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Closed Knight's Tours on $4 \times n$ Chessboards with Two Squares Removed

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Abstract It is known that the minimum numbers of square to be deleted from the $4 \times n$ chessboard so that it has a closed knight's tour is two. This article determines all positions of those two squares such that after being deleted from the $4 \times n$ chessboards, there exists a closed knight's tour on the deleted chessboard. The result solves Bi, Butler, DeGraaf and Doebel's conjecture which appeared in Knight's tours on boards with odd dimensions, Involve a Journal of Mathematics, 8(4), 2015, 615-627.

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Keywords: legal knight's move; closed knight's tour; open knight's tour; chessboard; Hamiltonian cycle; Hamiltonian path

1. Introduction

Originally, the chessboard is an array of squares arranged in eight rows and eight columns, each square colored black and white alternately. The extension of the chessboard is the $m \times n$ *chessboard* which we denote it by $\mathrm{CB}(m \times n)$. It is an array of squares arranged in m rows and n columns, each square colored black and white alternately. We usually label each square of the $m \times n$ chessboard by (i, j) in the matrix fashion. The interested chess piece is the knight. It can move one square vertically or one square horizontally and then two squares at 90 degrees angle. One classical problem is called a *closed knight's tour (CKT) problem*: Can a knight moves to visit every square on $\mathrm{CB}(m \times n)$ and return back to its starting position? Euler used to construct a CKT on $\mathrm{CB}(8 \times 8)$ and several other CKT on some other sizes of chessboard have been constructed, see [1] for details. Until 1991, Schwenk [2] obtained suffcient and necessary conditions for $\mathrm{CB}(m \times n)$ to admit a CKT.

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Theorem 1.1. [2] *CB($m \times n$) with $m \leq n$ admits a CKT unless one or more of the following conditions hold: (i) m and n are both odd or (ii) $m \in \{1, 2, 4\}$ or (iii) $m = 3$ and $n \in \{4, 6, 8\}$.*

However, for those CB($m \times n$) which do not admit a CKT, one can notice that if we ignore some squares of them, a CKT can be constructed on those deficient CB($m \times n$). In 2009, DeMaio and Hippchen [3] found $T(m, n)$, the minimum number of squares removal from CB($m \times n$), so that a CKT on the deficient CB($m \times n$) exists but they did not determine the exact position of each removal square. In particular, it is stated that (i) for $m, n \geq 3$ are odd and $(m, n) \neq (3, 5)$, $T(m, n) = 1$ and (ii) for $n \geq 3$, $T(4, n) = 2$.

Consequently, in 2013, Miller and Farnsworth [4] determined the exact position of the one square to be removed from CB($3 \times n$) where $n \neq 5$ so that a CKT exists. While, in 2015, Bi et al. [5] determined the exact position of the one square to be removed from CB($m \times n$) where $m, n \geq 3$ are odd and $(m, n) \neq (3, 5)$ so that a CKT exists. In [5], they also try to considered the exact positions of two squares be removed from CB($4 \times n$), where $n \geq 3$. One useful proposition is stated here for ease of reference. The first part was proved by [5] and the second part is from the fact that a knight's move always moves from black to white or white to black square.

Proposition 1.2. [5] *If two squares in CB($4 \times n$) are deleted and a CKT exists for the remaining board, then (i) neither square could come from the middle two rows and (ii) these two squares have different color.*

They also gave the following conjecture.

Conjecture 1 [5] Consider CB($4 \times n$) with $n \geq 7$. For any pair of squares, with one of each parity of color and neither coming from the middle two rows, there is a CKT on CB($4 \times n$) that avoids only these two squares.

Therefore, the aim of this article is to prove the Conjecture 1 in Section 4 and also determine the exact pair of squares removal from CB($4 \times n$) for $3 \leq n \leq 6$ in Section 2. If A is a set of two squares of CB($4 \times n$), then CB($4 \times n$) $- A$ is the deficient board after deleting these two squares. Actually, each square (i, j) of CB($m \times n$) can be regarded as a vertex (i, j) and there is an edge connects between two vertices if there is a knight's move between these two squares. The graph consists of all vertices (i, j) of CB($m \times n$) and all edges constructed by every possible knight's move is called the *knight graph* and it is denoted by $G(m \times n)$. In addition, if A is a set of two vertices of $G(m \times n)$, then we use $G(m \times n) - A$ to represent the knight graph after deleting these two vertices. Therefore, the existence of a CKT on CB($4 \times n$) $- A$ is simply the existence of a Hamiltonian cycle on $G(4 \times n) - A$. The fact about the existence of a Hamiltonian cycle and a Hamiltonian path on a graph G that we use in this article are the following theorem.

Theorem 1.3. *Let $G = (V, E)$ be a graph, S be a proper subset of V and $\omega(G - S)$ be the number of components of $G - S$.*

(a) If $\omega(G - S) > |S|$, then G does not contain any Hamiltonian cycle.
(b) If $\omega(G - S) > |S| + 1$, then G does not contain any Hamiltonian path.

To prove the Conjecture 1, some special open knight's tours (OKTs) on CB($4 \times n$)$-\{(i, j)\}$ for $n \geq 5$ are required. Actually the OKT is the Hamiltonian path on $G(4 \times n) - \{(i, j)\}$. These OKTs are constructed in Section 3. Finally, conclusion and discussion of our future works are given in Section 5.

2. CKTs on CB$(4 \times n) - A$ where $3 \leq n \leq 6$ and $|A| = 2$

For small n such that $3 \leq n \leq 6$, we can only remove some pairs of different color in the first and the fourth rows.

Lemma 2.1. *There exists a CKT on CB*$(4 \times 3) - A$ *if and only if* $A = \{(1,2),(4,2)\}$.

Proof. A CKT on CB$(4 \times 3) - \{(1,2),(4,2)\}$ is shown in Figure 1.

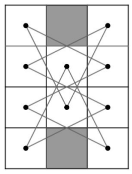

Figure 1. A CKT on CB$(4 \times 3) - \{(1,2),(4,2)\}$

Conversely, there are 4 cases, namely

(i) $A \in \{\{(1,1),(4,1)\},\{(1,3),(4,3)\}\}$,
(ii) $A \in \{\{(1,1),(4,3)\},\{(1,3),(4,1)\}\}$,
(iii) $A \in \{\{(1,1),(1,2)\},\{(1,2),(1,3)\},\{(4,1),(4,2)\},\{(4,2),(4,3)\}\}$, and
(iv) $A \neq \{(1,2),(4,2)\}$ and is not in cases (i) - (iii).

Figure 2 from left to right represents each case (i) - (iii) scenario according to their symmetry and also shows components of $(G(4 \times 3) - A) - S$, where the shaded squares are elements in A and the crossed squares are elements in S.

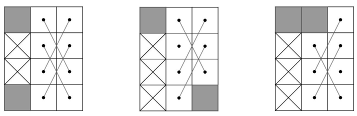

Figure 2. Components of $(G(4 \times 3) - A) - S$ in cases (i) - (iii)

It is clear from Theorem 1.3(a) that the CKT does not exist on CB$(4 \times 3) - A$, where A is in cases (i) - (iii) and Proposition 1.2 also implies that if A is in case (iv), then the CKT does not exist on CB$(4 \times 3) - A$. ∎

Lemma 2.2. *There exists a CKT on CB*$(4 \times 4) - A$ *if and only if* $A \in \{\{(1,1),(1,4)\}, \{(1,1),(4,1)\}, \{(1,4),(4,4)\}, \{(4,1),(4,4)\}\}$.

Proof. According to the symmetry, a CKT on CB$(4 \times 4) - \{(1,1),(1,4)\}$ is shown in Figure 3.

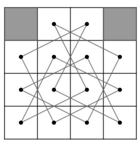

Figure 3. A CKT on CB$(4 \times 4) - \{(1,1),(1,4)\}$

If A is not in $\{\{(1,1),(1,4)\}, \{(1,1),(4,1)\}, \{(1,4),(4,4)\}, \{(4,1),(4,4)\}\}$, then Proposition 1.2 can be applied to conclude the nonexistence of CKTs on $\mathrm{CB}(4 \times 4) - A$.

If one of A is from the first row and another one is from the fourth row have the same parity color, then Proposition 1.2 can be applied immediately.

If one of A is from either the second or the third column, then the rotation make this square becomes on the middle two rows and the Proposition 1.2 can be applied. ∎

Lemma 2.3. *There exists a CKT on CB*$(4 \times 5) - A$ *if and only if* $A \in \{\{(1,1),(1,2)\}, \{(1,4),(1,5)\}, \{(4,1),(4,2)\}, \{(4,4),(4,5)\}, \{(1,1),(4,1)\}, \{(1,5),(4,5)\}, \{(1,1),(4,5)\}, \{(1,5),(4,1)\}\}$.

Proof. First, we consider 3 cases, namely

(i) $A \in \{\{(1,1),(1,2)\}, \{(1,4),(1,5)\}, \{(4,1),(4,2)\}, \{(4,4),(4,5)\}\}$,
(ii) $A \in \{\{(1,1),(4,1)\}, \{(1,5),(4,5)\}\}$, and
(iii) $A \in \{\{(1,1),(4,5)\}, \{(1,5),(4,1)\}\}$.

A CKT on $\mathrm{CB}(4 \times 5) - A$, where $A \in \{\{(1,1),(1,2)\}, \{(1,1),(4,1)\}, \{(1,1),(4,5)\}\}$, is shown in Figure 4. By its symmetry of each of Figure 4, a CKT on $\mathrm{CB}(4 \times 5) - A$ is obtained for each remaining A of each case, respectively.

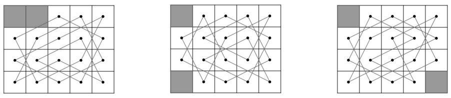

FIGURE 4. CKTs on $\mathrm{CB}(4 \times 5) - A$

Conversely, there are 7 cases, namely

(i) $A \in \{\{(1,2),(1,3)\}, \{(1,3),(1,4)\}, \{(4,2),(4,3)\}, \{(4,3),(4,4)\}\}$,
(ii) $A = \{(1,3),(4,3)\}$,
(iii) $A \in \{\{(1,1),(1,4)\}, \{(1,2),(1,5)\}, \{(4,1),(4,4)\}, \{(4,2),(4,5)\}\}$,
(iv) $A \in \{\{(1,2),(4,4)\}, \{(1,4),(4,2)\}\}$,
(v) $A \in \{\{(1,1),(4,3)\}, \{(1,3),(4,1)\}, \{(1,3),(4,5)\}, \{(1,5),(4,3)\}\}$,
(vi) $A \in \{\{(1,2),(4,2)\}, \{(1,4),(4,4)\}\}$, and
(vii) A is not in $\{\{(1,1),(1,2)\}, \{(1,4),(1,5)\}, \{(4,1),(4,2)\}, \{(4,4),(4,5)\}, \{(1,1),(4,1)\}, \{(1,5),(4,5)\}, \{(1,1),(4,5)\}, \{(1,5),(4,1)\}\}$ and is not in cases (i) - (vi).

Figure 5 from left to right of the first and the second rows represents each case (i) - (vi) scenario according to their symmetry and also shows components of $(G(4 \times 5) - A) - S$, where the shaded squares are elements in A and the crossed squares are elements in S. It is clear from Theorem 1.3(a) that the CKT does not exist on $\mathrm{CB}(4 \times 5) - A$, where A is in cases (i) - (vi).

In the case (vii), either one of A is from the middle two rows or the two squares of A are the same parity color. Then, Proposition 1.2 also implies the nonexistence of CKTs on $\mathrm{CB}(4 \times 5) - A$.

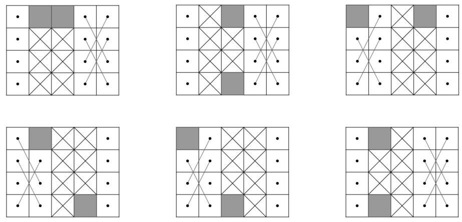

Figure 5. Components of $(G(4 \times 5) - A) - S$ in cases (i) - (vi)

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Lemma 2.4. *There exists a CKT on CB*$(4 \times 6) - A$ *if and only if* $A \in \{\{(1,1),(4,1)\}$, $\{(1,6),(4,6)\}$, $\{(1,1),(1,2)\}$, $\{(1,5),(1,6)\}$, $\{(4,1),(4,2)\}$, $\{(4,5),(4,6)\}$, $\{(1,2),(4,2)\}$, $\{(1,5),(4,5)\}$, $\{(1,1),(1,6)\}$, $\{(4,1),(4,6)\}$, $\{(1,2),(1,5)\}$, $\{(4,2),(4,5)\}$, $\{(1,1),(4,5)\}$, $\{(1,2),(4,6)\}$, $\{(1,5),(4,1)\}$, $\{(1,6),(4,2)\}\}$.

Proof. First, we consider 6 cases, namely

(i) $A \in \{\{(1,1),(4,1)\}, \{(1,6),(4,6)\}\}$,
(ii) $A \in \{\{(1,1),(1,2)\}, \{(1,5),(1,6)\}, \{(4,1),(4,2)\}, \{(4,5),(4,6)\}\}$,
(iii) $A \in \{\{(1,2),(4,2)\}, \{(1,5),(4,5)\}\}$,
(iv) $A \in \{\{(1,1),(1,6)\}, \{(4,1),(4,6)\}\}$,
(v) $A \in \{\{(1,2),(1,5)\}, \{(4,2),(4,5)\}\}$, and
(vi) $A \in \{\{(1,1),(4,5)\}, \{(1,2),(4,6)\}, \{(1,5),(4,1)\}, \{(1,6),(4,2)\}\}$.

A CKT on CB$(4 \times 6) - A$, where $A \in \{\{(1,1),(4,1)\}, \{(1,1),(1,2)\}, \{(1,2),(4,2)\}, \{(1,1),(1,6)\}, \{(1,2),(1,5)\}, \{(1,1),(4,5)\}\}$, is shown in Figure 6. By its symmetry of each of Figure 6, a CKT on CB$(4 \times 6) - A$ is obtained for each remaining A of each case, respectively.

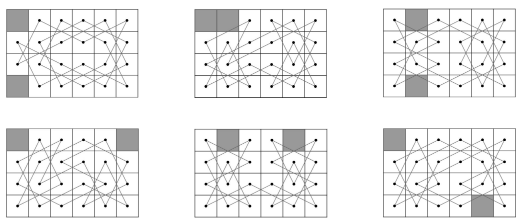

Figure 6. CKTs on CB$(4 \times 6) - A$

Conversely, there are 7 cases, namely

(i) $A \in \{\{(1,1),(4,3)\}, \{(1,3),(4,1)\}, \{(1,4),(4,6)\}, \{(1,6),(4,4)\}\}$,
(ii) $A \in \{\{(1,2),(4,4)\}, \{(1,3),(4,5)\}, \{(1,4),(4,2)\}, \{(1,5),(4,3)\}\}$,
(iii) $A \in \{\{(1,1),(1,4)\}, \{(1,3),(1,6)\}, \{(4,1),(4,4)\}, \{(4,3),(4,6)\}\}$,
(iv) $A \in \{\{(1,3),(4,3)\}, \{(1,4),(4,4)\}\}$,
(v) $A \in \{\{(1,2),(1,3)\}, \{(1,4),(1,5)\}, \{(4,2),(4,3)\}, \{(4,4),(4,5)\}\}$,
(vi) $A \in \{\{(1,3),(1,4)\}, \{(4,3),(4,4)\}\}$, and

(vii) A is not in $\{\{(1,1),(4,1)\}, \{(1,6),(4,6)\}, \{(1,1),(1,2)\}, \{(1,5),(1,6)\}, \{(4,1),(4,2)\}, \{(4,5),(4,6)\}, \{(1,2),(4,2)\}, \{(1,5),(4,5)\}, \{(1,1),(1,6)\}, \{(4,1),(4,6)\}, \{(1,2),(1,5)\}, \{(4,2),(4,5)\}, \{(1,1),(4,5)\}, \{(1,2),(4,6)\}, \{(1,5),(4,1)\}, \{(1,6),(4,2)\}\}$ and is not in cases (i) - (vi).

Figure 7 from left to right of the first and the second rows represents each case (i) - (vi) scenario according to their symmetry and also shows components of $(G(4 \times 6) - A) - S$, where the shaded squares are elements in A and the crossed squares are elements in S. It is clear from Theorem 1.3(a) that the CKT does not exist on $\text{CB}(4 \times 6) - A$, where A is in cases (i) - (vi).

In the case (vii), either one of A is from the middle two rows or the two squares of A are the same parity color. Then, Proposition 1.2 also implies the nonexistence of CKTs on $\text{CB}(4 \times 6) - A$. ∎

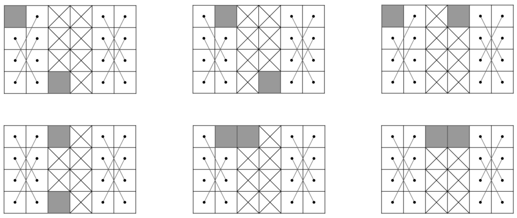

FIGURE 7. Components of $(G(4 \times 6) - A) - S$ in cases (i) - (vi)

3. Existence of some special OKTs on $\text{CB}(4 \times n) - \{(i,j)\}$ where $n \geq 5$

The following lemmas give necessary and sufficient conditions on the existence of special OKTs on $\text{CB}(4 \times n) - \{(i,j)\}$ where $n \geq 5$ and (i,j) is a square on $\text{CB}(4 \times n)$. These OKTs will be used to prove our main result. First, we consider the case where $n \geq 5$ and n is odd.

Lemma 3.1. *Let $n \geq 5$ and n is odd. Then,*

(a) $CB(4 \times n) - \{(i,j)\}$ contains an OKT from $(2,n)$ to $(4,n)$ if and only if ($i = 1$ and $i + j$ is even) or ($i = 4$ and $i + j$ is even).

(b) $CB(4 \times n) - \{(i,j)\}$ contains an OKT from $(1,n)$ to $(3,n)$ if and only if ($i = 1$ and $i + j$ is odd) or ($i = 4$ and $i + j$ is odd).

(c) $CB(4 \times n) - \{(i,j)\}$ contains an OKT from $(1,1)$ to $(3,1)$ if and only if ($i = 1$ and $i + j$ is odd) or ($i = 4$ and $i + j$ is odd).

(d) $CB(4 \times n) - \{(i,j)\}$ contains an OKT from $(2,1)$ to $(4,1)$ if and only if ($i = 1$ and $i + j$ is even) or ($i = 4$ and $i + j$ is even).

Proof. Let $n \geq 5$ and n is odd. We consider $\text{CB}(4 \times n) - \{(i,j)\}$ where (i,j) is a square on $\text{CB}(4 \times n)$.

(a) Assume that ($i = 1$ and $i + j$ is even) or ($i = 4$ and $i + j$ is even). Let $n = a + 4k$ where $k \in \mathbb{N} \cup \{0\}$ and $a \in \{5, 7\}$. We prove by the mathematical induction on k.

First, for $k = 0$, we construct the required OKTs from $(2,a)$ to $(4,a)$ on $\text{CB}(4 \times a) - \{(i,j)\}$ as shown in Figures 8 and 9.

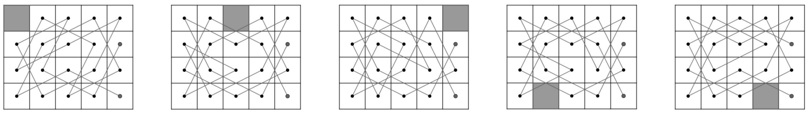

FIGURE 8. The required OKTs on $\mathrm{CB}(4 \times 5) - \{(i, j)\}$ where ($i = 1$ and $i + j$ is even) or ($i = 4$ and $i + j$ is even)

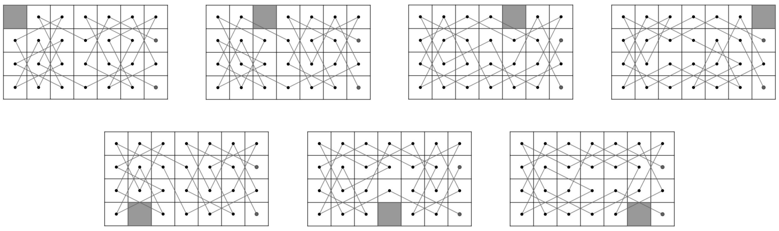

FIGURE 9. The required OKTs on $\mathrm{CB}(4 \times 7) - \{(i, j)\}$ where ($i = 1$ and $i + j$ is even) or ($i = 4$ and $i + j$ is even)

Next, let $k \geq 0$ be an integer. Assume that $\mathrm{CB}(4 \times (a + 4k)) - \{(i, j)\}$ contains an OKT from $(2, a+4k)$ to $(4, a+4k)$. Consider two cases of $\mathrm{CB}(4 \times (a+4(k+1))) - \{(i, j)\}$.

Case 1: $1 \leq j \leq a + 4k$. We separate $\mathrm{CB}(4 \times (a + 4(k + 1))) - \{(i, j)\}$ into two sub-boards, $\mathrm{CB}(4 \times (a+4k)) - \{(i, j)\}$ and $\mathrm{CB}(4 \times 4)$ as shown in Figure 10 with $(i, j) = (1, 1)$.

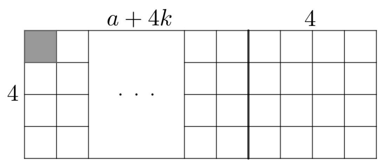

FIGURE 10. $\mathrm{CB}(4 \times (a + 4(k + 1))) - \{(1, 1)\}$ with two sub-boards

By the induction hypothesis, the sub-board $\mathrm{CB}(4 \times (a + 4k)) - \{(i, j)\}$ contains an OKT from $(2, a+4k)$ to $(4, a+4k)$. For the sub-board $\mathrm{CB}(4 \times 4)$, we construct two paths P_1 from $(2, 1)$ to $(4, 4)$ and P_2 from $(4, 1)$ to $(2, 4)$ as shown in Figure 11.

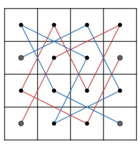

FIGURE 11. Two paths P_1 and P_2 on $\mathrm{CB}(4 \times 4)$

Then, we construct the required OKT on $\mathrm{CB}(4 \times (a + 4(k + 1))) - \{(i, j)\}$ by joining $(2, a + 4k)$ and $(4, a + 4k)$ of the OKT on the sub-board $\mathrm{CB}(4 \times (a + 4k)) - \{(i, j)\}$ to

$(4, 1)$ of P_2 and $(2, 1)$ of P_1 on the sub-board CB(4×4), respectively, as shown in Figure 12 with $(i, j) = (1, 1)$.

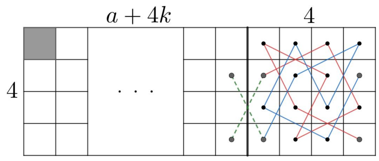

FIGURE 12. The required OKT from $(2, a+4(k+1))$ to $(4, a+4(k+1))$ on CB$(4 \times (a + 4(k + 1))) - \{(1, 1)\}$

Case 2: $a + 4k + 1 \le j \le a + 4(k + 1)$. We separate CB$(4 \times (a + 4(k + 1))) - \{(i, j)\}$ into two sub-boards, CB(4×4) and CB$(4 \times (a + 4k)) - \{(i, j)\}$ as shown in Figure 13 with $(i, j) = (1, a + 4(k + 1))$.

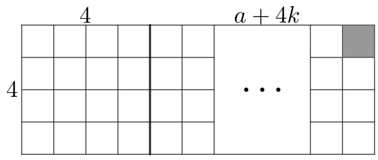

FIGURE 13. CB$(4 \times (a + 4(k + 1))) - \{(1, a + 4(k + 1))\}$ with two sub-boards

For the sub-board CB(4×4), we construct two paths P'_1 from $(1, 4)$ to $(2, 4)$ and P'_2 from $(3, 4)$ to $(4, 4)$ as shown in Figure 14.

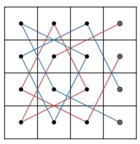

FIGURE 14. Two paths P'_1 and P'_2 on CB(4×4)

By the induction hypothesis, the sub-board CB$(4 \times (a + 4k)) - \{(i, j)\}$ contains an OKT from $(2, a + 4k)$ to $(4, a + 4k)$. Since $(1, 1)$ and $(4, 1)$ have degree 2 in $G(4 \times (a + 4k)) - \{(i, j)\}$, $(1, 1) - (3, 2)$ and $(2, 2) - (4, 1)$ are two edges of the OKT.

Then, we construct the required OKT by the following two steps:

(i) delete $(1, 1) - (3, 2)$ and $(2, 2) - (4, 1)$ of the OKT on the sub-board CB$(4 \times (a + 4k)) - \{(i, j)\}$;

(ii) join $(1, 4)$ and $(2, 4)$ of P'_1 to $(2, 2)$ and $(4, 1)$ of the OKT on the sub-board CB$(4 \times (a + 4k)) - \{(i, j)\}$, respectively and join $(3, 4)$ and $(4, 4)$ of P'_2 to $(1, 1)$ and $(3, 2)$ of the OKT on the sub-board CB$(4 \times (a + 4k)) - \{(i, j)\}$, respectively.

The required OKT is shown in Figure 15 with $(i, j) = (1, a + 4(k + 1))$.

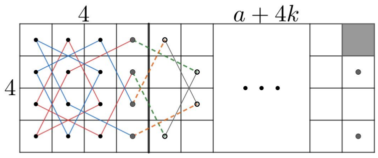

FIGURE 15. The required OKT on $\text{CB}(4 \times (a + 4(k + 1))) - \{(1, 4(k + 1))\}$

Hence, by the mathematical induction, if ($i = 1$ and $i + j$ is even) or ($i = 4$ and $i + j$ is even), then there exist an OKT on $\text{CB}(4 \times n) - \{(i, j)\}$ from $(2, n)$ to $(4, n)$.

Conversely, assume that ($i \neq 1$ or $i + j$ is odd) and ($i \neq 4$ or $i + j$ is odd) and $\text{CB}(4 \times n) - \{(i, j)\}$ contains an OKT from $(2, n)$ to $(4, n)$. Let (i, j) be the black square when $i + j$ is even and the white square when $i + j$ is odd.

If $(i, j) = (2, n)$ or $(i, j) = (4, n)$, then it contradicts with our assumption about the existence of the OKT from $(2, n)$ to $(4, n)$.

If $i + j$ is odd and $(i, j) \notin \{(2, n), (4, n)\}$, then the number of black squares is greater than the number of white squares on $\text{CB}(4 \times n) - \{(i, j)\}$. Since $\text{CB}(4 \times n) - \{(i, j)\}$ contains an OKT from $(2, n)$ to $(4, n)$, $(2, n)$ and $(4, n)$ must be black. However, $2 + n$ and $4 + n$ are odd, then $(2, n)$ and $(4, n)$ are white, a contradiction.

For ($i = 2$ and $i + j$ is even) or ($i = 3$ and $i + j$ is even), let $G_1 = G(4 \times n) - \{(i, j)\}$. Consider $G'_1 = G_1 - \{(2, n)\}$. Let $S = \{(2, s), (3, t) \mid s \text{ is even}, 2 \leq s \leq n - 1, t \text{ is odd and } 1 \leq t \leq n\} - \{(i, j)\}$. Then, $\omega(G'_1 - S) = n + 1 > n = |S| + 1$, see Figure 16 for the case $(i, j) = (3, 1)$ and $n = 9$. By Theorem 1.3(b), we have a contradiction.

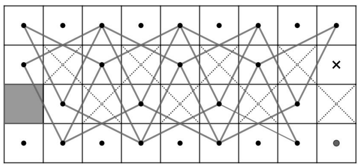

FIGURE 16. Components of $G'_1 - S$ where $(i, j) = (3, 1)$ and $n = 9$

(b) The required OKT can be obtained by horizontally flipping the OKT of $\text{CB}(4 \times n) - \{(i, j)\}$ in (a).

(c) The required OKT can be obtained by rotating 180 degrees of the OKT of $\text{CB}(4 \times n) - \{(i, j)\}$ in (a).

(d) The required OKT can be obtained by rotating 180 degrees of the OKT of $\text{CB}(4 \times n) - \{(i, j)\}$ in (b). ■

Next, we give the existence of the special OKT on $\text{CB}(4 \times n) - \{(i, j)\}$ for $n \geq 6$ and n is even.

Lemma 3.2. *Let $n \geq 6$ and n is even. Then,*

(a) $CB(4 \times n) - \{(i, j)\}$ contains an OKT from $(2, n)$ to $(4, n)$ if and only if ($i = 1$ and $i + j$ is odd) or ($i = 4$ and $i + j$ is odd).

(b) $CB(4 \times n) - \{(i, j)\}$ contains an OKT from $(1, n)$ to $(3, n)$ if and only if ($i = 1$ $i + j$ is even) or ($i = 4$ and $i + j$ is even).

Proof. Let $n \geq 6$ and n is even. We consider $\mathrm{CB}(4 \times n) - \{(i, j)\}$ where (i, j) is a square on $\mathrm{CB}(4 \times n)$.

(a) Assume that ($i = 1$ and $i + j$ is odd) or ($i = 4$ and $i + j$ is odd). Let $n = a + 4k$ where $k \in \mathbb{N} \cup \{0\}$ and $a \in \{6, 8\}$. We prove by the mathematical induction on k.

For $k = 0$, we construct the required OKTs from $(2, a)$ to $(4, a)$ on $\mathrm{CB}(4 \times a) - \{(i, j)\}$ as shown in Figures 17 and 18.

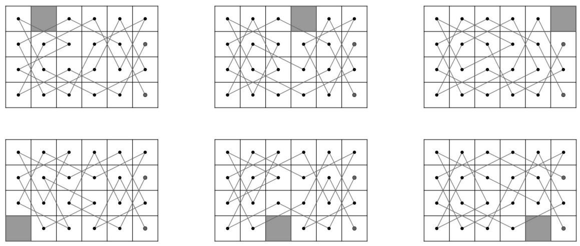

FIGURE 17. The required OKTs on $\mathrm{CB}(4 \times 6) - \{(i, j)\}$ where ($i = 1$ and $i + j$ is odd) or ($i = 4$ and $i + j$ is odd)

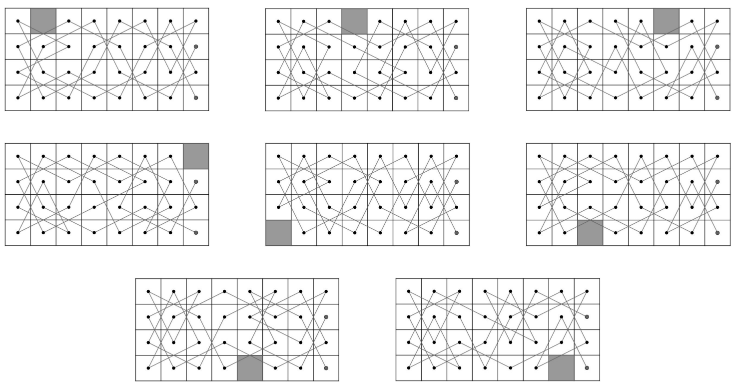

FIGURE 18. The required OKTs on $\mathrm{CB}(4 \times 8) - \{(i, j)\}$ where ($i = 1$ and $i + j$ is odd) or ($i = 4$ and $i + j$ is odd)

Next, let $k \geq 0$ be an integer. Assume that $\mathrm{CB}(4 \times (a + 4k)) - \{(i, j)\}$ contains an OKT from $(2, a+4k)$ to $(4, a+4k)$. Consider two cases of $\mathrm{CB}(4 \times (a+4(k+1))) - \{(i, j)\}$.

Case 1: $1 \leq j \leq a + 4k$. We separate $\mathrm{CB}(4 \times (a + 4(k + 1))) - \{(i, j)\}$ into two sub-boards, $\mathrm{CB}(4 \times (a+4k)) - \{(i, j)\}$ and $\mathrm{CB}(4 \times 4)$ as shown in Figure 19 with $(i, j) = (1, 2)$.

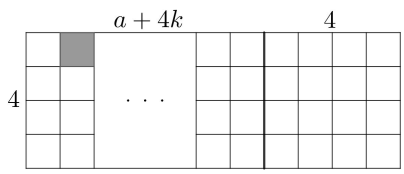

FIGURE 19. $\mathrm{CB}(4 \times (a + 4(k + 1))) - \{(1, 2)\}$ with two sub-boards

By the induction hypothesis, the sub-board $\mathrm{CB}(4\times(a+4k))-\{(i,j)\}$ contains an OKT from $(2,a+4k)$ to $(4,a+4k)$. Then, as shown in Figure 20 for $(i,j)=(1,2)$, we construct the required OKT by joining $(2,a+4k)$ and $(4,a+4k)$ of the OKT on the sub-board $\mathrm{CB}(4\times(a+4k))-\{(i,j)\}$ to $(4,1)$ of P_2 and $(2,1)$ of P_1 on the sub-board $\mathrm{CB}(4\times 4)$ (Figure 11), respectively.

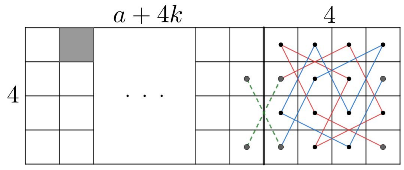

FIGURE 20. The required OKT from $(2,a+4(k+1))$ to $(4,a+4(k+1))$ on $\mathrm{CB}(4\times(a+4(k+1)))-\{(1,2)\}$

Case 2: $a+4k+1\le j\le a+4(k+1)$. We separate $\mathrm{CB}(4\times(a+4(k+1)))-\{(i,j)\}$ into two sub-boards, $\mathrm{CB}(4\times 4)$ and $\mathrm{CB}(4\times(a+4k))-\{(i,j)\}$ as shown in Figure 21 with $(i,j)=(1,a+4(k+1))$.

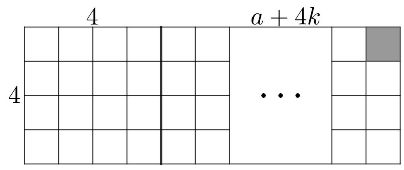

FIGURE 21. $\mathrm{CB}(4\times(a+4(k+1)))-\{(1,a+4(k+1))\}$ with two sub-boards

By the induction hypothesis, the sub-board $\mathrm{CB}(4\times(a+4k))-\{(i,j)\}$ contains an OKT from $(2,a+4k)$ to $(4,a+4k)$. Since $(1,1)$ and $(4,1)$ have degree 2 in $G(4\times(a+4k))-\{(i,j)\}$, $(1,1)-(3,2)$ and $(2,2)-(4,1)$ are two edges of the OKT.

Then, we construct the required OKT by the following two steps:

(i) delete $(1,1)-(3,2)$ and $(2,2)-(4,1)$ of the OKT on the sub-board $\mathrm{CB}(4\times(a+4k))-(i,j)$;

(ii) by using P_1' and P_2' on $\mathrm{CB}(4\times 4)$ shown in Figure 14, we join $(1,4)$ and $(2,4)$ of P_1' on the sub-board $\mathrm{CB}(4\times 4)$ to $(2,2)$ and $(4,1)$ of the sub-board $\mathrm{CB}(4\times(a+4k))-\{(i,j)\}$, respectively and join $(3,4)$ and $(4,4)$ of P_2' on the sub-board $\mathrm{CB}(4\times 4)$ to $(1,1)$ and $(3,2)$ of the sub-board $\mathrm{CB}(4\times(a+4k))-\{(i,j)\}$, respectively.

The required OKT is shown in Figure 22.

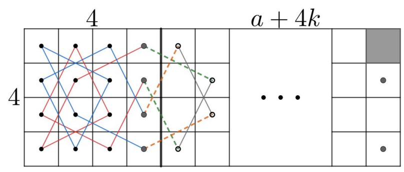

FIGURE 22. The required OKT on $\mathrm{CB}(4\times(a+4(k+1)))-\{(1,4(k+1))\}$

Hence, by the mathematical induction, if ($i = 1$ and $i + j$ is odd) or ($i = 4$ and $i + j$ is odd), then there exist an OKT on $\mathrm{CB}(4 \times n) - \{(i, j)\}$ from $(2, n)$ to $(4, n)$.

Conversely, assume that ($i \neq 1$ or $i + j$ is even) and ($i \neq 4$ or $i + j$ is even) and $\mathrm{CB}(4 \times n) - \{(i, j)\}$ contains an OKT from $(2, n)$ to $(4, n)$.

Let (i, j) be the black square when $i + j$ is even and the white square when $i + j$ is odd.

If $(i, j) = (2, n)$ or $(i, j) = (4, n)$, then it contradicts with our assumption about the existence of the OKT from $(2, n)$ to $(4, n)$.

If $i+j$ is even and $(i, j) \notin \{(2, n), (4, n)\}$, then the number of black squares is less than the number of white squares on $\mathrm{CB}(4 \times n) - \{(i, j)\}$. Since $\mathrm{CB}(4 \times n) - \{(i, j)\}$ contains an OKT from $(2, n)$ to $(4, n)$, $(2, n)$ and $(4, n)$ must be white. Since $2 + n$ and $4 + n$ are even, $(2, n)$ and $(4, n)$ are black, a contradiction.

For ($i = 2$ and $i + j$ is odd) or ($i = 3$ and $i + j$ is odd), let $G_1 = G(4 \times n) - \{(i, j)\}$. Consider $G_1' = G_1 - \{(2, n)\}$. Let $S = \{(2, s), (3, t) \mid s$ is odd, $1 \leq s \leq n - 1, t$ is even and $2 \leq t \leq n\} - \{(i, j)\}$. Then, $\omega(G_1' - S) = n + 1 > n = |S| + 1$, see Figure 23 for the case $(i, j) = (2, 1)$ and $n = 10$. By Theorem 1.3(b), we have a contradiction.

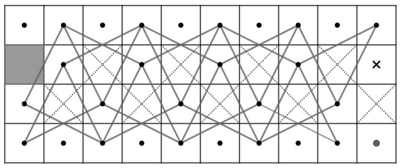

FIGURE 23. Components of $G_1' - S$ where $(i, j) = (2, 1)$ and $n = 10$

(b) The required OKT can be obtained by horizontally flipping the OKT of $\mathrm{CB}(4 \times n) - \{(i, j)\}$ in (a). ■

4. CKTs on $\mathrm{CB}(4 \times n) - A$ where $n \geq 7$ and $|A| = 2$

By mainly using the mathematical induction and using special OKTs constructed in Section 3 in some cases, we can prove our main result which is the Conjecture 1 as follows.

Theorem 4.1. *Consider CB$(4 \times n)$ with $n \geq 7$. For any pair of squares, with one of each parity of color and neither coming from the middle two rows, there is a CKT on the board that avoids only these two squares.*

Proof. Let $n \geq 7$ and

$S_n = \{\{(x, y), (z, w)\} \mid (x, z \in \{1, 4\}, 1 \leq y, w \leq n, (x, y) \neq (z, w))$ and $((x+y$ is odd and $z + w$ is even) or ($x + y$ is even and $z + w$ is odd))$\}$.

Now, we consider $\mathrm{CB}(4 \times n) - A$ with $n \geq 7$ and $A \in S_n$. Let $n = a + 3k$ where $a \in \{7, 8, 9\}$ and $k \in \mathbb{N} \cup \{0\}$. We prove by mathematical induction on k.

First, for $k = 0$, we construct the CKTs on $\mathrm{CB}(4 \times a) - A$ for some $A \in S_a$ as shown in Figures 24, 25 and 26. Note that actually the CKTs on $\mathrm{CB}(4 \times a) - A$ for all $A \in S_a$ can be obtained from the diagrams represented in Figures 24, 25 and 26 according to its symmetry.

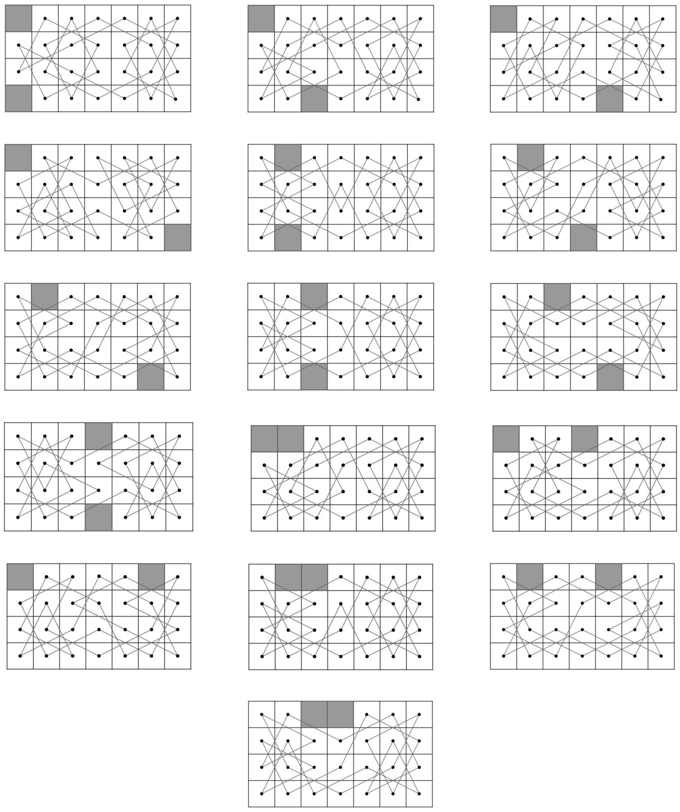

FIGURE 24. CKTs on $\mathrm{CB}(4 \times 7) - A$ for some $A \in S_7$

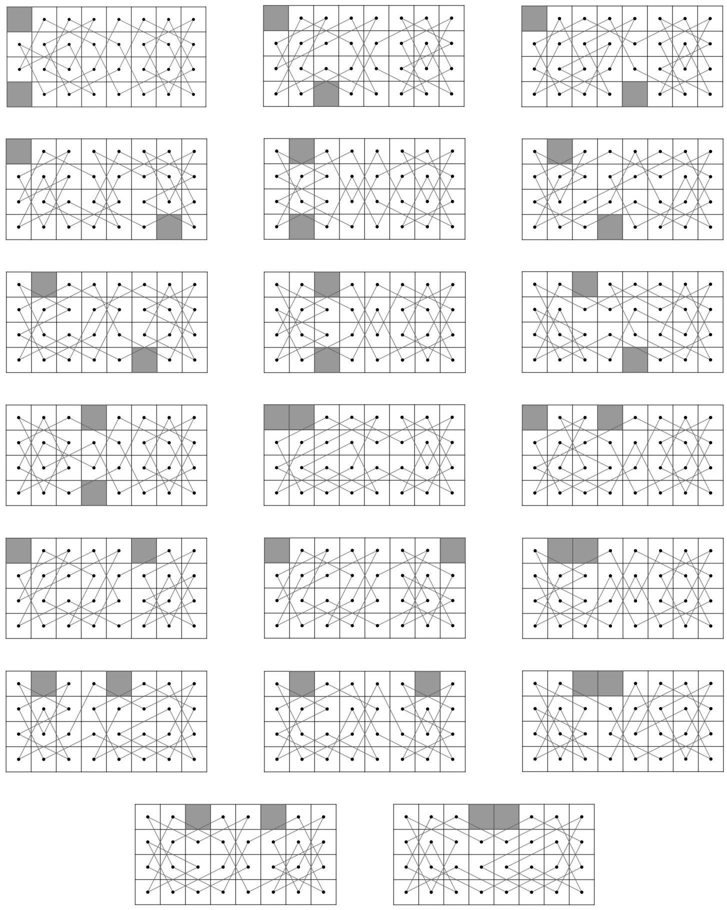

FIGURE 25. CKTs on $\text{CB}(4 \times 8) - A$ for some $A \in S_8$

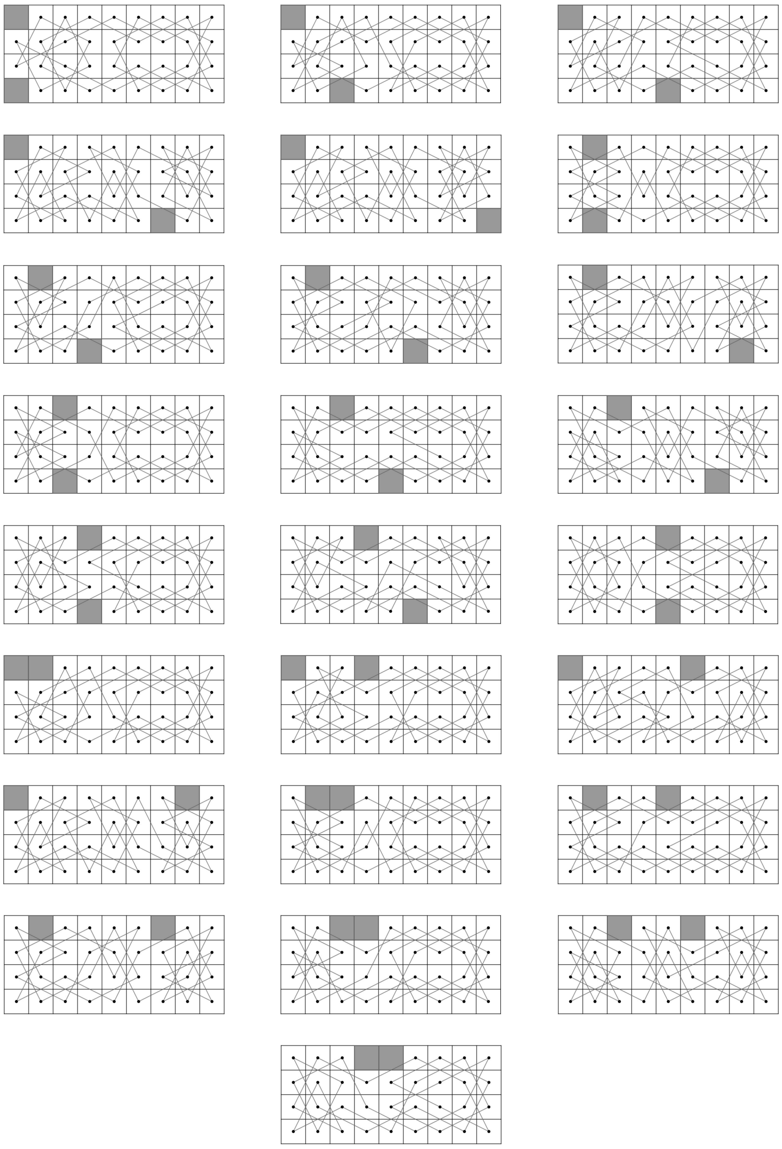

FIGURE 26. CKTs on CB(4×9) $- A$ for some $A \in S_9$

Next, let $k \geq 0$ be an integer. Assume that CB($4 \times (a + 3k)$) $- B$ contains a CKT for all $B \in S_{a+3k}$. Let $A = \{(x, y), (z, w)\} \in S_{a+3(k+1)}$.

Case 1: $1 \leq y, w \leq a + 3k - 2$.

We separate CB($4 \times (a + 3(k + 1))$) $- A$ into two sub-boards, CB($4 \times (a + 3k)$) $- A$ and CB(4×3) as shown in Figure 27 with $A = \{(1, 1), (4, 1)\}$.

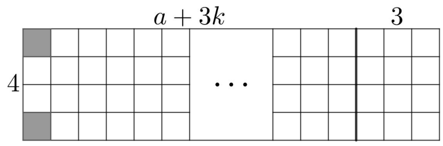

FIGURE 27. $\mathrm{CB}(4 \times (a+3(k+1))) - \{(1,1),(4,1)\}$ with two sub-boards

Since $A \in S_{a+3k}$, by the induction hypothesis, the sub-board $\mathrm{CB}(4 \times (a+3k)) - A$ contains a CKT. For the sub-board $\mathrm{CB}(4 \times 3)$, we construct two cycles C_1 and C_2 as shown in Figure 28.

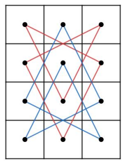

FIGURE 28. Two cycles C_1 and C_2 on $\mathrm{CB}(4 \times 3)$

Since $(1, a+3k)$ and $(4, a+3k)$ have degree 2 in $G(4 \times (a+3k)) - A$, $(1, a+3k) - (3, a+3k-1)$ and $(2, a+3k-1) - (4, a+3k)$ are two edges of the CKT on the sub-board $\mathrm{CB}(4 \times (a+3k)) - A$.

Then, we construct the required CKT by

(i) delete $(1, a+3k) - (3, a+3k-1)$ and $(2, a+3k-1) - (4, a+3k)$ of the CKT on the sub-board $\mathrm{CB}(4 \times (a+3k)) - A$ and delete $(1,1) - (3,2)$ and $(2,2) - (4,1)$ of C_1 and C_2 on the sub-board $\mathrm{CB}(4 \times 3)$, respectively;

(ii) join $(1, a+3k)$ and $(3, a+3k-1)$ of the CKT on the sub-board $\mathrm{CB}(4 \times (a+3k)) - A$ to $(2,2)$ and $(4,1)$ of C_2 on the sub-board $\mathrm{CB}(4\times3)$, respectively and join $(2, a+3k-1)$ and $(4, a+3k)$ of the CKT on the sub-board $\mathrm{CB}(4 \times (a+3k)) - A$ to $(1,1)$ and $(3,2)$ of C_1 on the sub-board $\mathrm{CB}(4 \times 3)$, respectively.

The constructed CKT is shown in Figure 29 with $A = \{(1,1),(4,1)\}$.

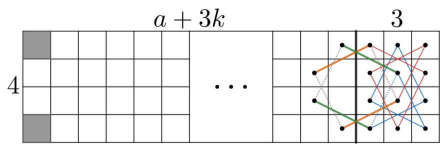

FIGURE 29. A CKT on $\mathrm{CB}(4 \times (a+3(k+1))) - \{(1,1),(4,1)\}$

Case 2: $a+3k-1 \le y, w \le a+3(k+1)$.

The required CKT can be obtained by rotating 180 degrees of the suitable CKT on $\mathrm{CB}(4 \times (a+3(k+1))) - A$ in Case 1.

Case 3: $x+y$ is odd, $z+w$ is even, $1 \le y \le 5$ and $a+3k-1 \le w \le a+3(k+1)$.

We separate $\mathrm{CB}(4 \times (a+3(k+1))) - \{(x,y),(z,w)\}$ into two sub-boards, $\mathrm{CB}(4 \times (a+3k-2)) - (x,y)$ and $\mathrm{CB}(4 \times 5) - (z,w)$ as shown in Figure 30 with $(x,y) = (1,2)$ and $(z,w) = (4, a+3(k+1))$.

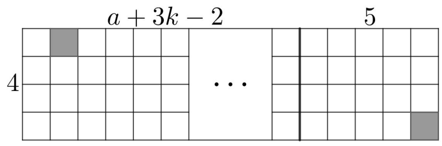

FIGURE 30. $\mathrm{CB}(4 \times (a + 3(k + 1))) - \{(1, 2), (4, a + 3(k + 1))\}$ with two sub-boards

Case 3.1: (k is even and $a \in \{7, 9\}$) or (k is odd and $a = 8$).

In this case, we have $a + 3k - 2 \geq 5$ is odd. Since $x + y$ is odd, by Lemma 3.1(b), the sub-board $\mathrm{CB}(4 \times (a + 3k - 2)) - \{(x, y)\}$ contains an OKT from $(1, a + 3k - 2)$ to $(3, a + 3k - 2)$.

If we regard (z, w) as the square of $\mathrm{CB}(4 \times (a+3(k+1)))$, then $z+w$ is even. However, if we regard (z, w) as the square of the sub-board $\mathrm{CB}(4 \times 5)$, then $z + w$ is odd. By Lemma 3.1(c), the sub-board $\mathrm{CB}(4 \times 5) - \{(z, w)\}$ contains an OKT from $(1, 1)$ to $(3, 1)$.

Then, as shown in Figure 31 with $(x, y) = (1, 2)$ and $(z, w) = (4, a + 3(k + 1))$, we construct the required CKT on $\mathrm{CB}(4 \times (a + 3(k + 1))) - \{(x, y), (z, w)\}$ by joining $(1, a + 3k - 2)$ and $(3, a + 3k - 2)$ of the OKT on the sub-board $\mathrm{CB}(4 \times (a + 3k - 2)) - \{(x, y)\}$ to $(3, 1)$ and $(1, 1)$ of the OKT on the sub-board $\mathrm{CB}(4 \times 5) - \{(z, w)\}$, respectively.

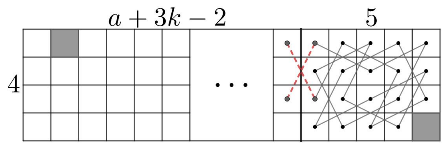

FIGURE 31. The required CKT on $\mathrm{CB}(4 \times (a+3(k+1))) - \{(1, 2), (4, a+3(k + 1))\}$

Case 3.2: (k is odd and $a \in \{7, 9\}$) or (k is even and $a = 8$).

In this case, we have $a + 3k - 2 \geq 6$ is even. Since $x + y$ is odd, by Lemma 3.2(a), the sub-board $\mathrm{CB}(4 \times (a + 3k - 2)) - \{(x, y)\}$ contains an OKT from $(2, a + 3k - 2)$ to $(4, a + 3k - 2)$.

If we regard (z, w) as the square of $\mathrm{CB}(4 \times (a+3(k+1)))$, then $z+w$ is even. Similarly, if we regard (z, w) as the square of the sub-board $\mathrm{CB}(4 \times 5)$, then $z+w$ is even. By Lemma 3.1(d), the sub-board $\mathrm{CB}(4 \times 5) - \{(z, w)\}$ contains an OKT from $(2, 1)$ to $(4, 1)$.

Then, as shown in Figure 32 with $(x, y) = (1, 2)$ and $(z, w) = (1, a + 3(k + 1))$, we construct the required CKT on $\mathrm{CB}(4 \times (a + 3(k + 1))) - \{(x, y), (z, w)\}$ by joining $(2, a + 3k - 2)$ and $(4, a + 3k - 2)$ of the OKT on the sub-board $\mathrm{CB}(4 \times (a + 3k - 2)) - \{(x, y)\}$ to $(4, 1)$ and $(2, 1)$ of the OKT on the sub-board $\mathrm{CB}(4 \times 5) - \{(z, w)\}$, respectively.

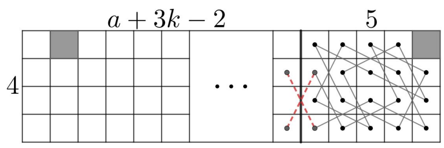

FIGURE 32. The required CKT on $\mathrm{CB}(4 \times (a+3(k+1))) - \{(1, 2), (1, a+3(k + 1))\}$

Case 4: $x + y$ is even, $z + w$ is odd, $1 \leq y \leq 5$ and $a + 3k - 1 \leq w \leq a + 3(k + 1)$.
The required CKT can be obtained by horizontally or vertically flipping of the suitable CKT on CB$(4 \times (a + 3(k + 1))) - A$ in Case 3.

Hence, in every cases, CB$(4 \times (a+3(k+1))) - A$ contains a CKT for all $A \in S_{a+3(k+1)}$. Thus, by the mathematical induction, we obtain the CKT on CB$(4 \times n) - A$ for all $A \in S_n$. ■

5. Conclusion and Discussion

The main result of this paper is to find all positions of 2 squares on CB$(4 \times n)$ so that after deleting these squares, then there exists a CKT on the deficient board. This result for $n \geq 7$ proves the Conjecture 1. However, this CKT is constructed using the legal knight's move. In 2005, Chia and Ong [6] defined the generalized knight's move or (a, b)-knight's move for which the knight moves a squares vertically or horizontally and then b squares at 90 degrees angle. Especially, they gave the existence of a CKT using the $(2, 3)$-knight's move on some CB$(m \times n)$. After that, there are some researchers [7] studied the nonexistence of CKTs using the (a, b)-knight's move on some CB$(m \times n)$. Therefore, as a future research, if we consider some CB$(m \times n)$ for which a CKT from the generalized knight's move does not exist, then we can investigate the minimum numbers of square to be removed and a CKT from the generalized knight's move exists on the deficient board as well as the exact positions of these squares to be removed.

Acknowledgements

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