a line of symmetry l,*

(1) if G_0 contains no vertices lying on l, then Strategy 2 is a drawing strategy for Player II;

(2) if G_0 contains at least one vertex lying on l, then Strategy 2 is a winning strategy for Player II.

Proof. Assume that G_0 is a linear symmetric plane graph containing no edges passing across or lying on a line of symmetry l. Then, each edge of G_0 is distinct from its shadow. Suppose that Player II plays according to Strategy 2. Then, for each edge $e \in E(G_0)$, Player II removes $\partial_l e$ from G_i when Player I removes e from G_{i-1}; $i \in \{1, 2, 3, \ldots, t-1\}$.

Case 1. G_0 contains no vertices lying on l. Then, each vertex of G_0 is distinct from its shadow. By Definition 4.2, v is an endpoint of e if and only if $\partial_l v$ is an endpoint of $\partial_l e$. Then, for each vertex $v \in V(G_0)$, the degree of $\partial_l v$ is decreased by 1 in a Player II's turn from G_i when the degree of v is decreased by 1 in a Player I's turn from G_{i-1}. Then, Player II gets a score point from $\partial_l v$ in the turn from G_i when Player I gets a score point from v in the turn from G_{i-1}. Hence, the number of score points of both players are equal, i.e., both players draw.

Case 2. G_0 contains at least one vertex lying on l. Then, vertices which do not lie on l can be considered similarly to Case 1. It is enough to consider only vertices lying on l. Let v_l be a vertex lying on l. By Definition 4.2, v_l is an endpoint of e if and only if $v_l = \partial_l v_l$ is an endpoint of $\partial_l e$. Since G_0 contains no edges lying on l, $deg_{G_0}(v_l)$ is even. Then, the degree of v_l is decreased by 1 in a Player II's turn from G_i when the degree of v_l is decreased by 1 in a Player I's turn from G_{i-1}. Then, Player II gets a score point from v_l. Hence, the number of score points of Player II is greater than Player I by the number of vertices lying on l, i.e., Player II wins. ∎

Theorem 4.4. *For playing an Unshackle game that starts on a linear symmetric plane graph containing exactly one edge e_0 passing across or lying on a line of symmetry l,*

(1) if the degree of the endpoints of e_0 of G_0 are both at least 2 and G_0 contains no vertices lying on l, then Strategy 2 is a drawing strategy for Player I;

(2) if the degree of at least one of the endpoints of e_0 of G_0 is 1 or G_0 contains at least one vertex lying on l, then Strategy 2 is a winning strategy for Player I.

Proof. Assume that G_0 is a linear symmetric plane graph containing exactly one edge e_0 passing across or lying on a line of symmetry l. Suppose that Player I plays according to Strategy 2. Then, Player I removes e_0 from G_0.

Case 1. The degree of the endpoints of e_0 of G_0 are both at least 2 and G_0 contains no vertices lying on l. Then, G_1 contains no isolated vertices. Clearly, G_1 is a linear symmetric plane graph containing no edges passing across or lying on l and no vertices lying on l. Then, G_1 can be considered as the initial graph of a new game such that the first turn is Player II's. By Theorem 4.3 (1), both players draw in the game that starts on G_1. Consequently, both players draw.

Case 2. The degree of at least one of the endpoints of e_0 of G_0 is 1. Then, Player I gets at least 1 score point from the endpoints of e_0 in the turn from G_0. Let I be the set of isolated vertices of G_1. Clearly, $G_1 - I$ is a linear symmetric plane graph containing no edges passing across or lying on l. Then, $G_1 - I$ can be considered as the initial graph of a new game such that the first turn is Player II's. By Theorem 4.3, Player I wins or both players draw in the game that starts on $G_1 - I$. Consequently, Player I wins.

Case 3. G_0 contains at least one vertex lying on l. Let I be the set of isolated vertices of G_1.

Case 3.1. $I = \emptyset$. Clearly, G_1 is a linear symmetric plane graph containing no edges passing across or lying on l but at least one vertex lying on l. Then, G_1 can be considered as the initial graph of a new game such that the first turn is Player II's. By Theorem 4.3 (2), Player I wins in the game that starts on G_1. Consequently, Player I wins.

Case 3.2. $I \neq \emptyset$. Then, Player I gets at least 1 score point from vertices in I in the turn from G_0. Clearly, $G_1 - I$ is a linear symmetric plane graph containing no edges passing across or lying on l. Similar to Case 2, Player I wins or both players draw in the game that starts on $G_1 - I$. Consequently, Player I wins. ■

Strategy 2 is a strategy of playing an Unshackle game that starts on a linear symmetric plane graph containing at most one edge passing across or lying on a line of symmetry, and Theorem 4.3 and Theorem 4.4 show outcomes of all games such that each game starts on a linear symmetric plane graph which satisfies the initial conditions of the strategy. Clearly, an $m \times n$ grid is a linear symmetric plane graph with at least two lines of symmetry. Then, terminologies in Definition 4.2 can be used with the $m \times n$ grid, and Theorem 4.3 and Theorem 4.4 can be applied to the game that starts on an $m \times n$ grid containing at most one edge passing across or lying on a line of symmetry.

Theorem 4.5. *For playing an Unshackle game that starts on an $n \times n$ grid where $n \geq 2$, Strategy 2 is a winning strategy for Player II.*

Proof. Assume that G_0 is an $n \times n$ grid where $n \geq 2$. Then, G_0 is a linear symmetric plane graph. Choose a line of symmetry l as $y = x$. Then, G_0 contains no edges passing across or lying on l but n vertices lying on l. By Theorem 4.3 (2), Strategy 2 is a winning strategy for Player II. ■

Figure 5 shows an example of an $n \times n$ grid with a line of symmetry $y = x$.

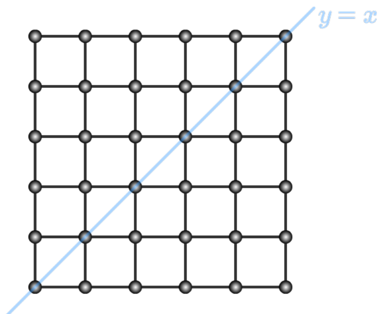


FIGURE 5. A 6×6 grid with a line of symmetry $y = x$

Theorem 4.6. *For playing an Unshackle game that starts on either a $2 \times n$ grid or an $n \times 2$ grid where $n \geq 3$ is odd, Strategy 2 is a winning strategy for Player I.*

Proof. It is enough to consider only the game that starts on a $2 \times n$ grid where $n \geq 3$ is odd. Assume that G_0 is a $2 \times n$ grid where $n \geq 3$ is odd. Then, G_0 is a linear symmetric plane graph. Choose a line of symmetry l as $y = \frac{n+1}{2}$. Then, G_0 contains exactly one edge lying on l and two vertices lying on l. By Theorem 4.4 (2), Strategy 2 is a winning strategy for Player I. ■

Figure 6 shows an example of a $2 \times n$ grid where n is odd with a line of symmetry $y = \frac{n+1}{2}$.

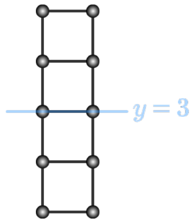


FIGURE 6. A 2×5 grid with a line of symmetry $y = 3$

Unlike an Unshackle game that starts on a $2 \times n$ grid or an $n \times 2$ grid where $n \geq 3$ is odd or an $n \times n$ grid where $n \geq 2$, Strategy 2 can not be used in all other cases because the initial graph contains more than one edge passing across or lying on l for all its lines of symmetry l. However, an Unshackle game that starts on either a 2×4 grid or an 4×2 grid can be considered case-by-case as follows.

Theorem 4.7. *For playing an Unshackle game that starts on either a 2×4 grid or a 4×2 grid, there is a winning strategy for Player II.*

Proof. It is enough to consider only the game that starts on a 2×4 grid. Assume that G_0 is a 2×4 grid. Then, G_0 is a point symmetric plane graph with the central point of symmetry $o(\frac{3}{2}, \frac{5}{2})$ shown in Figure 7.

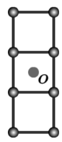


FIGURE 7. A 2×4 grid with the central point of symmetry $o(\frac{3}{2}, \frac{5}{2})$

Let e_1 be an edge joining $(1, 1)$ and $(2, 1)$ and let e_2 be an edge joining $(1, 4)$ and $(2, 4)$.
Case 1. Player I removes $e_0 \in E(G_0) \setminus \{e_1, e_2\}$ from G_0. Suppose that Player II removes $\partial_o e_0$ from G_1. Then, there are 4 cases for G_2 shown in Figure 8.

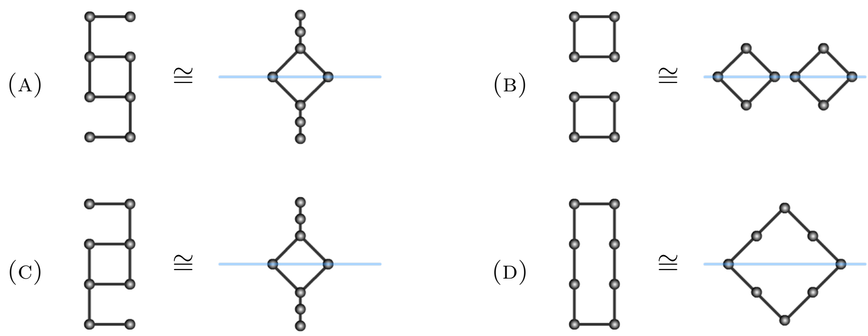


FIGURE 8. Four cases for G_2 in Case 1

Clearly, G_2 in each case has an isomorphic graph which is a linear symmetric plane graph containing no edges passing across or lying on a line of symmetry l and at least two vertices lying on l. Then, G_2 can be considered as the initial graph of a new game such that the first turn is Player I's. Suppose that Player II plays according to Strategy 2 in the game that starts on G_2. By Theorem 4.3 (2), Player II wins in the game that starts on G_2. Consequently, Player II wins.

Case 2. Player I removes e_1 or e_2 from G_0. Up to isomorphism, it is enough to consider only the case that Player I removes e_1. Suppose that Player II removes an edge joining $(1, 1)$ and $(1, 2)$ from G_1. Then, Player II gets a score point from $(1, 1)$ in the turn from G_2, and G_2 is a graph shown in Figure 9.

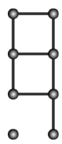


FIGURE 9. A graph G_2 in Case 2

Suppose that Player II plays a turn from G_3 as follows.

(1) If Player I removes an edge joining $(1, 2)$ and $(1, 3)$ from G_2, then Player II removes an edge joining $(2, 2)$ and $(2, 3)$ from G_3.

(2) If Player I removes an edge joining $(1, 3)$ and $(1, 4)$ from G_2, then Player II removes an edge joining $(2, 2)$ and $(2, 3)$ from G_3.

(3) If Player I removes an edge joining $(1, 2)$ and $(2, 2)$ from G_2, then Player II removes an edge joining $(1, 3)$ and $(2, 3)$ from G_3.

(4) If Player I removes an edge joining $(1, 3)$ and $(2, 3)$ from G_2, then Player II removes an edge joining $(2, 1)$ and $(2, 2)$ from G_3.

(5) If Player I removes an edge joining $(1, 4)$ and $(2, 4)$ from G_2, then Player II removes an edge joining $(2, 3)$ and $(2, 4)$ from G_3.

(6) If Player I removes an edge joining $(2, 1)$ and $(2, 2)$ from G_2, then Player II removes an edge joining $(1, 3)$ and $(2, 3)$ from G_3.

(7) If Player I removes an edge joining $(2, 2)$ and $(2, 3)$ from G_2, then Player II removes an edge joining $(1, 3)$ and $(2, 3)$ from G_3.

(8) If Player I removes an edge joining $(2, 3)$ and $(2, 4)$ from G_2, then Player II removes an edge joining $(1, 4)$ and $(2, 4)$ from G_3.

By (1), Player I and Player II get no score points in the turns from G_2 and G_3. Then, the number of score points of Player II is greater than Player I by 1, and $G_4 - \{(1, 1)\}$ is a graph shown in Figure 10 (A).

By (2), Player I and Player II get no score points in the turns from G_2 and G_3. Then, the number of score points of Player II is greater than Player I by 1, and $G_4 - \{(1, 1)\}$ is a graph shown in Figure 10 (B).

By (3), Player I and Player II get no score points in the turns from G_2 and G_3. Then, the number of score points of Player II is greater than Player I by 1, and $G_4 - \{(1, 1)\}$ is a graph shown in Figure 10 (C).

By (4), Player I gets no score points in the turn from G_2 and Player II gets a score point from $(2, 1)$ in the turn from G_3. Then, the number of score points of Player II is greater than Player I by 2, and $G_4 - \{(1, 1), (2, 1)\}$ is a graph shown in Figure 10 (D).

By (5), Player I gets no score points in the turn from G_2 and Player II gets a score point from $(2,4)$ in the turn from G_3. Then, the number of score points of Player II is greater than Player I by 2, and $G_4 - \{(1,1),(2,4)\}$ is a graph shown in Figure 10 (E).

By (6), Player I gets a score point from $(2,1)$ in the turn from G_2 and Player II gets no score points in the turn from G_3. Then, the number of score points of both players are equal, and $G_4 - \{(1,1),(2,1)\}$ is a graph shown in Figure 10 (F).

By (7), Player I and Player II get no score points in the turns from G_2 and G_3. Then, the number of score points of Player II is greater than Player I by 1, and $G_4 - \{(1,1)\}$ is a graph shown in Figure 10 (G).

By (8), Player I gets no score points in the turn from G_2 and Player II gets a score point from $(2,4)$ in the turn from G_3. Then, the number of score points of Player II is greater than Player I by 2, and $G_4 - \{(1,1),(2,4)\}$ is a graph shown in Figure 10 (H).

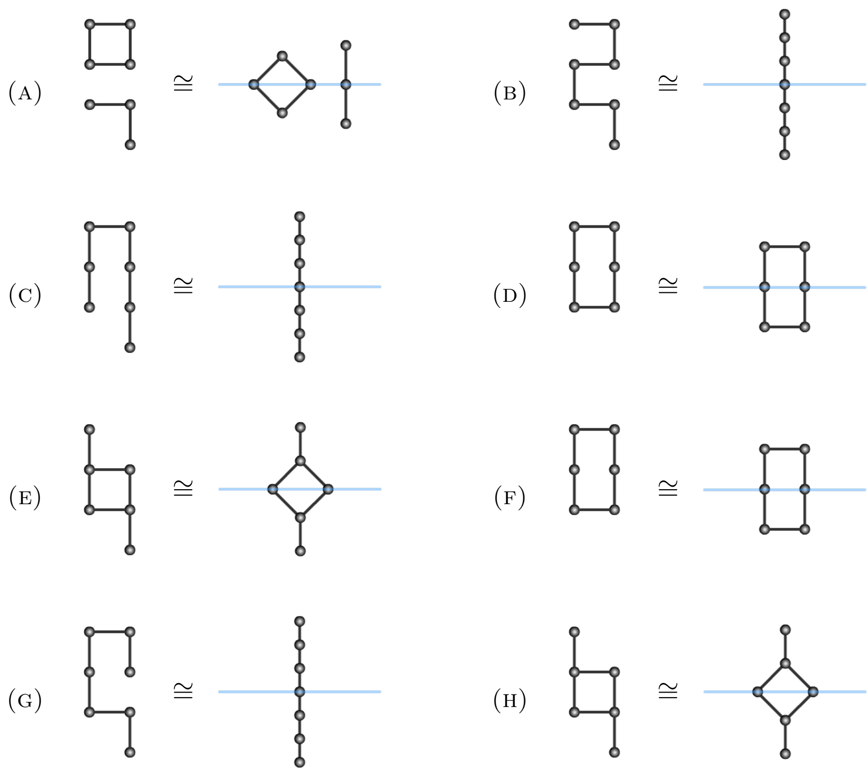


FIGURE 10. Eight cases for $G_4 - I$ in Case 2

Now, the number of score points of Player II is greater than or equal to Player I. Let I be the set of isolated vertices of G_4. Clearly, $G_4 - I$ in each case has an isomorphic graph which is a linear symmetric plane graph containing no edges passing across or lying on a line of symmetry l and at least one vertex lying on l. Then, $G_4 - I$ can be considered as the initial graph of a new game such that the first turn is Player I's. Suppose that Player II plays according to Strategy 2 in the game that starts on $G_4 - I$. By Theorem 4.3 (2), Player II wins in the game that starts on $G_4 - I$. Consequently, Player II wins. ∎

Table 2 shows the improving outcomes of the games such that each game starts on an $m \times n$ grid, that are proved in Theorem 4.1, Theorem 4.5, Theorem 4.6 and Theorem 4.7.

TABLE 2. Improving outcomes of Unshackle games on an $m \times n$ grid

m \ n	1	2	3	4	5	6	7	8	9	10	$\cdots$
1		I_1	II_1	I_1	II_1	I_1	II_1	I_1	II_1	I_1	$\cdots$
2	I_1	II_1	I_1	II_1	I_1	II_0	I_1	II_0	I_1	II_0	$\cdots$
3	II_1	I_1	II_1	I_0	II_1	I_0	II_1	I_0	II_1	I_0	$\cdots$
4	I_1	II_1	I_0	II_1	I_0	II_0	I_0	II_0	I_0	II_0	$\cdots$
5	II_1	I_1	II_1	I_0	II_1	I_0	II_1	I_0	II_1	I_0	$\cdots$
6	I_1	II_0	I_0	II_0	I_0	II_1	I_0	II_0	I_0	II_0	$\cdots$
7	II_1	I_1	II_1	I_0	II_1	I_0	II_1	I_0	II_1	I_0	$\cdots$
8	I_1	II_0	I_0	II_0	I_0	II_0	I_0	II_1	I_0	II_0	$\cdots$
9	II_1	I_1	II_1	I_0	II_1	I_0	II_1	I_0	II_1	I_0	$\cdots$
10	I_1	II_0	I_0	II_0	I_0	II_0	I_0	II_0	I_0	II_1	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

I_1 : There is a winning strategy for Player I.
I_0 : There is a drawing strategy for Player I.
II_1 : There is a winning strategy for Player II.
II_0 : There is a drawing strategy for Player II.

5. Conclusion and Discussion

In conclusion, the outcomes "Player I wins" of the games on a 1×2 grid and a 2×1 grid are trivial, the outcomes "Player I wins" of the games on a $1 \times n$ grid and an $n \times 1$ grid for all even integers $n \geq 4$ are obtained by using the point reflection shadow strategy in all turns except the first turn, the outcomes "Player I wins" of the games on a $2 \times n$ grid and an $n \times 2$ grid for all odd integers $n \geq 3$ are obtained by using the linear reflection shadow strategy, the outcomes "Player II wins" of the games on an $m \times n$ grid for all odd integers $m \neq n$ are obtained by using the point reflection shadow strategy, the outcomes "Player II wins" of the games on an $n \times n$ grid for all even integers $n \geq 2$ are obtained by using the linear reflection shadow strategy, the outcomes "Player II wins" of the games on an $n \times n$ grid for all odd integers $n \geq 3$ are obtained by using either the point reflection shadow strategy or the linear reflection shadow strategy, and the outcomes "Player II wins" of the games on a 2×4 grid and a 4×2 grid are obtained by considering case-by-case and using the point reflection shadow strategy, the linear reflection shadow strategy and others. Is it possible to obtain outcomes "Player II wins" of games on a 2×6 grid and a 6×2 grid by using other strategies? Moreover, is it possible to obtain outcomes "Player II wins" of games on a $2 \times n$ grid and an $n \times 2$ grid for all even integers $n \geq 8$? This article poses the following conjecture.

Conjecture. For playing an Unshackle game that starts on either a $2 \times n$ grid or an $n \times 2$ grid where $n \geq 6$ is even, there is a winning strategy for Player II.

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