Thai Journal of Mathematics
Special Issue (2022): Annual Meeting in Mathematics 2021
Pages 53 – 63
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

Pancyclicity of Generalized Prisms over Specific Types of Skirted Graphs

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Abstract A skirted graph is a planar graph G which is a union of a rooted tree $T \neq P_2$, where the root a of T is a vertex of degree at least two and all other vertices, except the leaves, are of degree at least three, and a path whose vertices are all leaves of T. A graph of order n is said to be pancyclic if it contains a cycle of each length l for $3 \leq l \leq n$. A graph of order n is almost pancyclic if it contains a cycle of each length l for $3 \leq l \leq n$ except possibly for a single even length. It is known that a skirted graph of order n is almost pancyclic. Moreover, if a skirted graph G contains no cycle of even length m, $3 < m \leq n$, then G contains a skirted subgraph of order $2m-1$ with specific types. In this paper, we are interested in these skirted subgraphs and prove that the cartesian product between the skirted subgraphs of each type and a path of arbitrary length is pancyclic.

MSC: 05C30; 05C38; 05C45
Keywords: skirted graph; cycle; pancyclic; almost pancyclic; cartesian product

1. Introduction

We consider finite, undirected, simple and connected graphs G with the vertex set $V(G)$ and the edge set $E(G)$. An (s,t)-path is a path in G from vertex s to vertex t, denoted by $P(s,t)$. Then, the path $P(t,s)$ denotes the reversed path of $P(s,t)$. A cycle of G is a *hamiltonian cycle* if it contains all the vertices of G. A graph G is said to be *hamiltonian* if it contains a hamiltonain cycle. A graph G of order n is said to be *pancyclic* if it contains a cycle of each length l for $3 \leq l \leq n$. A graph G of order n is *almost pancyclic* [1] if it contains a cycle of each length l for $3 \leq l \leq n$ except possibly for a single even length. We use the term *m-almost pancyclic* for an almost panyclic graph without a cycle of even length m.

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Let G and H be two graphs. The *cartesian product* of G with H, denoted by $G\Box H$, is defined as a graph with the vertex set $V(G) \times V(H)$ and an edge $\{(u_1, v_1), (u_2, v_2)\}$ presents in the product whenever $u_1 = u_2$ and $v_1v_2 \in E(H)$ or symmetrically $v_1 = v_2$ and $u_1u_2 \in E(G)$. The path graph P_k is a graph with k vertices, $v_1, v_2, v_3, \ldots, v_k$, where v_i is adjacent to v_{i+1} for $1 \leq i \leq k-1$. The *prism over a graph* G, denoted by $G\Box P_2$, is defined as the cartesian product of a graph G with P_2; that is, take two disjoint copies of G and add a matching joining the corresponding vertices in the two copies. For $n \geq 3$, we call a graph $G\Box P_n$, a *generalized prism over a graph* G.

Before giving a definition of a skirted graph, we introduce a definition of rooted trees as follows.

A *rooted tree* is a tree T, $T \neq P_2$, where the root of T is a vertex of degree at least two and all other vertices, except the leaves, are of degree at least three.

A *skirted graph* is a planar graph [2], denoted by $G(a, u_0, u_\alpha)$. It is a union $T \cup P$ of a rooted tree T, where a is the root of T, and a path $P = u_0u_1u_1 \cdots u_\alpha$ whose vertices are all leaves of T. A vertex of a skirted graph is called an *outer vertex* if it is on the path P or a shortest (a, u_0) or (a, u_α)-path.

In 1971, Bondy [3] posed a metaconjecture: almost any nontrivial condition on a graph which implies that the graph is hamiltonian also implies that the graph is pancyclic (There may be a simple family of exceptional graphs). There are a number of results that correspond to this conjecture. For example: in 1960, Ore [4] proved that if G is a graph of order n with $d(u) + d(v) \geq n$ for each pair of non-adjacent vertices u, v in G, then G is hamiltonian. After that, Bondy [5] showed that a graph G with the same condition as in [4] is pancyclic or G is isomorphic to the complete bipartite graph which each partite set contains $n/2$ vertices, $K_{n/2,n/2}$.

However, there exists a simple family of exceptional graphs such as skirted graphs which are hamiltonian, but they are just almost pancyclic [1]. For any skirted graph $G(a, b, c)$, we denote the path P of length α by $u_0u_1u_2 \cdots u_\alpha$, and the (a, c)-path of length β and (a, b)-path of length γ in T by $v_0v_1v_2 \cdots v_\beta$ and $w_0w_1w_2 \cdots w_\gamma$, respectively. Thus, $v_0 = w_0 = a$, $u_0 = w_\gamma = b$, and $u_\alpha = v_\beta = c$ (see Fig. 1).

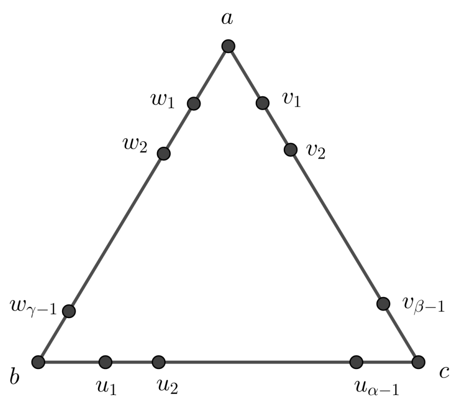

FIGURE 1. Paths $u_0u_1u_2 \cdots u_\alpha$, (a, c)-path and (a, b)-path of $G(a, b, c)$

For any skirted graph $G(a, b, c)$ of order n, let $a_1, a_2, a_3, \ldots, a_r$ be the neighbours of the root a. For $1 \le i \le r$, each a_i is the root of a skirted subgraph $G_i = G_i(a_i, b_i, c_i)$. Let $\alpha_i, \beta_i, \gamma_i$ and n_i be the analogues for G_i of α, β, γ and n, respectively (see Fig. 2).

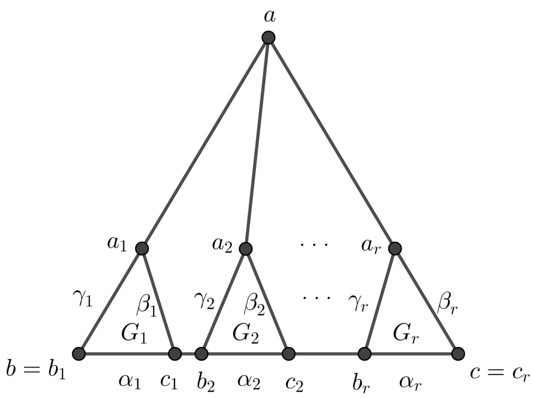

FIGURE 2. A skirted graph $G(a, b, c)$

Lemma 1.1 (Bondy and Lovász [1]). *Let $G = G(a, b, c)$ be a skirted graph of order n. Then, G contains:*

(i) an (a, c)-path of each length l for $\alpha + \gamma \le l \le n - 1$;
(ii) a (b, c)-path of each length l for $\alpha \le l \le n - 1$.

Remark 1.2. Following from Lemma 1.1, we obtain that

1. Lemma 1.1(i) gives an (a, b)-path of each length ℓ for $\alpha + \beta \le \ell \le n - 1$ by the symmetry of $G(a, b, c)$.
2. To track down the path from each skirted subgraph of $G(a, b, c)$, a (b, c)-path of length $n - 2$ (without the root vertex a) can be obtained by Lemma 1.1(ii).

Meanwhile, for the prism over a graph G, there are some hamiltonian and pancyclicity results. For example: Paulraja [6] proved in 1993 that if G is a 3-connected 3-regular graph, then the prism $G\Box P_2$ is hamiltonian. In 2001, Wayne [7] showed that if G is a 3-connected 3-regular graph that contains a triangle, then the prism $G\Box P_2$ is pancyclic.

The following section provides our preliminary results on hamiltonicity and pancyclicity as well as the motivation of our main results.

2. Preliminary results and motivation

For a skirted graph $G = G(a, b, c)$ and $P_n = v_0v_1v_2 \cdots v_{n-1}$, we denote the vertices of $G\Box P_n$ shortly by $u^{(j)} = (u, v_j)$ where $u \in V(G)$, $v_j \in V(P_n)$ for $0 \le i \le n - 1$.

The following theorem is an immediate observation about the existence of a hamiltonian cycle over the generalized prism over any skirted graphs.

Theorem 2.1. *The generalized prism over any skirted graphs is hamiltonian.*

Proof. Let $G = G(a, b, c)$ be a skirted graph of order m and P_n be a path of order n. We show that $G\Box P_n$ is hamiltonian by finding a cycle of length mn in $G\Box P_n$. Let $a_1, a_2, a_3, \ldots, a_r$ be the neighbours of the rooted a in G. To show that $G\Box P_n$ contains a cycle of length mn, we give the following paths and then link together with edges joining each copy of G.

- The first and the last copies of G contain paths $P(a^{(0)}, c^{(0)})$ and $P(a^{(n-1)}, c^{(n-1)})$, respectively, of length $m-1$ by Lemma 1.1(i) (see Figs. 3(a) and 3(c)). Also, a path $P(a^{(n-1)}, b^{(n-1)})$ of the last copy of G exists by the symmetry of G in Remark 1.2.
- The remaining $n-2$ copies of G contain a path $P(b^{(i)}, c^{(i)})$ of length $m-2$ (without the root vertex $a^{(i)}$) for $1 \le i \le n-2$, which exists by Remark 1.2.
- The path $P(a^{(n-1)}, a^{(0)}) = a^{(n-1)}a^{(n-2)}a^{(n-3)} \cdots a^{(0)}$ is a path of $G\Box P_n$ from the last copy to the first copy of G.

Now, we link each path by edge $x_i = b^{(i)}b^{(i+1)}$ when i is odd and edge $y_i = c^{(i)}c^{(i+1)}$ when i is even. The cycle of length mn is

$$P(a^{(0)}, c^{(0)})y_0P(c^{(1)}, b^{(1)})x_1P(b^{(2)}, c^{(2)})y_2 \cdots x_{n-2}P(b^{(n-1)}, a^{(n-1)})P(a^{(n-1)}, a^{(0)})$$

when n is odd or

$$P(a^{(0)}, c^{(0)})y_0P(c^{(1)}, b^{(1)})x_1P(b^{(2)}, c^{(2)})y_2 \cdots y_{n-2}P(c^{(n-1)}, a^{(n-1)})P(a^{(n-1)}, a^{(0)})$$

when n is even.

This completes the proof. ∎

By linking paths $P(a^{(0)}, c^{(0)})$ and $P(a^{(1)}, c^{(1)})$ of length $m-1$ of the first and the second copies of G and edges $c^{(0)}c^{(1)}$ and $a^{(0)}a^{(1)}$, $G\Box P_2$ also contains a hamiltonian cycle.

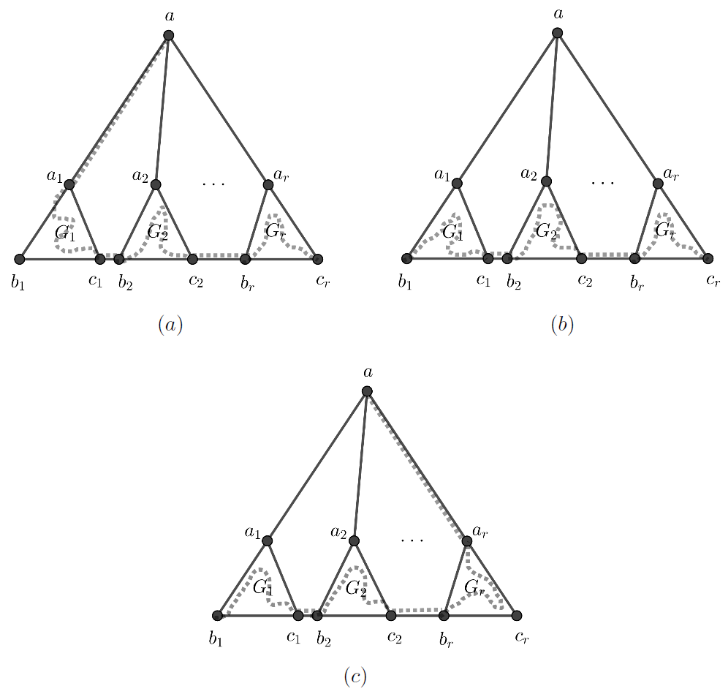

FIGURE 3. (a) (a, c)-path, (b) (b, c)-path and (c) (a, b)-path

The next result is a preliminary result for a skirted graph of order 7.

Theorem 2.2. *Let $G = G(a, b, c)$ be a skirted graph of order 7 such that G contains no cycle of length 4. Then, $G \square P_n$ is pancyclic for $n \geq 2$.*

Proof. Let $G = G(a, b, c)$ be a skirted graph of order 7 such that G contains no cycle of length 4 (see Fig. 4). We show that the generalized prism over G is pancyclic by using the mathematical induction on the order of P_n. It is easy to see that $G \square P_2$ contains a cycle of each length l for $3 \leq l \leq 14$. Thus, $G \square P_2$ is pancyclic.

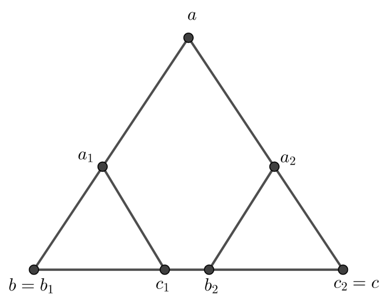

FIGURE 4. A skirted graph of order 7 containing no cycle of length 4

For $n = 3$, since $G \square P_2$ is a subgraph of $G \square P_3$ and $G \square P_2$ is pancyclic, $G \square P_3$ contains a cycle of each length l for $3 \leq l \leq 14$. It suffices to show that $G \square P_3$ contains a cycle of each length l for $15 \leq l \leq 21$. Two steps are shown. The first one is finding a cycle of each length l for $17 \leq l \leq 21$ and the second one is finding cycles of lengths 15 and 16.

Step 1 : To show that $G \square P_3$ contains cycles of such lengths, we give the following paths and then link them together with edges joining each copy of G.

- The first copy and the last copy of G contain paths $P(a^{(0)}, c^{(0)})$ and $P(a^{(2)}, c^{(2)})$, respectively, of each length l for $5 \leq l \leq 6$ by Lemma 1.1(i). Also, for the last copy of G, a path $P(a^{(2)}, b^{(2)})$ of each length l for $5 \leq l \leq 6$ exists by the symmetry of G in Remark 1.2.

- The middle copy of G contains a path $P(b^{(1)}, c^{(1)})$ of each length l for $3 \leq l \leq 5$ (without the root vertex $a^{(1)}$), which exists by Remark 1.2.

- The path $P(a^{(2)}, a^{(0)}) = a^{(2)}a^{(1)}a^{(0)}$ of length 2 is a path of $G \square P_3$ from the last copy to the first copy of G.

Now, we link each path (maybe of different sizes) by edge $x_1 = b^{(1)}b^{(2)}$ and $y_0 = c^{(0)}c^{(1)}$. The cycle of length l for $17 \leq l \leq 21$ is

$$P(a^{(0)}, c^{(0)})y_0P(c^{(1)}, b^{(1)})x_1P(b^{(2)}, a^{(2)})P(a^{(2)}, a^{(0)}).$$

Step 2 : To show that $G \square P_3$ contains cycles of length 15 and 16, we refer to the cycle of length 17 from Step 1 where $P(a^{(0)}, c^{(0)})$ and $P(b^{(2)}, a^{(2)})$ have length 5 and $P(b^{(1)}, c^{(1)})$ has length 3. In this case, let $P(a^{(0)}, c^{(0)}) = a^{(0)}a_1^{(0)}b^{(0)}c_1^{(0)}b_2^{(0)}c^{(0)}$ and $P(b^{(2)}, a^{(2)}) = b^{(2)}c_1^{(2)}b_2^{(2)}c^{(2)}a_2^{(2)}a^{(2)}$. Then, removing vertex $b^{(0)}$ (respectively $b^{(0)}$ and $c^{(2)}$) make the cycle of length 17 to the cycle of length 16 (respectively the cycle of length 15).

Therefore, $G\Box P_3$ is pancyclic.

For $n \geq 4$, suppose that $G\Box P_{n-1}$ is pancyclic, i.e., $G\Box P_{n-1}$ contains a cycle of each length l for $3 \leq l \leq 7(n-1)$. We shall find a cycle of each length l for $7(n-1)+1 \leq l \leq 7n$ in $G\Box P_n$.

To show that $G\Box P_n$ contains cycles of such lengths, we give the following paths and then link them together with edges joining each copy of G.

- The first copy and the last copy of G contain paths $P(a^{(0)}, c^{(0)})$ and $P(a^{(n-1)}, c^{(n-1)})$, respectively, of each length l for $5 \leq l \leq 6$ by Lemma 1.1(i). Also, for the last copy of G a path $P(a^{(n-1)}, b^{(n-1)})$ of each length l for $5 \leq l \leq 6$ exists by the symmetry of G in Remark 1.2.
- The remaining $n-2$ copies of G contain a path $P(b^{(i)}, c^{(i)})$ of each length l for $3 \leq l \leq 5$ (without the root vertex $a^{(i)}$) for $1 \leq i \leq n-2$, which exists by Remark 1.2.
- The path $P(a^{(n-1)}, a^{(0)}) = a^{(n-1)}a^{(n-2)}a^{(n-3)} \cdots a^{(0)}$ of length $n-1$ is a path of $G\Box P_n$ from the last copy to the first copy of G.

Now, we link each path (maybe of different sizes) by edge $x_i = b^{(i)}b^{(i+1)}$ when i is odd and edge $y_i = c^{(i)}c^{(i+1)}$ when i is even. The cycle of length l for $5n+2 \leq l \leq 7n$ is

$$P(a^{(0)}, c^{(0)})y_0P(c^{(1)}, b^{(1)})x_1P(b^{(2)}, c^{(2)})y_2 \cdots x_{n-2}P(b^{(n-1)}, a^{(n-1)})P(a^{(n-1)}, a^{(0)})$$

when n is odd or

$$P(a^{(0)}, c^{(0)})y_0P(c^{(1)}, b^{(1)})x_1P(b^{(2)}, c^{(2)})y_2 \cdots y_{n-2}P(c^{(n-1)}, a^{(n-1)})P(a^{(n-1)}, a^{(0)})$$

when n is even.

Since $5n+2 \leq 7(n-1)+1$ for $n \geq 4$, $G\Box P_n$ contains a cycle of each length l for $7(n-1)+1 \leq l \leq 7n$.

Therefore, $G\Box P_n$ is pancyclic. ■

Bondy and Lovász [1] showed in 1985 that a skirted graph of order n contains cycles of each length l for $3 \leq l \leq n$, except, possibly, for one even value of l. Moreover, if it contains no cycle of even length m, where $3 < m \leq n$, then it contains a subgraph which is also a skirted graph of order $2m-1$ of type I, II or III (see Fig. 5).

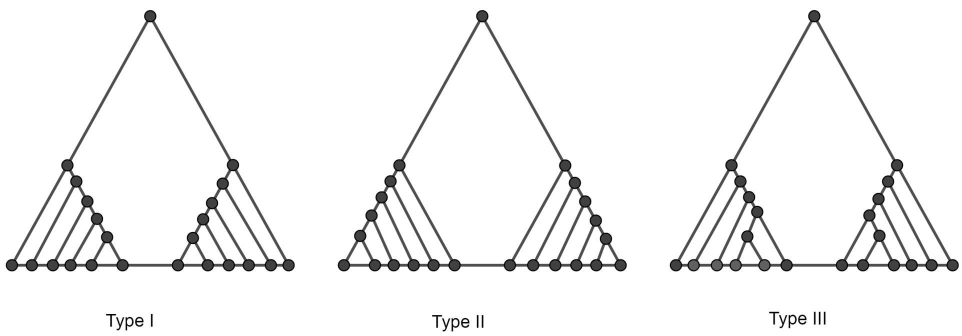

FIGURE 5. A skirted graph of order $2m-1$ of type I, II and III

Note that, the types I and III contain $\alpha = m-1, \beta = 2$ and $\gamma = 2$, while $\alpha = m-1, \beta = m/2$ and $\gamma = m/2$ for the type II.

This motivates us to wonder that for a skirted graph of order $2m-1$ of type I, II or III which are m-almost pancyclic, does the generalized prism over this specific type of a skirted graph turn into pancyclic for $n \geq 2$? From Fig. 5, α, β and γ of the types I and III are the same, while another type has different values of β and γ. Thus, to study pancyclicity of generalized prisms over these three types of graphs, we separate it into two sections. In Section 3, we prove the pancyclicity results for the generalized prism over skirted graphs of type I or III by using Lemma 1.1 and the mathematical induction on the order of a path P_n. In Section 4, by using the similar idea, we can also prove the pancyclicity for the generalized prism over skirted graphs of type II. Finally, conclusion and discussion about the possibly future research are given in Section 5.

3. Pancyclicity of generalized prisms over skirted graphs of type I or III

We already know that a skirted graph $G = G(a, b, c)$ of type I or III of order $2m-1$ is an m-almost pancyclic, i.e., G contains a cycle of each length l for $3 \leq l \leq 2m-1$ except for a cycle of even length m. Since G is a subgraph of $G \square P_2$, $G \square P_2$ contains such cycles of length l for $3 \leq l \leq 2m-1$ except possibly $l = m$. To show that $G \square P_2$ is pancyclic, we first show that $G \square P_2$ contains a cycle of length m.

Lemma 3.1. *Let $G = G(a, b, c)$ be a skirted graph of order $2m-1$, where m is an even integer such that $m \geq 6$ and G is of type I or III. Then, $G \square P_2$ contains a cycle of each length l where l is an even integer ranging from 4 to $2m+6$.*

Proof. Since G is of type I or III, it contains $m+3$ consecutive outer vertices, called $u_0, u_1, u_2, \ldots, u_{m+2}$, respectively. We define a sequence of $m+2$ cycles on $G \square P_2$ as follows.

$$u_1^{(0)} u_0^{(0)} u_0^{(1)} u_1^{(1)} u_1^{(0)},$$
$$u_2^{(0)} u_1^{(0)} u_0^{(0)} u_0^{(1)} u_1^{(1)} u_2^{(1)} u_2^{(0)},$$
$$u_3^{(0)} u_2^{(0)} u_1^{(0)} u_0^{(0)} u_0^{(1)} u_1^{(1)} u_2^{(1)} u_3^{(1)} u_3^{(0)},$$
$$\cdots,$$
$$u_{m+2}^{(0)} u_{m+1}^{(0)} u_m^{(0)} u_{m-1}^{(0)} \cdots u_1^{(0)} u_0^{(0)} u_0^{(1)} u_1^{(1)} \cdots u_m^{(1)} u_{m+1}^{(1)} u_{m+2}^{(1)} u_{m+2}^{(0)}.$$

The length of each cycle in the sequence increases as an arithmetic sequence with the common difference 2. Then, the last cycle

$$u_{m+2}^{(0)} u_{m+1}^{(0)} u_m^{(0)} u_{m-1}^{(0)} \cdots u_1^{(0)} u_0^{(0)} u_0^{(1)} u_1^{(1)} \cdots u_m^{(1)} u_{m+1}^{(1)} u_{m+2}^{(1)} u_{m+2}^{(0)}$$

of this sequence has length $2m+6$. Since the first cycle $u_1^{(0)} u_0^{(0)} u_0^{(1)} u_1^{(1)} u_1^{(0)}$ is a cycle of length 4, the lengths of the cycles are even integers ranging from 4 to $2m+6$. ■

By Lemma 3.1, we can see that if G is a skirted graph of order $2m-1$, where $m \geq 6$ is an even integer and G is of type I or III, then $G \square P_2$ contains a cycle of length m. Next, we need the following lemma to show that the generalized prism over a skirted graph of order $2m-1$ of type I or III is pancyclic.

Lemma 3.2. *Let $G = G(a, b, c)$ be a skirted graph of order $2m-1$, where m is an even integer such that $m \geq 6$, and G is of type I or III. Then, $G \square P_2$ is pancyclic.*

Proof. By the result of Bondy and Lovász in [1] that G is an m-almost pancyclic and Lemma 3.1, $G\Box P_2$ contains a cycle of each length l for $3 \leq l \leq 2m-1$. It suffices to show that the prism over G contains a cycle of each length l for $2m \leq l \leq 4m-2$.

For $0 \leq i \leq 1$, the ith copy of G contains a path $P(b^{(i)}, c^{(i)})$ of length l for $m-1 \leq l \leq 2m-2$, by Lemma 1.1(ii). We link each path $P(b^{(i)}, c^{(i)})$ (maybe of different sizes) for $0 \leq i \leq 1$ together with edges $b^{(0)}b^{(1)}$ and $c^{(0)}c^{(1)}$. The cycle of each length l for $2m \leq l \leq 4m-2$ is $P(b^{(0)}, c^{(0)})c^{(0)}c^{(1)}P(c^{(1)}, b^{(1)})b^{(1)}b^{(0)}$.

Therefore, $G\Box P_2$ is pancyclic. ■

By using Lemma 3.2 as a basic step, we can use the mathematical induction to establish the following result.

Theorem 3.3. *Let $G = G(a, b, c)$ be a skirted graph of order $2m-1$, where m is an even integer such that $m \geq 6$, and of type I or III. Then, $G\Box P_n$ is pancyclic for $n \geq 2$.*

Proof. We prove by the mathematical induction on the order of P_n. The basic step is already done in Lemma 3.2 for $n = 2$. For $n \geq 3$, suppose that $G\Box P_{n-1}$ is pancyclic, i.e., $G\Box P_{n-1}$ contains a cycle of each length l for $3 \leq l \leq (n-1)(2m-1)$. We shall find a cycle of each length l for $(n-1)(2m-1)+1 \leq l \leq n(2m-1)$.

To show that $G\Box P_n$ contains cycles of such length, we give the following paths and then link them together with edges joining each copy of G.

- The first copy and the last copy of G contain paths $P(a^{(0)}, c^{(0)})$ and $P(a^{(n-1)}, c^{(n-1)})$, respectively, of each length l for $m+1 \leq l \leq 2m-2$ by Lemma 1.1(i). Also, for the last copy of G a path $P(a^{(n-1)}, b^{(n-1)})$ of each length l for $m+1 \leq l \leq 2m-2$ exists by the symmetry of G in Remark 1.2.

- The remaining $n-2$ copies of G contain a path $P(b^{(i)}, c^{(i)})$ of each length l for $m-1 \leq l \leq 2m-3$ (without the root vertex $a^{(i)}$) for $1 \leq i \leq n-2$, which exists by Remark 1.2.

- The path $P(a^{(n-1)}, a^{(0)}) = a^{(n-1)}a^{(n-2)}a^{(n-3)}\cdots a^{(0)}$ of length $n-1$ is a path of $G\Box P_n$ from the last copy to the first copy of G.

Now, we link each path (maybe of different sizes) by edge $x_i = b^{(i)}b^{(i+1)}$ when i is odd and edge $y_i = c^{(i)}c^{(i+1)}$ when i is even. The cycle of length l for $mn+n+2 \leq l \leq n(2m-1)$ is

$$P(a^{(0)}, c^{(0)})y_0P(c^{(1)}, b^{(1)})x_1P(b^{(2)}, c^{(2)})y_2\cdots x_{n-2}P(b^{(n-1)}, a^{(n-1)})P(a^{(n-1)}, a^{(0)})$$

when n is odd or

$$P(a^{(0)}, c^{(0)})y_0P(c^{(1)}, b^{(1)})x_1P(b^{(2)}, c^{(2)})y_2\cdots y_{n-2}P(c^{(n-1)}, a^{(n-1)})P(a^{(n-1)}, a^{(0)})$$

when n is even.

We can conclude that $G\Box P_n$ is pancyclic if $mn+n+2 \leq (n-1)(2m-1)+1$, that is, $n \geq 2m/(m-2)$. Since $3 \geq 2m/(m-2)$ for all $m \geq 6$, $n \geq 2m/(m-2)$ for all $n \geq 3$.

Therefore, $G\Box P_n$ is pancyclic. ■

4. Pancyclicity of Generalized Prisms over Skirted Graphs of Type II

We already know that a skirted graph $G = G(a, b, c)$ of type II of order $2m-1$ is an m-almost pancyclic, i.e., G contains a cycle of each length l for $3 \leq l \leq 2m-1$ except for a cycle of even length m. Since G is subgraph of $G\Box P_2$, $G\Box P_2$ contains such cycles of length l for $3 \leq l \leq 2m-1$ except possibly $l = m$. To show that $G \square P_2$ is pancyclic, we first show that $G \square P_2$ contains a cycle of length m.

Lemma 4.1. *Let $G = G(a,b,c)$ be a skirted graph of order $2m-1$, where m is an even integer such that $m \geq 6$ and G is of type II. Then, $G \square P_2$ contains a cycle of each length l where l is an even integer ranging from 4 to $4m-2$.*

Proof. Since G is of type II, it contains $2m-1$ consecutive outer vertices, called $u_0, u_1, u_2, \ldots, u_{2m-2}$, respectively. We define a sequence of $2m-2$ cycles on $G \square P_2$ as follows.

$$u_1^{(0)} u_0^{(0)} u_0^{(1)} u_1^{(1)} u_1^{(0)},$$
$$u_2^{(0)} u_1^{(0)} u_0^{(0)} u_0^{(1)} u_1^{(1)} u_2^{(1)} u_2^{(0)},$$
$$u_3^{(0)} u_2^{(0)} u_1^{(0)} u_0^{(0)} u_0^{(1)} u_1^{(1)} u_2^{(1)} u_3^{(1)} u_3^{(0)},$$
$$\cdots,$$
$$u_{2m-2}^{(0)} u_{2m-3}^{(0)} u_{2m-4}^{(0)} u_{2m-5}^{(0)} \cdots u_1^{(0)} u_0^{(0)} u_0^{(1)} u_1^{(1)} \cdots u_{2m-4}^{(1)} u_{2m-3}^{(1)} u_{2m-2}^{(1)} u_{2m-2}^{(0)}.$$

The length of each cycle in the sequence increases as an arithmetic sequence with the common difference 2. Then, the last cycle

$$u_{2m-2}^{(0)} u_{2m-3}^{(0)} u_{2m-4}^{(0)} u_{2m-5}^{(0)} \cdots u_1^{(0)} u_0^{(0)} u_0^{(1)} u_1^{(1)} \cdots u_{2m-4}^{(1)} u_{2m-3}^{(1)} u_{2m-2}^{(1)} u_{2m-2}^{(0)}$$

of this sequence has length $4m-2$. Since the first cycle $u_1^{(0)} u_0^{(0)} u_0^{(1)} u_1^{(1)} u_1^{(0)}$ is a cycle of length 4, the lengths of the cycles are even integers ranging from 4 to $4m-2$. ∎

By Lemma 4.1, we can see that if G is a skirted graph of order $2m-1$, where $m \geq 6$ is an even integer and G is of type II, then $G \square P_2$ contains a cycle of length m. Next, we need the following lemmas to show that the generalized prism over a skirted graph of order $2m-1$ of type II is pancyclic.

Lemma 4.2. *Let $G = G(a,b,c)$ be a skirted graph of order $2m-1$, where m is an even integer such that $m \geq 6$, and G is of type II. Then, $G \square P_2$ is pancyclic.*

Proof. By the result of Bondy and Lovász in [1] that G is an m-almost pancyclic and Lemma 4.1, $G \square P_2$ contains a cycle of each length l, $3 \leq l \leq 2m-1$. It suffices to show that the prism over G contains a cycle of each length l for $2m \leq l \leq 4m-2$.

For $0 \leq i \leq 1$, the ith copy of G contains a path $P(b^{(i)}, c^{(i)})$ of length l for $m-1 \leq l \leq 2m-2$, by Lemma 1.1(ii). We link each path $P(b^{(i)}, c^{(i)})$ (maybe of different sizes) for $0 \leq i \leq 1$ together with edges $b^{(0)}b^{(1)}$ and $c^{(0)}c^{(1)}$. The cycle of each length l for $2m \leq l \leq 4m-2$ is $P(b^{(0)}, c^{(0)})c^{(0)}c^{(1)}P(c^{(1)}, b^{(1)})b^{(1)}b^{(0)}$.

Therefore, $G \square P_2$ is pancyclic. ∎

Lemma 4.3. *Let $G = G(a,b,c)$ be a skirted graph of order $2m-1$, where m is an even integer such that $m \geq 6$ and G is of type II. Then, $G \square P_3$ is pancyclic.*

Proof. By Lemma 4.2, $G \square P_3$ contains a cycle of each length l for $3 \leq l \leq 4m-2$. It suffices to show that $G \square P_3$ contains a cycle of each length l for $4m-1 \leq l \leq 6m-3$. Two steps are shown. The first one is finding a cycle of each length l for $4m+1 \leq l \leq 6m-3$ and the second one is finding cycles of length $4m-1$ and $4m$.

Step 1 : To show that $G \square P_3$ contains cycles of such length, we give the following paths and then link them together with edges joining each copy of G.

- The first copy of G contains a path $P(a^{(0)}, c^{(0)})$, of each length l for $(3m-2)/2 \leq l \leq 2m-2$ by Lemma 1.1(i). Also, for the last copy of G, a path $P(a^{(2)}, b^{(2)})$ of each length l for $(3m-2)/2 \leq l \leq 2m-2$ exists by the symmetry of G in Remark 1.2.
- The middle copy of G contains a path $P(b^{(1)}, c^{(1)})$ of each length l for $m-1 \leq l \leq 2m-3$ (without the root vertex $a^{(1)}$), which exists by Remark 1.2.
- The path $P(a^{(2)}, a^{(0)}) = a^{(2)}a^{(1)}a^{(0)}$ of length 2 is a path of $G\Box P_3$ from the last copy to the first copy of G.

Now, we link each path (maybe of different sizes) by edges $x_1 = b^{(1)}b^{(2)}$ and $y_0 = c^{(0)}c^{(1)}$. The cycle of length l for $4m+1 \leq l \leq 6m-3$ is

$$P(a^{(0)}, c^{(0)})y_0P(c^{(1)}, b^{(1)})x_1P(b^{(2)}, a^{(2)})P(a^{(2)}, a^{(0)}).$$

Step 2 : To show that $G\Box P_3$ contains cycles of lengths $4m-1$ and $4m$, we modify the cycle of length $(4m+1)$ from Step 1, where $P(a^{(0)}, c^{(0)})$ and $P(b^{(2)}, a^{(2)})$ have length $(3m-2)/2$ and $P(c^{(1)}, b^{(1)})$ has length $m-1$. In this case, let $P(a^{(0)}, c^{(0)})$ be the path containing all consecutive outer vertices from $a^{(0)}$ to $c^{(0)}$ passing through $b^{(0)}$. Also, let $P(b^{(2)}, a^{(2)})$ be the path containing all consecutive outer vertices from $b^{(2)}$ to $a^{(2)}$ passing through $c^{(2)}$. Then, removing vertex $b^{(0)}$ (respectively $b^{(0)}$ and $c^{(2)}$) make the cycle of length $4m+1$ to the cycle of length $4m$ (respectively the cycle of length $4m-1$).

Therefore, $G\Box P_3$ is pancyclic. ∎

We see that, in the proof of Lemma 4.3, the cartesian product of G over a path of order 3, we have to consider the special case as shown in Step 2. However, there is no special case when we show that $G\Box P_n$ is pancyclic for $n \geq 4$.

By using Lemmas 4.2 and 4.3 as a basic step, we can use the mathematical induction to establish the following result.

Theorem 4.4. *Let $G = G(a, b, c)$ be a skirted graph of order $2m-1$, where m is an even integer such that $m \geq 6$ and G is of type II. Then, $G\Box P_n$ is pancyclic for $n \geq 2$.*

Proof. We prove by the mathematical induction on the order of P_n. The basic step is already done in Lemmas 4.2 and 4.3 for $n = 2, 3$. For $n \geq 4$, suppose that $G\Box P_{n-1}$ is pancyclic, i.e., $G\Box P_{n-1}$ contains a cycle of each length l for $3 \leq l \leq (n-1)(2m-1)$. We shall find a cycle of each length l for $(n-1)(2m-1)+1 \leq l \leq n(2m-1)$.

To show that $G\Box P_n$ contains cycles of such lengths, we give the following paths and then link them together with edges joining each copy of G.
- The first copy and the last copy of G contain paths $P(a^{(0)}, c^{(0)})$ and $P(a^{(n-1)}, c^{(n-1)})$, respectively, of each length l for $(3m-2)/2 \leq l \leq 2m-2$ by Lemma 1.1(i). Also, for the last copy of G a path $P(a^{(n-1)}, b^{(n-1)})$ of each length l for $(3m-2)/2 \leq l \leq 2m-2$ exists by the symmetry of G in Remark 1.2.
- The remaining $n-2$ copies of G contain a path $P(b^{(i)}, c^{(i)})$ of each length l for $m-1 \leq l \leq 2m-3$ (without the root vertex $a^{(i)}$) for $1 \leq i \leq n-2$, which exists by Remark 1.2.
- The path $P(a^{(n-1)}, a^{(0)}) = a^{(n-1)}a^{(n-2)}a^{(n-3)} \cdots a^{(0)}$ of length $n-1$ is a path of $G\Box P_n$ from the last copy to the first copy of G.

Now, we link each path (maybe of different sizes) by edge $x_i = b^{(i)}b^{(i+1)}$ when i is odd and edge $y_i = c^{(i)}c^{(i+1)}$ when i is even. The cycle of length l for $mn+m+n-2 \leq l \leq n(2m-1)$ is

$$P(a^{(0)}, c^{(0)})y_0P(c^{(1)}, b^{(1)})x_1P(b^{(2)}, c^{(2)})y_2 \cdots x_{n-2}P(b^{(n-1)}, a^{(n-1)})P(a^{(n-1)}, a^{(0)})$$

when n is odd or

$$P(a^{(0)}, c^{(0)})y_0P(c^{(1)}, b^{(1)})x_1P(b^{(2)}, c^{(2)})y_2 \cdots y_{n-2}P(c^{(n-1)}, a^{(n-1)})P(a^{(n-1)}, a^{(0)})$$

when n is even.

We can conclude that $G\square P_n$ is pancyclic if $mn+m+n-2 \leq (n-1)(2m-1)+1$, that is, $n \geq (3m-4)/(m-2)$. Since $4 > (3m-4)/(m-2)$ for all $m \geq 6$, $n \geq (3m-4)/(m-2)$ for all $n \geq 4$.

Therefore, $G\square P_n$ is pancyclic. ∎

5. Conclusion and Discussion

In this paper, we can prove that the generalized prism over the m-almost pancyclic graph, namely, skirted graphs of type I, II and III, is pancyclic. Moreover, since the cartesian product of a graph over a path P_n is a subgraph of the cartesian product of the graph over a cycle C_n and the cartesian product of the graph over a complete graph K_n, the results of this paper can be concluded in the similar way when P_n is replaced by C_n or K_n for $n \geq 3$. However, the technique that we use migth not be directly applied to any skirted graphs other than these 3 types since we do not know the exact configuration of their vertices and edges. Thus, our future research is to develop a technique to overcome this difficulty.

Acknowledgements

We would like to thank the referees for their comments and suggestions on the manuscript. This work was supported by Science Achievement Scholarship of Thailand.

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