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Some Properties of Tensor Products of Ternary Semimodules

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Abstract We introduce tensor products of ternary semimodules over ternary semifields and prove the universal mapping property of the tensor products. Moreover, we introduce the exact sequences of ternary semimodules and flat ternary semimodules. We later provide a condition for preserving the flatness of ternary semimodules with tensor products of ternary semimodule homomorphisms.

MSC: 15A69; 16D10; 16D20; 12K10

Keywords: tensor product; ternary semimodule; ternary semifield

1. Introduction

The subject of a ternary algebra was started by D. H. Lehmer [1]. He studied some concepts of ternary systems called triplexes, which obtained the generalization of abelian groups. Afterward, Los [2] mentioned this algebraic structure studied by Banach and and showed an example of a ternary semigroup which is not a semigroup. In 1971, W. G. Lister [3] studied the abstract structure of a ternary ring which is a ternary product on abelian groups. In 2003, R. Intarawong [4] studied and provided the universal mapping property of tensor products of modules over semifields. T. K. Dutta and S. Kar [5] studied a ternary semiring and gave some properties of ternary semifields. In addition, H. J. M. Al-Thani [6] studied the flat semimodules which constructed by the exact sequences of semimodules and tensor products of semimodule homormorphisms.

The universal mapping property in the branch of modules over rings gives that there exists a unique group homomorphism from $M \otimes_R N$ to a module A which is composed with a bilinear map $M \times N \to M \otimes_R N$ is a bilinear map $M \times N \to A$. The structure of the ternary semimodules also leads us to derive the similar result.

The goal of this research is to investigate some properties of ternary semimodules over ternary semifields. We also study the universal mapping property of tensor products of

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ternary semimodules and investigate some types of ternary semimodules defined by tensor products of homomorphisms on ternary semimodules and exact sequences.

2. Preliminaries

The following familiar definitions and theorems from [7], [8] and [9] regarding the notion of free abelian groups are needed to define tensor products of ternary semimodules over ternary semifields.

A *ternary semifield* K is a system $(K, +, \cdot)$ with a binary operation $(+)$ and a ternary operation $(\cdot)$ satisfying the following conditions for all $u, v, w, s, t \in K$

(1) $(K, +)$ is a commutative semigroup with identity 0,
(2) $(u \cdot v \cdot w) \cdot s \cdot t = u \cdot (v \cdot w \cdot s) \cdot t = u \cdot v \cdot (w \cdot s \cdot t)$,
(3) $(u + v) \cdot w \cdot s = u \cdot w \cdot s + v \cdot w \cdot s$,
(4) $u \cdot (v + w) \cdot s = u \cdot v \cdot s + u \cdot w \cdot s$, and
(5) $u \cdot v \cdot (w + s) = u \cdot v \cdot w + a \cdot v \cdot s$.
(6) $u \cdot v \cdot w = v \cdot u \cdot w = w \cdot v \cdot u = u \cdot w \cdot v$,
(7) $\exists 0 \in K\ \forall u, v \in K,\ 0 + u = u$ and $0 \cdot u \cdot v = u \cdot 0 \cdot v = u \cdot v \cdot 0 = 0$, and
(8) $\forall u \in K \setminus \{0\}\ \exists v \in K\ \forall t \in K,\ u \cdot v \cdot t = v \cdot u \cdot t = t \cdot u \cdot v = t \cdot v \cdot u = t$.
(An element v is called an *inverse* of u. In addition, the inverse of u is unique and u^{-1} denotes the inverse of u.)

For convenience, let uvw denote $u \cdot v \cdot w$ for all $u, v, w \in K$.

Let $(R, +, \cdot)$ be a commutative ring. For a nonempty set S of R, a subring S of R is said to be a *positive cone* if $S \cup (-S) = R$ and $S \cap (-S) = \{0\}$. $-S$ is called a *negative cone* of S.

It is clear that every negative cone of a cone S of R is a ternary semiring. For example, $\mathbb{Z}_0^-$ is a natural example of ternary semiring. Moreover, $\mathbb{Q}_0^-$ and $\mathbb{R}_0^-$ become ternary semifields.

A group $\mathcal{F}$ is a *free abelian group* if $\mathcal{F}$ is an abelian group, and for every nonzero element g of $\mathcal{F}$, there exist unique nonzero integers $\alpha_1, \alpha_2, ..., \alpha_n$ and unique distinct $x_1, x_2, ..., x_n$ in $X \subseteq \mathcal{F}$ such that $g = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$. We sometimes call X a *basis* for $\mathcal{F}$.

For a nonempty set X, let

$$\mathcal{FA}(X) = \{f : X \to \mathbb{Z} \,|\, \exists F \subseteq X \text{ such that } |F| < \infty \text{ and } f(x) = 0 \text{ for all } x \in X \setminus F\}.$$

Define $+$ on $\mathcal{FA}(X)$ by for any $f, g \in \mathcal{FA}(X)$,

$$(f + g)(x) = f(x) + g(x) \quad \text{for all } x \in X.$$

Then $(\mathcal{FA}(X), +)$ is an abelian group. For any $x \in X$, define $f_x : X \to \mathbb{Z}$ by

$$f_x(y) = \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{otherwise.} \end{cases}$$

Then $f_x \in \mathcal{FA}(X)$ for all $x \in X$. A group $(\mathcal{FA}(X), +)$ is a free abelian group on $\{f_x \,|\, x \in X\}$ $(\subseteq \mathcal{FA}(X))$. We sometimes say instead that $\mathcal{FA}(X)$ is a free abelian group on X.

Let K be a ternary semifield. A *left K-ternary semimodule* or *left ternary semimodule over K* is an additive abelian group M together with a function from $K \times K \times M$ into M, defined by $(k_1, k_2, m) \mapsto k_1 k_2 m$ called ternary scalar multiplication, which satisfies the following conditions for all $m, m_1, m_2 \in M$ and $k_1, k_2, k_3, k_4 \in K$,

(1) $k_1k_2(m_1+m_2) = k_1k_2m_1 + k_1k_2m_2$,
(2) $k_1(k_2+k_3)m = k_1k_2m + k_1k_3m$,
(3) $(k_1+k_2)k_3m = k_1k_3m + k_2k_3m$,
(4) $(k_1k_2k_3)k_4m = k_1(k_2k_3k_4)m = k_1k_2(k_3k_4m)$.

A *right K-ternary semimodule* or *right ternary semimodule over K* is defined similarly via a function $M \times K \times K$ into M and satisfies the obvious analogues of (1) - (4).

For convenience, we simply write ${}_KM$ $[M_K]$ as M is a left [right] ternary semimodule over a ternary semifield K.

From now on, unless specified otherwise, "K-ternary semimodule ${}_KM$ $[M_K]$" means "left [right] K-ternary semimodule M". Moreover, "K-ternary semimodule" means "left K-ternary semimodule".

Example 2.1. Let $F[0,1] = \{f | f : [0,1] \to \mathbb{Q}_0^-\}$ with the operations $+$ and $\cdot$, defined by

$$(f+g)(x) = f(x) + g(x) \text{ and } (\alpha \cdot \beta \cdot f)(x) = \alpha \cdot \beta \cdot f(x) \text{ for all } x \in [0,1],$$

where $f, g \in F[0,1]$ and $\alpha, \beta \in \mathbb{Q}_0^-$. Then $F[0,1]$ is a left $\mathbb{Q}_0^-$-ternary semimodule.

Example 2.2. If $n \in \mathbb{N}$ and $M_1,\ M_2,\ \ldots,\ M_n$ are ternary semimodules over a ternary semifield K, then $M_1 \times M_2 \times \cdots \times M_n$ is a ternary semimodules over K under usual componentwise addition and scalar multiplication.

Let M be a left [right] ternary semimodule over a ternary semifield K. A *left* [*right*] *ternary subsemimodule* of M is a subset of M which is, itself, a left [right] ternary semimodule over K with the addition and ternary scalar multiplication of M.

Let K and S be ternary semifields. An additive abelian group M is a (K,S)*-ternary bisemimodule* if M is a left K-ternary semimodule and also a right S-ternary semimodule, and $k_1k_2(ms_1s_2) = (k_1k_2m)s_1s_2$ for all $k_1, k_2 \in K, m \in M$, and $s_1, s_2 \in S$. We write ${}_KM_S$ for a (K,S)-ternary bisemimodule M.

3. Universal Mapping Properties

For given ternary semimodules M and N over a ternary semifield K, it is known from Example 2.2 that $M \times N$ is a ternary semimodule over K. To find another ternary semimodule over K arising from M and N which is different from $M \times N$, the tensor product of M and N is the case. The notion of free abelian groups plays a major role for constructing the tensor product of ternary semimodules over K.

Let M_K and ${}_KN$ be ternary semimodules over a ternary semifield K. For each $m \in M$ and $n \in N$, a function $f_{(m,n)} : M \times N \to \mathbb{Z}$ is defined by

$$f_{(m,n)}(x,y) = \begin{cases} 1, & \text{if } (x,y) = (m,n), \\ 0, & \text{otherwise} \end{cases}$$

for all $x \in M_K$ and $y \in {}_KN$. Then $f_{(m,n)} \in \mathcal{FA}(M \times N)$ and $\mathcal{FA}(M \times N)$ is a free abelian group on a basis $\{f_{(m,n)} \mid m \in M,\ n \in N\}$.

Definition 3.1. Let M_K and ${}_KN$ be ternary semimodules over a ternary semifield K and let $\mathcal{F}$ be the free abelian group on $M \times N$, that is $\mathcal{F} = \mathcal{FA}(M \times N)$. Let L be the subgroup of $\mathcal{F}$ generated by elements of the following forms:

(1) $f_{(m+\tilde{m},n)} - f_{(m,n)} - f_{(\tilde{m},n)}$,
(2) $f_{(m,n+\tilde{n})} - f_{(m,n)} - f_{(m,\tilde{n})}$,
(3) $f_{(m\alpha\beta,n)} - f_{(m,\alpha\beta n)}$

where $\alpha, \beta \in K$, $m, \tilde{m} \in M$ and $n, \tilde{n} \in N$. We call $\mathcal{F}/L$ the *tensor product* of M and N, and denoted by $M \otimes_K N$.

Recall that $\mathcal{F} = \mathcal{FA}(M \times N)$ and $\mathcal{F}/L = \{f + L \mid f \in \mathcal{F}\}$. For any $m \in M$ and $n \in N$, we write $m \otimes n$ for $f_{(m,n)} + L$. The following property can be derived directly from the definition.

Let M_K and ${}_KN$ be ternary semimodules over a ternary semifield K. Then $m\alpha\beta \otimes n = m \otimes \alpha\beta n$ and $m \otimes 0 = 0 \otimes n = 0 \otimes 0 = 0$ for all $\alpha, \beta \in K$, $m \in M$ and $n \in N$.

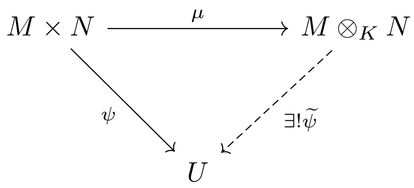
Let M_K and ${}_KN$ be ternary semimodules over a ternary semifield K and A is an additive abelian group. A *middle linear map* (over K) from $M \times N$ to A is a function $\tau : M \times N \to A$ such that for all $m, \tilde{m} \in M$, $n, \tilde{n} \in N$, and $\alpha, \beta \in K$,

(1) $\tau(m + \tilde{m}, n) = \tau(m, n) + \tau(\tilde{m}, n)$,
(2) $\tau(m, n + \tilde{n}) = \tau(m, n) + \tau(m, \tilde{n})$ and
(3) $\tau(m\alpha\beta, n) = \tau(m, \alpha\beta n)$.

Let M_K and ${}_KN$ be ternary semimodules over a ternary semifield K. The map $\mu : M \times N \to M \otimes_K N$ defined by $\mu(m, n) = m \otimes n$ is called the *canonical middle linear map*. The function $\pi : \mathcal{FA}(M \times N) \to \mathcal{FA}(M \times N)/L$ defined by $\pi(x) = x + L$ for all $x \in \mathcal{FA}(M \times N)$ is an epimorphism of groups, which called the *canonical projection.*

By making use of Theorem 5.6 [9, p. 43], we get the following theorem.

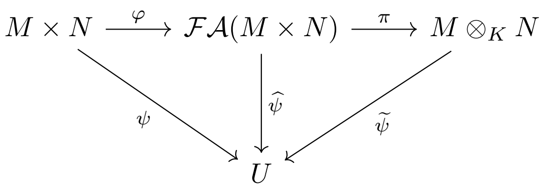
Theorem 3.2. *Let M_K and ${}_KN$ be ternary semimodules over a ternary semifield K and $\mu : M \times N \to M \otimes_K N$ be the canonical middle linear map. For any additive abelian group U over K and any middle linear map $\psi : M \times N \to U$, there exists a unique group homomorphism $\widetilde{\psi} : M \otimes_K N \to U$ such that $\psi = \widetilde{\psi} \circ \mu$, i.e., the following diagram commutes.*



Proof. Let $(U, +)$ be an abelian group. By Theorem 1.1 [9, p. 71], there exists a unique group homomorphism $\widehat{\psi} : \mathcal{FA}(M \times N) \to U$ such that $\psi = \widehat{\psi} \circ \varphi$.

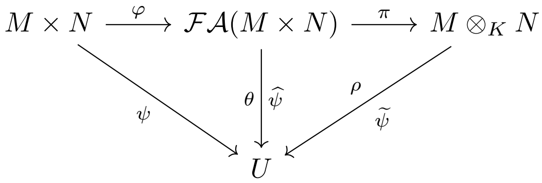
Let L be the subgroup defined in Definition 3.1. Then L is a subgroup of $\ker \widehat{\psi}$ because ψ is a middle linear map and $\psi = \widehat{\psi} \circ \varphi$.

Let $\pi : \mathcal{FA}(M \times N) \to \mathcal{FA}(M \times N)/L$ be the canonical projection. Since L is a subgroup of $\ker \widehat{\psi}$, by Theorem 5.6 [9, p. 43], there exists a unique group homomorphism $\widetilde{\psi} : \mathcal{FA}(M \times N)/L \to U$ such that $\widehat{\psi} = \widetilde{\psi} \circ \pi$. Now we obtain a group homomorphism $\widetilde{\psi} : M \otimes_K N \to U$. Next, we will consider the diagram



For each $m \in M$ and $n \in N$, the canonical middle linear map $\mu : M \times N \to M \otimes_K N$ satisfies $\mu(m, n) = m \otimes n = \varphi(m, n) + L = \pi(\varphi(m, n)) = \pi \circ \varphi(m, n)$. That is, $\mu = \pi \circ \varphi$.

Hence, $\widetilde{\psi}\circ\mu = \widetilde{\psi}\circ(\pi\circ\varphi) = (\widetilde{\psi}\circ\pi)\circ\varphi = \widehat{\psi}\circ\varphi = \psi$. Last, let $\rho : M\otimes_K N \to U$ be a group homomorphism such that $\psi = \rho\circ\mu$ and let $\theta = \rho\circ\pi$. Consider the following diagram.



We have that $\theta\circ\varphi = (\rho\circ\pi)\circ\varphi = \rho\circ(\pi\circ\varphi) = \rho\circ\mu = \psi$. So $\theta = \widehat{\psi}$ because of the uniqueness of $\widehat{\psi}$. Moreover, $\rho\circ\pi = \theta = \widehat{\psi} = \widetilde{\psi}\circ\pi$. By the uniqueness of $\widetilde{\psi}$, we obtain that $\rho = \widetilde{\psi}$. ∎

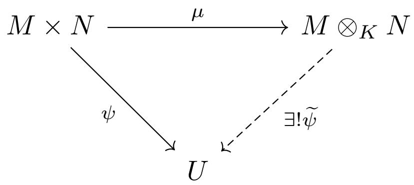
Let M_K, ${}_KN$, and ${}_KU$ be ternary semimodules over a ternary semifield K. A *bilinear map* over K from $M\times N$ to U is a function $T : M\times N \to U$ such that for all $m, \tilde{m}\in M$, $n, \tilde{n}\in N$ and $\alpha, \beta\in K$,

(1) $T(m+\tilde{m}, n) = T(m, n) + T(\tilde{m}, n)$,
(2) $T(m, n+\tilde{n}) = T(m, n) + T(m, \tilde{n})$, and
(3) $T(m\alpha\beta, n) = \alpha\beta T(m, n) = T(m, \alpha\beta n)$.

The map $\mu : M\times N \to M\otimes_K N$ given by $\mu(m, n) = m\otimes n$ is called the *canonical bilinear map* (over a ternary semifield K).

By Theorem 3.2, the following result is derived.

Corollary 3.3. *Let ${}_KM_K$, ${}_KN$ and ${}_KU$ be ternary semimodules over a ternary semifield K and $\mu : M\times N \to M\otimes_K N$ the canonical bilinear map. For any bilinear map $\psi : M\times N \to U$, there exists a unique K-ternary semimodule homomorphism $\widetilde{\psi} : M\otimes_K N \to U$ such that $\psi = \widetilde{\psi}\circ\mu$, i.e., the following diagram commutes.*



Example 3.4. We will apply Corollary 3.3 to show that $\mathbb{Q}_0^- \otimes_{\mathbb{Q}_0^-} F[0,1] \cong F[0,1]$.

Let $\mu : \mathbb{Q}_0^- \times F[0,1] \to \mathbb{Q}_0^- \otimes_{\mathbb{Q}_0^-} F[0,1]$ be a canonical bilinear map. That is, $\mu(a, f) = a\otimes f$ for all $a\in\mathbb{Q}_0^-$ and $f\in F[0,1]$.

Define $\psi : \mathbb{Q}_0^- \times F[0,1] \to F[0,1]$ by $\psi(a, f) = (-1)af$ where $a\in\mathbb{Q}_0^-$ and $f\in F[0,1]$. We can show that ψ is a bilinear map.

By the universal mapping property, there exists a unique ternary homomorphism $\widetilde{\psi} : \mathbb{Q}_0^- \otimes_{\mathbb{Q}_0^-} F[0,1] \to F[0,1]$ such that $\psi = \widetilde{\psi}\circ\mu$. That is, $\widetilde{\psi}(a\otimes f) = \widetilde{\psi}(\mu(a, f)) = \psi(a, f) = (-1)af$ for all $a\in\mathbb{Q}_0^-$ and $f\in F[0,1]$.

We will show that $\widetilde{\psi}$ is an isomorphism. Let $f\in F[0,1]$. We have $-1\otimes f \in \mathbb{Q}_0^- \otimes_{\mathbb{Q}_0^-} F[0,1]$ and $\widetilde{\psi}(-1\otimes f) = (-1)(-1)f = f$. So $\widetilde{\psi}$ is surjective.

Next, we will show that $\ker\widetilde{\psi} = \{0\}$. Let $a\otimes f \in \mathbb{Q}_0^- \otimes_{\mathbb{Q}_0^-} F[0,1]$. Consider $a\otimes f = (-1)(-1)a\otimes f = -1\otimes(-1)af$. Since $F[0,1]$ is a left $\mathbb{Q}_0^-$-ternary semimodule, $a\otimes f = -1\otimes g$ for some $g\in F[0,1]$.

Let $-1 \otimes g \in \ker \widetilde{\psi}$. Then $\widetilde{\psi}(-1 \otimes g) = 0$. Thus $g = (-1)(-1)g = 0$, i.e., $a \otimes f = -1 \otimes 0 = 0$. Hence, $\ker \widetilde{\psi} = \{0\}$, so $\widetilde{\psi}$ is injective. Consequently, $\widetilde{\psi}$ is an isomorphism.

By Theorem 3.2, the following results about the tensor products of two ternary semimodule homomorphism are derived.

Proposition 3.5. *Let M_K, M'_K, ${}_KN$, and ${}_KN'$ be ternary semimodules over a ternary semifield K. If $f : M \to M'$ and $g : N \to N'$ are right and left K-ternary semimodule homomorphisms, respectively, then there exists a unique group homomorphism h from $M \otimes_K N$ into $M' \otimes_K N'$ such that for all $m \in M$ and $n \in N$, $h(m \otimes n) = f(m) \otimes g(n)$.*

The unique group homomorphism h in Proposition 3.5 is denoted by $f \otimes g : M \otimes_K N \to M' \otimes_K N'$.

Theorem 3.6. *Let K and S be ternary semifields, and ${}_SM_K$, ${}_SM'_K$, ${}_KN$, ${}_KN'$ be ternary semimodules as indicated. If $f : M \to M'$ is a right K-ternary semimodule homomorphism and $g : N \to N'$ is a left K-ternary semimodule homomorphism, then $f \otimes g$ is a left S-ternary semimodule homomorphism.*

Proposition 3.7. *Let K be a ternary semifield, and M_K, M'_K, M''_K, ${}_KN$, ${}_KN'$ and ${}_KN''$ be ternary semimodules. If $f : M \to M'$ and $f' : M' \to M''$ are right K-ternary semimodule homomorphisms, $g : N \to N'$ and $g' : N' \to N''$ are left K-ternary semimodule homomorphisms then*

$$(f' \otimes g') \circ (f \otimes g) = (f' \circ f) \otimes (g' \circ g) : M \otimes_K N \to M'' \otimes_K N''$$

is a group homomorphism. If f and g are right and left K-ternary semimodule isomorphisms, respectively, then $f \otimes g$ is a group isomorphism and $(f \otimes g)^{-1} = f^{-1} \otimes g^{-1}$ is also a group isomorphism.

4. Exact Sequences

For the last section, we investigate short exact sequences of ternary semimodules and flat ternary semimodules. By the way, we follow the definitions of short exact sequence and flat semimodules from [6] and [10] but we change the semimodules to the ternary semimodules.

Theorem 4.1. *Let K be a ternary semifield, M be a left K-ternary semimodule and $\{V_i \mid i \in I\}$ be a family of right K-ternary semimodules. If each V_i for all $i \in I$ is M-flat then the direct sum V_i is M-flat.*

Proof. Let id_M be the identity function on a K-ternary semimodule M. Consider the following diagram

$$\begin{array}{ccccc}
V_i \otimes_K N & & & & \left(\bigoplus_{i\in I} V_i\right) \otimes_K N \\
\Big\downarrow{\scriptstyle \mathrm{id}_{V_i} \otimes \mathrm{id}_N} & & & & \Big\downarrow{\scriptstyle \mathrm{id}_{\bigoplus_{i\in I} V_i} \otimes \mathrm{id}_N} \\
V_i \otimes_K M & \underset{\pi_i}{\overset{\iota_i}{\rightleftarrows}} & \bigoplus_{i\in I} (V_i \otimes_K M) & \underset{\psi}{\overset{\psi^{-1}}{\rightleftarrows}} & \left(\bigoplus_{i\in I} V_i\right) \otimes_K M
\end{array} \qquad (*)$$

where $\pi_i : \bigoplus_{i\in I}(V_i \otimes_K M) \to V_i \otimes_K M$ given by $\pi_i(\{v_i \otimes m\}) = v_i \otimes m$ for all $i \in I$ and $\iota_i : V_i \otimes_K M \to \bigoplus_{i\in I}(V_i \otimes_K M)$ given by $\iota_i(v_i \otimes m) = \{v_j \otimes m\}$ where

$$v_j \otimes m = \begin{cases} v_i \otimes m & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j \end{cases}$$

for all $i, j \in I$. There exists a group isomorphism $\psi : \left(\bigoplus_{i\in I} V_i\right) \otimes_K M \to \bigoplus_{i\in I}(V_i \otimes_K M)$ such that $\psi(\{v_i\} \otimes m) = \{v_i \otimes m\}$.

Assume that V_i is M-flat for each $i \in I$. Then

$$0 = (\mathrm{id}_{V_i} \otimes \mathrm{id}_N)(v_i \otimes n) = \mathrm{id}_{V_i}(v_i) \otimes \mathrm{id}_N(n) = v_i \otimes m$$

for some $m \in M$. By diagram (*), we have that

$$0 = \psi^{-1}(\iota_i(v_i \otimes m)) = \psi^{-1}(\{v_i \otimes m\}) = \{v_i\} \otimes m.$$

So $\left(\mathrm{id}_{\bigoplus_{i\in I} V_i} \otimes \mathrm{id}_N\right)(\{v_i\} \otimes n) = \mathrm{id}_{\bigoplus_{i\in I} V_i}(\{v_i\}) \otimes \mathrm{id}_N(n) = \{v_i\} \otimes m = 0.$

That is, $\ker\left(\mathrm{id}_{\bigoplus_{i\in I} V_i} \otimes \mathrm{id}_N\right) = 0$. Hence, $\bigoplus_{i\in I} V_i$ is M-flat. ∎

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