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A New Concept of Non-deterministic Hypersubstitutions for Algebraic Systems

Sorasak Leeratanavalee[1] **and Sarawut Phuapong**[2,*]

[1] *Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand*
e-mail : sorasak.l@cmu.ac.th (S. Leeratanavalee)

[2] *Department of Mathematics, Faculty of Science and Agricultural Technology, Rajamangala University of Technology Lanna, Chiang Mai 50300, Thailand*
e-mail : Phuapong.sa@gmail.com (S. Phuapong)

Abstract The algebraic systems are well-established structures of classical universal algebras. An algebraic system is a triple consisting a nonempty set with the collection of operations and the collection of relations. The purpose of this paper, we extend the concepts related to non-deterministic hypersubstitutions (nd-hypersubstitutions) for universal algebras to algebraic systems and obtain that the set of all non-deterministic hypersubstitutions for algebraic systems forms a monoid with a binary operation defined on this set.

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1. Introduction

It is well-known that terms is one of powerful tools in the study of universal algebra for classifying algebras into a collection, for example the class of semigroup is refer to algebras of type (2) satisfying the associative law. In fact, both sides of the associative law can be regarded as terms. In combinatoric, terms can be reformulated as a tree diagram, called a semantic tree. Applying this idea, elementary and advance topics in theoretical computer science are connected with terms. Basically, a collection of terms is called tree languages which was introduced by Gécseg and Steinby in 1984, see [1]. It has various applications in the study of non-deterministic transformations and automata theory. In particular, Denecke and his groups are studied tree languages in several points of view, for instance semigroups of tree languages and a mapping which takes any operation symbols to tree laguages. For more information, see [2]. Current developments in tree laguages can be found, for example, in [3–5].

*Corresponding author.

One of outstanding structures in mathematics is an algebraic system which was introduced by Mal'cev, the Russian mathematician, in 1973 [6]. It is a nonempty set together with a collection of operations and a collection of relations on this set. This system is one of the basic mathematical concepts. Its general theory has been deeply developed. The ordered group is an example, which is widely applied in many branches of mathematics. The concept of formulas is first introduced by Mal'cev [6]. Denecke and Phusanga [7] introduced the superposition of formulas R^n_m and the concept of a hypersubstitution for algebraic systems. They studied formulas over tree languages by applying the idea in [8] to define formulas over tree languages. For more research direction in this area, see [9–13]. Our first aim is to construct the many-sorted algebra as in the same situation of $\mathcal{P}-\text{Formclone}(\tau,\tau')$, and then define the superposition operation $\hat{R}^n_m$ on the power set of all n-ary formulas of type (τ,τ'), i.e., $P\left(\mathcal{F}_{(\tau,\tau')}\left(W_\tau(X_n)\right)\right)$. Furthermore, the canonical concept of a non-deterministic hypersubstitution for algebraic systems is introduced.

2. Preliminaries

First of all, we recall some essential concepts that we will use in this work. To define to terms, the set of variables and the set of operation symbols are needed. Let $X_n := \{x_1,\ldots,x_n\}$ be a finite set of alphabets and its element called variables. While the symbol $X := \{x_1,\ldots,x_n,\ldots\}$, we refer to an infinite set of variables. By $(f_i)_{i\in I}$, we denote a set of operation symbols which each f_i having the arity n_i. The type τ is refer to the sequence of all arities of operation symbols. As a conseguence, an n-ary term of type τ is defined inductively as follows:

(i) The variables $x_1,\ldots,x_n$ are n-ary term of type τ.
(ii) If $t_1,\ldots,t_{n_i}$ are n-ary terms of type τ, then $f_i(t_1,\ldots,t_{n_i})$ is an n-ary term of type τ.

The symbols $W_\tau(X_n)$ and $W_\tau(X)$ we mean the set of all n-ary terms of type τ and the set of all terms of type τ, respectively.

Now, we recall the concept of a superposition operation of terms. Let $m,n\geq 1$. The superposition operation is a many-sorted mapping

$$S^n_m : W_\tau(X_n)\times W_\tau(X_m)^n \to W_\tau(X_m)$$

defined by

(i) $S^n_m(x_j,t_1,\ldots,t_n) := t_j$ if $x_j\in X_n$ is a variable and $t_1,\ldots,t_n\in W_\tau(X_m)$.
(ii) $S^n_m(f_i(s_1,\ldots,s_{n_i}),t_1,\ldots,t_n) := f_i(S^n_m(s_1,t_1,\ldots,t_n),\ldots,S^n_m(s_{n_i},t_1,\ldots,t_n))$ if $f_i(s_1,\ldots,s_{n_i})$ is a composite term.

Then *the clone of all terms of type* τ can be defined by

$$\text{clone}\tau := \left((W_\tau(X_n))_{n\in\mathbb{N}^+},(S^n_m)_{m,n\in\mathbb{N}^+},(x_i)_{i\leq n\in\mathbb{N}^+}\right).$$

In this case, the variables $x_1,x_2,\ldots,x_n$ act as the nullary operations. It in accordance with the identities (C1),(C2),(C3) (see [14]). For more detial concerned algebraic systems, see [6, 15]. Let J be a nonempty indexed set and $(\gamma_j)_{j\in J}$ be a sequence of relation symbols. Let $\tau' := (n_j)_{j\in J}$ where n_j is the arity of γ_j for every $j\in J$. The pair (τ,τ') is called *the type* of an algebraic system.

Definition 2.1. [6] An algebraic system of type (τ,τ') is a triple consisting of a nonempty set A together with a sequence $(f_i^{\mathcal{A}})_{i\in I}$ of n_i-ary operations on A and a sequence $(\gamma_j^{\mathcal{A}})_{j\in J}$ of n_j-ary relations on A, i.e., $\mathcal{A} := \left(A,(f_i^{\mathcal{A}})_{i\in I},(\gamma_j^{\mathcal{A}})_{j\in J}\right)$.

Next, we recall the definition of formulas of type (τ, τ').

Definition 2.2. [7] For a natual number n, an n-ary formula of type (τ, τ') is defined by the following way:

(i) The expression $t_1 \approx t_2$ is an n-ary formula of type (τ, τ') where t_1, t_2 are n-ary terms of type τ.

(ii) The expression $\gamma_j\left(t_1, \ldots, t_{n_j}\right)$ is an n-ary formula of type (τ, τ') where $t_1, \ldots, t_{n_j}$ are n-ary terms of type τ and γ_j is an n_j-ary relation symbol.

(iii) The expression $\neg F$ is an n-ary formula of type (τ, τ') where F is an n-ary formula of type (τ, τ').

(iv) The expression $F_1 \vee F_2$ is an n-ary formula of type (τ, τ') where F_1 and F_2 are n-ary formulas of type (τ, τ').

(v) The expression $\exists x_i(F)$ is an n-ary formula of type (τ, τ').

By $\mathcal{F}_{(\tau,\tau')}\left(W_\tau(X_n)\right)$ we denote the set of all n-ary formulas of type (τ, τ') and

$$\mathcal{F}_{(\tau,\tau')}\left(W_\tau(X)\right) := \bigcup_{n\geq 1}^{\infty} \mathcal{F}_{(\tau,\tau')}\left(W_\tau(X_n)\right)$$

the set of all formulas of type (τ, τ').

It is well known that the superposition operation on the sets of formulas

$$R^n_m : \mathcal{F}_{(\tau,\tau')}\left(W_\tau(X_n)\right) \times \left(W_\tau(X_m)\right)^n \longrightarrow \mathcal{F}_{(\tau,\tau')}\left(W_\tau(X_m)\right)$$

is defined in the following way.

(i) $R^n_m\left(t_1 \approx t_2, s_1, \ldots, s_n\right)$ is the formula $S^m_n\left(t_1, s_1, \ldots, s_n\right) \approx S^m_n\left(t_2, s_1, \ldots, s_n\right)$ if $t_1 \approx t_2$ is an n-ary formula of type (τ, τ') and $s_1, \ldots, s_n$ are m-ary formulas of type (τ, τ').

(ii) $R^n_m\left(\gamma_j\left(t_1, \ldots, t_{n_j}\right), s_1, \ldots, s_n\right)$ is the formula $\gamma_j\left(S^m_n\left(t_1, s_1, \ldots, s_n\right), \ldots, S^m_n\left(t_{n_i}, s_1, \ldots, s_n\right)\right)$ if $\gamma_j\left(t_1, \ldots, t_{n_j}\right)$ is an n-ary formula of type (τ, τ') and $s_1, \ldots, s_n$ are m-ary formulas of type (τ, τ').

(iii) $R^n_m\left(\neg F, s_1, \ldots, s_n\right)$ is the formula $\neg R^n_m\left(F, s_1, \ldots, s_n\right)$ if $\neg F$ is an n-ary formula of type (τ, τ') and $s_1, \ldots, s_n$ are m-ary formulas of type (τ, τ').

(iv) $R^n_m\left(F_1 \vee F_2, s_1, \ldots, s_n\right)$ is the formula $R^n_m\left(F_1, s_1, \ldots, s_n\right) \vee R^n_m\left(F_2, s_1, \ldots, s_n\right)$ if $F_1 \vee F_2$ is an n-ary formula of type (τ, τ') and $s_1, \ldots, s_n$ are m-ary formulas of type (τ, τ').

(v) $R^n_m\left(\exists x_i(F), s_1, \ldots, s_n\right)$ is the formula $\exists x_i\left(R^n_m\left(F, s_1, \ldots, s_n\right)\right)$ if $\exists x_i(F)$ is an n-ary formula of type (τ, τ') and $s_1, \ldots, s_n$ are m-ary formulas of type (τ, τ').

Next, we form the algebra
$Formclone\,(\tau, \tau') := \left(\left(W_\tau(X_n) \cup \mathcal{F}_{(\tau,\tau')}\left(W_\tau(X_n)\right)\right)_{n\geq 1}, (R^n_m)_{m,n\geq 1}, (x_i)_{1\leq i\leq n, i,n\in\mathbb{N}}\right)$.
This algebraic structure is called the *clone of formulas* of type (τ, τ').

In 2008, the concept of non-deterministic superposition of terms of type τ was introduced as a superposition of sets of terms of type τ (of tree languages) as follows [8]. Let $P(W_\tau(X_n))$ be the power set of $W_\tau(X_n)$. Then the operations

$$\hat{S}^n_m : P(W_\tau(X_n)) \times P(W_\tau(X_m))^n \longrightarrow P(W_\tau(X_m))$$

for $m, n \in \mathbb{N}^+$ is defined as follows:

(i) If $B = \{x_j\}$ for $1 \leq j \leq n$, then $\hat{S}^n_m\left(B, B_1, \ldots, B_n\right) := B_j$.

(ii) If $B = \{f_i(t_1, \ldots, t_{n_i})\}$ and assume that $\hat{S}^n_m(\{t_j\}, B_1, \ldots, B_n)$ for $1 \le j \le n$ are already defined, then
$\hat{S}^n_m(B, B_1, \ldots, B_n) := \{f_i(r_1, \ldots, r_{n_i}) \mid r_j \in \hat{S}^n_m(\{t_j\}, B_1, \ldots, B_n)\}$.

(iii) If B is an arbitrary non-singleton subset of $W_\tau(X_n)$ and $B \neq \emptyset$, then
$\hat{S}^n_m(B, B_1, \ldots, B_n) := \bigcup\limits_{b \in B} \hat{S}^n_m(\{b\}, B_1, \ldots, B_n)$.

If one of the sets $B, B_1, \ldots, B_n$ is empty, then $\hat{S}^n_m(B, B_1, \ldots, B_n) := \emptyset$.

Then we consider the heterogeneous algebra

$$\mathcal{P} - \text{clone}\tau := \left((P(W_\tau(X_n)))_{n \in \mathbb{N}^+}; \left(\hat{S}^n_m\right)_{m,n \in \mathbb{N}^+}, (\{x_i\})_{1 \le i \le m, n \in \mathbb{N}^+}\right)$$

where $\mathcal{P} - \text{clone}\tau$ is called the *power clone* of type τ [8]. The power clone of type τ satisfies the following conditions:

(PC1) $\hat{S}^p_m(B, \hat{S}^n_m(A_1, B_1, \ldots, B_n), \ldots, \hat{S}^n_m(A_p, B_1, \ldots, B_n))$
$= \hat{S}^n_m(\hat{S}^p_n(B, A_1, \ldots, A_p), B_1, \ldots, B_n)$

where $A_1, \ldots, A_p \in P(W_\tau(X_n)), B_1, \ldots, B_n \in P(W_\tau(X_m))$.

(PC2) $\hat{S}^n_m(\{x_i\}, B_1, \ldots, B_n) = B_i$ where $B_1, \ldots, B_n \in P(W_\tau(X_m))$.

(PC3) $\hat{S}^n_n(B, \{x_1\}, \ldots, \{x_n\}) = B$.

3. Superposition of set of formulas

First of all, we generalize the definition of superposition of sets of terms to tree languages of formulas. The superposition operations from single formulas to sets of formulas, i.e. tree languages of formulas, are extended. Let $P\left(\mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))\right)$ be the power set of $\mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))$. Then the operation

$$\hat{R}^n_m : P\left(\mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))\right) \times P(W_\tau(X_m))^n \longrightarrow P\left(\mathcal{F}_{(\tau,\tau')}(W_\tau(X_m))\right)$$

for $m, n \in \mathbb{N}^+$ and $B_1, \ldots, B_n \in P(W_\tau(X_m))$ is defined as follows:

(i) $\hat{R}^n_m(\{s \approx t\}, B_1, \ldots, B_n) := \{r_1 \approx r_2 \mid r_1 \in \hat{S}^n_m(\{s\}, B_1, \ldots, B_n)$ and $r_2 \in \hat{S}^n_m(\{t\}, B_1, \ldots, B_n)\}$.

(ii) $\hat{R}^n_m\left(\{\gamma_j\left(t_1, \ldots, t_{n_j}\right)\}, B_1, \ldots, B_n\right) :=$
$\{\gamma_j\left(v_1, \ldots, v_{n_j}\right) \mid v_k \in \hat{S}^n_m(\{t_k\}, B_1, \ldots, B_n)$ for all $1 \le k \le n_j\}$.

(iii) $\hat{R}^n_m(\{\neg F\}, B_1, \ldots, B_n) := \{\neg Q \mid Q \in \hat{R}^n_m(\{F\}, B_1, \ldots, B_n)\}$.

(iv) $\hat{R}^n_m(\{F_1 \vee F_2\}, B_1, \ldots, B_n) := \{Q_1 \vee Q_2 \mid Q_1 \in \hat{R}^n_m(\{F_1\}, B_1, \ldots, B_n)$ and $Q_2 \in \hat{R}^n_m(\{F_2\}, B_1, \ldots, B_n)\}$.

(v) $\hat{R}^n_m(\{\exists x_i(F)\}, B_1, \ldots, B_n) := \{\exists x_i(Q) \mid Q \in \hat{R}^n_m(\{F\}, B_1, \ldots, B_n)\}$.

(vi) If B is an arbitrary non-singleton subset of $\mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))$ and $B \neq \emptyset$, then
$\hat{R}^n_m(B, B_1, \ldots, B_n) := \bigcup\limits_{b \in B} \hat{R}^n_m(\{b\}, B_1, \ldots, B_n)$.

If one of the sets $B, B_1, \ldots, B_n$ is empty, then $\hat{R}^n_m(B, B_1, \ldots, B_n) := \emptyset$.

We now provide a concrete example of computation with this power operation.

Example 3.1. For a binary operation symbol f and a binary relation symbol γ of type $(\tau, \tau') = (2, 2)$. Consider
$\hat{R}^3_4 : P\left(\mathcal{F}_{(2,2)}\left(W_{(2)}(X_3)\right)\right) \times P(W_{(2)}(X_4))^3 \longrightarrow P\left(\mathcal{F}_{(2,2)}\left(W_{(2)}(X_4)\right)\right)$.
Then we have

(i) $\hat{R}^3_4(\{x_1 \approx x_3\}, \{x_4\}, \{f(x_2, x_4)\}, \{x_3, f(x_4, x_1)\})$
$= \{r_1 \approx r_2 \mid r_1 \in \hat{S}^3_4(\{x_1\}, \{x_4\}, \{f(x_2, x_4)\}, \{x_3, f(x_4, x_1)\})$ and
$r_2 \in \hat{S}^3_4(\{x_3\}, \{x_4\}, \{f(x_2, x_4)\}, \{x_3, f(x_4, x_1)\})\}$
$= \{r_1 \approx r_2 \mid r_1 \in \{x_4\}$ and $r_2 \in \{x_3, f(x_4, x_1)\}\}$
$= \{x_4 \approx x_3, x_4 \approx f(x_4, x_1)\}$.

(ii) $\hat{R}^3_4(\{\gamma(x_2, x_3)\}, \{x_4\}, \{f(x_2, x_4)\}, \{x_3, f(x_4, x_1)\})$
$= \{\gamma(v_1, v_2) \mid v_1 \in \hat{S}^3_4(\{x_2\}, \{x_4\}, \{f(x_2, x_4)\}, \{x_3, f(x_4, x_1)\})$ and
$v_2 \in \hat{S}^3_4(\{x_3\}, \{x_4\}, \{f(x_2, x_4)\}, \{x_3, f(x_4, x_1)\})\}$
$= \{\gamma(v_1, v_2) \mid v_1 \in \{f(x_2, x_4)\}$ and $v_2 \in \{x_3, f(x_4, x_1)\}\}$
$= \{\gamma(f(x_2, x_4), x_3), \gamma(f(x_2, x_4), f(x_4, x_1))\}$.

(iii) $\hat{R}^3_4(\{\neg(x_1 \approx x_3)\}, \{x_4\}, \{f(x_2, x_4)\}, \{x_3, f(x_4, x_1)\})$
$= \{\neg Q \mid Q \in \hat{R}^3_4(\{x_1 \approx x_3\}, \{x_4\}, \{f(x_2, x_4)\}, \{x_3, f(x_4, x_1)\})\}$
$= \{\neg Q \mid Q \in \{x_4 \approx x_3, x_4 \approx f(x_4, x_1)\}\}$
$= \{\neg(x_4 \approx x_3), \neg(x_4 \approx f(x_4, x_1))\}$.

(iv) $\hat{R}^3_4(\{(x_1 \approx x_3) \vee \gamma(x_2, x_3)\}, \{x_4\}, \{f(x_2, x_4)\}, \{x_3, f(x_4, x_1)\})$
$= \{Q_1 \vee Q_2 \mid Q_1 \in \hat{R}^3_4(\{x_1 \approx x_3\}, \{x_4\}, \{f(x_2, x_4)\}, \{x_3, f(x_4, x_1)\})$ and
$Q_2 \in \hat{R}^3_4(\{\gamma(x_2, x_3)\}, \{x_4\}, \{f(x_2, x_4)\}, \{x_3, f(x_4, x_1)\})\}$
$= \{Q_1 \vee Q_2 \mid Q_1 \in \{x_4 \approx x_3, x_4 \approx f(x_4, x_1)\}$ and
$Q_2 \in \{\gamma(f(x_2, x_4), x_3), \gamma(f(x_2, x_4), f(x_4, x_1))\}\}$
$= \{(x_4 \approx x_3) \vee (\gamma(f(x_2, x_4), x_3)), (x_4 \approx x_3) \vee (\gamma(f(x_2, x_4), f(x_4, x_1))), (x_4 \approx f(x_4, x_1)) \vee (\gamma(f(x_2, x_4), x_3)), (x_4 \approx f(x_4, x_1)) \vee (\gamma(f(x_2, x_4), f(x_4, x_1)))\}$.

Next, we form the algebra

$$\mathcal{P} - \text{Formclone}(\tau, \tau') := \Big((P(W_\tau(X_n)))_{n \in \mathbb{N}^+}, \left(P\left(\mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))\right)\right)_{n \in \mathbb{N}^+}; \left(\hat{S}^n_m\right)_{m,n \in \mathbb{N}^+}, \left(\hat{R}^n_m\right)_{m,n \in \mathbb{N}^+}, (\{x_i\})_{1 \le i \le m, n \in \mathbb{N}^+} \Big).$$

The following theorem is a primary result that describes properties of $\mathcal{P}-$Formclone(τ, τ').

Theorem 3.2. *The algebra $\mathcal{P} -$ Formclone(τ, τ') satisfies the equations (PFC1), (PFC3) where*

(PFC1) $\hat{R}^n_m\left(\hat{R}^p_n(G, A_1, \ldots, A_p), B_1, \ldots, B_n\right)$
$$= \hat{R}^p_n\left(G, \hat{R}^n_m(A_1, B_1, \ldots, B_n), \ldots, \hat{R}^n_m(A_1, B_1, \ldots, B_n)\right)$$
where $G \in P\left(\mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))\right)$, $A_1, \ldots, A_p \in P(W_\tau(X_n))$ *and* $B_1, \ldots, B_n \in P(W_\tau(X_m))$.

(PFC3) $\hat{R}^n_m(G, \{x_1\}, \ldots, \{x_n\}) = G$ *where* $G \in P\left(\mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))\right)$.

Proof. (PFC1): Let $G \in P\left(\mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))\right)$. If G is empty, then the proof is clearly true. If G is non-empty, then we can prove by the definition of a formula G of tree languages.

(i) If $G = \{s \approx t\}$, then
$\hat{R}^n_m\left(\hat{R}^p_n(\{s \approx t\}, A_1, \ldots, A_p), B_1, \ldots, B_n\right)$
$= \hat{R}^n_m(\{r_1 \approx r_2 \mid r_1 \in \hat{S}^p_n(\{s\}, A_1, \ldots, A_p)$ and
$r_2 \in \hat{S}^p_n(\{t\}, A_1, \ldots, A_p)\}, B_1, \ldots, B_n)$
$= \{r_3 \approx r_4 \mid r_3 \in \hat{S}^n_m\left(\hat{S}^p_n(\{s\}, A_1, \ldots, A_p), B_1, \ldots, B_n\right)$ and
$r_4 \in \hat{S}^n_m\left(\hat{S}^p_n(\{t\}, A_1, \ldots, A_p), B_1, \ldots, B_n\right)\}$

$= \{r_3 \approx r_4 \mid r_3 \in \hat{S}_n^p \left(\{s\}, \hat{S}_m^n\left(A_1, B_1, \ldots, B_n\right), \ldots, \hat{S}_m^n\left(A_p, B_1, \ldots, B_n\right)\right)$ and
$r_4 \in \hat{S}_n^p \left(\{t\}, \hat{S}_m^n\left(A_1, B_1, \ldots, B_n\right), \ldots, \hat{S}_m^n\left(A_p, B_1, \ldots, B_n\right)\right)\}$
$= \hat{R}_n^p \left(\{s \approx t\}, \hat{R}_m^n\left(A_1, B_1, \ldots, B_n\right), \ldots, \hat{R}_m^n\left(A_p, B_1, \ldots, B_n\right)\right).$

(ii) If $G = \{\gamma_j(t_1, \ldots, t_{n_j})\}$, then
$\hat{R}_m^n \left(\hat{R}_n^p\left(\{\gamma_j(t_1, \ldots, t_{n_j})\}, A_1, \ldots, A_p\right), B_1, \ldots, B_n\right)$
$= \hat{R}_m^n \left(\{\gamma_j(v_1, \ldots, v_{n_j}) \mid v_k \in \hat{S}_n^p\left(\{t_k\}, A_1, \ldots, A_p\right) \text{ for all } 1 \leq k \leq n_j\}, B_1, \ldots, B_n\right)$
$= \{\gamma_j(u_1, \ldots, u_{n_j}) \mid u_k \in \hat{S}_m^n \left(\hat{S}_n^p\left(\{t_k\}, A_1, \ldots, A_p\right), B_1, \ldots, B_n\right)$
for all $1 \leq k \leq n_j\}$
$= \{\gamma_j(u_1, \ldots, u_{n_j}) \mid u_k \in \hat{S}_n^p \left(\{t_k\}, \hat{S}_m^n(A_1, B_1, \ldots, B_n), \ldots, \hat{S}_m^n(A_p, B_1, \ldots, B_n)\right)\}$
$= \hat{R}_n^p \left(\{\gamma_j(t_1, \ldots, t_{n_j})\}, \hat{R}_m^n(A_1, B_1, \ldots, B_n), \ldots, \hat{R}_m^n(A_p, B_1, \ldots, B_n)\right).$

(iii) If $G = \{\neg F\}$ and assume that (PFC1) satisfied for $\{F\}$, then
$\hat{R}_m^n \left(\hat{R}_n^p\left(\{\neg F\}, A_1, \ldots, A_p\right), B_1, \ldots, B_n\right)$
$= \hat{R}_m^n \left(\{\neg Q \mid Q \in \hat{R}_n^p\left(\{F\}, A_1, \ldots, A_p\right)\}, B_1, \ldots, B_n\right)$
$= \{\neg M \mid M \in \hat{R}_m^n \left(\hat{R}_n^p\left(\{F\}, A_1, \ldots, A_p\right), B_1, \ldots, B_n\right)\}$
$= \{\neg M \mid M \in \hat{R}_m^n \left(\{F\}, \hat{R}_m^n\left(A_1, B_1, \ldots, B_n\right), \ldots, \hat{R}_m^n\left(A_p, B_1, \ldots, B_n\right)\right)\}$
(by (i),(ii))
$= \hat{R}_n^p \left(\{\neg F\}, \hat{R}_m^n(A_1, B_1, \ldots, B_n), \ldots, \hat{R}_m^n(A_p, B_1, \ldots, B_n)\right).$

(iv) If $G = \{F_1 \vee F_2\}$ and assume that $\{F_i\}$ satisfies (PFC1); $i \in \{1, 2\}$, then
$\hat{R}_m^n \left(\hat{R}_n^p\left(\{F_1 \vee F_2\}, A_1, \ldots, A_p\right), B_1, \ldots, B_n\right)$
$= \hat{R}_m^n \left(\{Q_1 \vee Q_2 \mid Q_j \in \hat{R}_n^p\left(\{F_j\}, A_1, \ldots, A_p\right); j = 1, 2\}, B_1, \ldots, B_n\right)$
$= \{M_1 \vee M_2 \mid M_j \in \hat{R}_m^n \left(\hat{R}_n^p\left(\{F_j\}, A_1, \ldots, A_p\right), B_1, \ldots, B_n\right); j = 1, 2\}$
$= \{M_1 \vee M_2 \mid M_j \in \hat{R}_n^p \left(\{F_j\}, \hat{R}_m^n\left(A_1, B_1, \ldots, B_n\right), \ldots, \hat{R}_m^n\left(A_p, B_1, \ldots, B_n\right)\right);$
$j = 1, 2\}$
$= \hat{R}_n^p \left(\{F_1 \vee F_2\}, \hat{R}_m^n(A_1, B_1, \ldots, B_n), \ldots, \hat{R}_m^n(A_p, B_1, \ldots, B_n)\right).$

(v) If $G = \{\exists x_i(F)\}$ and assume that a set $\{F\}$ satisfies (PFC1), then
$\hat{R}_m^n \left(\hat{R}_n^p\left(\{\exists x_i(F)\}, A_1, \ldots, A_p\right), B_1, \ldots, B_n\right)$
$= \hat{R}_m^n \left(\{\exists x_i(Q) \mid Q \in \hat{R}_n^p\left(\{F\}, A_1, \ldots, A_p\right)\}, B_1, \ldots, B_n\right)$
$= \{\exists x_i(M) \mid M \in \hat{R}_m^n \left(\hat{R}_n^p\left(\{F\}, A_1, \ldots, A_p\right), B_1, \ldots, B_n\right)\}$
$= \{\exists x_i(M) \mid M \in \hat{R}_n^p \left(\{F\}, \hat{R}_m^n\left(A_1, B_1, \ldots, B_n\right), \ldots, \hat{R}_m^n\left(A_p, B_1, \ldots, B_n\right)\right)\}$
(by (i),(ii))
$= \hat{R}_n^p \left(\{\exists x_i(F)\}, \hat{R}_m^n(A_1, B_1, \ldots, B_n), \ldots, \hat{R}_m^n(A_p, B_1, \ldots, B_n)\right).$

(vi) If G is an arbitrary non-singleton subset of $\mathcal{F}_{(\tau,\tau')}\left(W_\tau(X_n)\right)$ and $G \neq \emptyset$, then
$\hat{R}_m^n \left(\hat{R}_n^p\left(G, A_1, \ldots, A_p\right), B_1, \ldots, B_n\right)$

$$
\begin{aligned}
&= \hat{R}_m^n\left(\bigcup_{b\in G}\hat{R}_n^p\left(\{b\}, A_1, \ldots, A_p\right), B_1, \ldots, B_n\right)\\
&= \bigcup_{b\in G}\hat{R}_m^n\left(\hat{R}_n^p\left(\{b\}, A_1, \ldots, A_p\right), B_1, \ldots, B_n\right)\\
&= \bigcup_{b\in G}\hat{R}_n^p\left(\{b\}, \hat{R}_m^n\left(A_1, B_1, \ldots, B_n\right), \ldots, \hat{R}_m^n\left(A_p, B_1, \ldots, B_n\right)\right) \quad \text{(by (i),(ii))}\\
&= \hat{R}_n^p\left(G, \hat{R}_m^n\left(A_1, B_1, \ldots, B_n\right), \ldots, \hat{R}_m^n\left(A_p, B_1, \ldots, B_n\right)\right).
\end{aligned}
$$

(PFC3): Let $G \in P\left(\mathcal{F}_{(\tau,\tau')}\left(W_\tau(X_n)\right)\right)$. If G is empty, then the proof is clearly true. If G is non-empty, then we can prove by the definition of a formula G of tree languages.

(i) If $G = \{s \approx t\}$, then
$$
\begin{aligned}
&\hat{R}_m^n\left(\{s \approx t\}, \{x_1\}, \ldots, \{x_n\}\right)\\
&= \{r_1 \approx r_2 \mid r_1 \in \hat{S}_m^n\left(\{s\}, \{x_1\}, \ldots, \{x_n\}\right) \text{ and } r_2 \in \hat{S}_m^n\left(\{t\}, \{x_1\}, \ldots, \{x_n\}\right)\}\\
&= \{r_1 \approx r_2 \mid r_1 \in \{s\}, r_2 \in \{t\}\}\\
&= \{s \approx t\}.
\end{aligned}
$$

(ii) If $G = \{\gamma_j(t_1, \ldots, t_{n_j})\}$, then
$$
\begin{aligned}
&\hat{R}_m^n\left(\{\gamma_j(t_1, \ldots, t_{n_j})\}, \{x_1\}, \ldots, \{x_n\}\right)\\
&= \{\gamma_j(v_1, \ldots, v_{n_j}) \mid v_k \in \hat{S}_m^n\left(\{t_k\}, \{x_1\}, \ldots, \{x_n\}\right) \text{ for all } 1 \le k \le n_j\}\\
&= \{\gamma_j(v_1, \ldots, v_{n_j}) \mid v_k \in \{t_k\} \text{ for all } 1 \le k \le n_j\}\\
&= \{\gamma_j(t_1, \ldots, t_{n_j})\}.
\end{aligned}
$$

(iii) If $G = \{\neg F\}$, then
$$
\begin{aligned}
\hat{R}_m^n\left(\{\neg F\}, \{x_1\}, \ldots, \{x_n\}\right) &= \{\neg Q \mid Q \in \hat{R}_m^n\left(\{F\}, \{x_1\}, \ldots, \{x_n\}\right)\}\\
&= \{\neg Q \mid Q \in \{F\}\}\\
&= \{\neg F\}.
\end{aligned}
$$

(iv) If $G = \{F_1 \vee F_2\}$, then
$$
\begin{aligned}
\hat{R}_m^n\left(\{F_1 \vee F_2\}, \{x_1\}, \ldots, \{x_n\}\right) &= \{Q_1 \vee Q_2 \mid Q_j \in \hat{R}_m^n\left(\{F_j\}, \{x_1\}, \ldots, \{x_n\}\right); j = 1, 2\}\\
&= \{Q_1 \vee Q_2 \mid Q_j \in \{F_j\}; j = 1, 2\}\\
&= \{F_1 \vee F_2\}.
\end{aligned}
$$

(v) If $G = \{\exists x_i(F)\}$, then
$$
\begin{aligned}
\hat{R}_m^n\left(\{\exists x_i(F)\}, \{x_1\}, \ldots, \{x_n\}\right) &= \{\exists x_i(Q) \mid Q \in \hat{R}_m^n\left(\{F\}, \{x_1\}, \ldots, \{x_n\}\right)\}\\
&= \{\exists x_i(Q) \mid Q \in \{F\}\}\\
&= \{\exists x_i(F)\}.
\end{aligned}
$$

(vi) If G is an arbitrary non-singleton subset of $\mathcal{F}_{(\tau,\tau')}\left(W_\tau(X_n)\right)$ and $G \neq \emptyset$, then
$$
\hat{R}_m^n\left(G, \{x_1\}, \ldots, \{x_n\}\right) = \bigcup_{b\in G}\hat{R}_m^n\left(\{b\}, \{x_1\}, \ldots, \{x_n\}\right) = G.
$$
■

4. Non-deterministic Hypersubstitutions for Algebraic Systems

Non-deterministic hypersubstitutions map operation symbols to sets of terms and were considered in [8]. First, we will introduce the concept of a non-deterministic hypersubstitution for algebraic systems.

Definition 4.1. Any mapping

$$
\sigma^{nd} : \{f_i \mid i \in I\} \cup \{\gamma_j \mid j \in J\} \longrightarrow P(W_\tau(X_n)) \cup P\left(\mathcal{F}_{(\tau,\tau')}\left(W_\tau(X_n)\right)\right)
$$

which maps operation symbols to tree languages and maps relational symbols to set of formulas and preserves arities, is called a non-deterministic hypersubstitution for algebraic systems of type (τ,τ') (for short, nd-hypersubstitution for algebraic systems). Let $Hyp^{nd}(\tau,\tau')$ be the collection of all nd-hypersubstitutions for algebraic systems of type (τ,τ').

We define the extension of an nd-hypersubstitution σ^{nd} for algebraic systems

$$\hat{\sigma}^{nd} : P(W_\tau(X_n)) \cup P\left(\mathcal{F}_{(\tau,\tau')}\left(W_\tau(X_n)\right)\right) \longrightarrow P(W_\tau(X_n)) \cup P\left(\mathcal{F}_{(\tau,\tau')}\left(W_\tau(X_n)\right)\right)$$

by the following definition.

Definition 4.2. Let $\sigma^{nd} \in Hyp^{nd}(\tau,\tau')$. Then we define $\hat{\sigma}^{nd}$ as following:

(i) $\hat{\sigma}^{nd}[\emptyset] := \emptyset$.

(ii) $\hat{\sigma}^{nd}[\{x_k\}] := \{x_k\}$ for every variable $x_k \in X_n$.

(iii) For $t = f_i(t_1,\ldots,t_{n_i}) \in W_\tau(X_n)$ we let

$$\hat{\sigma}^{nd}\left[\{f_i(t_1,\ldots,t_{n_i})\}\right] := \hat{S}^{n_i}_n\left(\sigma^{nd}(f_i), \hat{\sigma}^{nd}[\{t_1\}],\ldots,\hat{\sigma}^{nd}[\{t_{n_i}\}]\right)$$

if we inductively assume that $\hat{\sigma}^{nd}[\{t_j\}]$, $1 \le j \le n_i$ are already defined.

(iv) $\hat{\sigma}^{nd}[B] := \bigcup\{\hat{\sigma}^{nd}[\{t\}] \mid t \in B\}$ if B is an arbitrary non-singleton subset of $W_\tau(X_n)$ and $B \neq \emptyset$.

(v) $\hat{\sigma}^{nd}[\{s \approx t\}] := \left\{u \approx v \mid u \in \hat{\sigma}^{nd}[\{s\}], v \in \hat{\sigma}^{nd}[\{t\}]\right\}$.

(vi) $\hat{\sigma}^{nd}\left[\{\gamma_j(t_1,\ldots,t_{n_j})\}\right] := \hat{R}^{n_j}_n\left(\sigma^{nd}\left(\gamma_j\right), \hat{\sigma}^{nd}[\{t_1\}],\ldots,\hat{\sigma}^{nd}\left[\{t_{n_j}\}\right]\right)$

if we inductively assume that $\hat{\sigma}^{nd}[\{t_j\}]$, $1 \le j \le n_i$ are already defined.

(vii) $\hat{\sigma}^{nd}[\{\neg F\}] := \{\neg Q \mid Q \in \hat{\sigma}^{nd}[\{F\}]\}$.

(viii) $\hat{\sigma}^{nd}[\{F_1 \vee F_2\}] := \{Q_1 \vee Q_2 \mid Q_1 \in \hat{\sigma}^{nd}[\{F_1\}], Q_2 \in \hat{\sigma}^{nd}[\{F_2\}]\}$.

(ix) $\hat{\sigma}^{nd}[\{\exists x_i(F)\}] := \{\exists x_i(Q) \mid Q \in \hat{\sigma}^{nd}[\{F\}]\}$.

(x) $\hat{\sigma}^{nd}[B] := \bigcup\limits_{b\in B} \hat{\sigma}^{nd}[\{b\}]$ if B is an arbitrary non-singleton subset of $\mathcal{F}_{(\tau,\tau')}\left(W_\tau(X_n)\right)$ and $B \neq \emptyset$.

Next, we give an example of an nd-hypersubstitution for algebraic systems.

Example 4.3. For a ternary operation symbol f and a ternary relation symbol γ of type $(\tau,\tau') = (3,3)$. Let $\sigma^{nd} : \{f\} \cup \{\gamma\} \longrightarrow P(W_{(3)}(X_3)) \cup P\left(\mathcal{F}_{(3,3)}\left(W_{(3)}(X_3)\right)\right)$ where
$\sigma^{nd}(f) = \{f(x_1,x_2,x_2), f(x_2,x_3,x_3)\}$ and
$\sigma^{nd}(\gamma) = \{x_1 \approx x_3, \neg(\gamma(x_1,x_3,x_1) \vee (x_3 \approx x_2))\}$. Then we have

$$\begin{aligned}
&\hat{\sigma}^{nd}[\{f(x_3,x_3,x_1) \approx x_3\}]\\
&= \{u \approx v \mid u \in \hat{\sigma}^{nd}[\{f(x_3,x_3,x_1)\}] \text{ and } v \in \hat{\sigma}^{nd}[\{x_3\}]\}\\
&= \{u \approx v \mid u \in \hat{S}^3(\sigma^{nd}(f), \hat{\sigma}^{nd}[\{x_3\}], \hat{\sigma}^{nd}[\{x_3\}], \hat{\sigma}^{nd}[\{x_1\}]) \text{ and } v \in \{x_3\}\}\\
&= \{u \approx v \mid u \in \hat{S}^3(\{f(x_1,x_2,x_2), f(x_2,x_3,x_3)\}, \{x_3\}, \{x_3\}, \{x_1\}) \text{ and } v \in \{x_3\}\}\\
&= \{u \approx v \mid u \in \{\hat{S}^3(\{f(x_1,x_2,x_2)\}, \{x_3\}, \{x_3\}, \{x_1\}) \cup\\
&\quad \hat{S}^3(\{f(x_2,x_3,x_3)\}, \{x_3\}, \{x_3\}, \{x_1\})\} \text{ and } v \in \{x_3\}\}\\
&= \{u \approx v \mid u \in \{f(x_3,x_3,x_3), f(x_3,x_1,x_1)\} \text{ and } v \in \{x_3\}\\
&= \{f(x_3,x_3,x_3) \approx x_3, f(x_3,x_1,x_1) \approx x_3\}.
\end{aligned}$$

We can prove that those extensions are endomorphisms of $\mathcal{P}-\text{Formclone}(\tau,\tau')$.

Theorem 4.4. *Let $\sigma^{nd} \in Hyp^{nd}(\tau,\tau')$. Then the following assertions hold:*

(1) $\hat{\sigma}^{nd}\left[\hat{S}^n_m(A,B_1,\ldots,B_n)\right] = \hat{S}^n_m\left(\hat{\sigma}^{nd}[A], \hat{\sigma}^{nd}[B_1],\ldots,\hat{\sigma}^{nd}[B_n]\right)$,

(2) $\hat{\sigma}^{nd}\left[\hat{R}^n_m(G, B_1, \ldots, B_n)\right] = \hat{R}^n_m\left(\hat{\sigma}^{nd}[G], \hat{\sigma}^{nd}[B_1], \ldots, \hat{\sigma}^{nd}[B_n]\right)$

where $A \in P(W_\tau(X_n)$, $B_1, \ldots, B_n \in P(W_\tau(X_m)$ *and* $G \in P\left(\mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))\right)$.

Proof. (1) For any $A \in P(W_\tau(X_n)$. The equation

$$\hat{\sigma}^{nd}\left[\hat{S}^n_m(A, B_1, \ldots, B_n)\right] = \hat{S}^n_m\left(\hat{\sigma}^{nd}[A], \hat{\sigma}^{nd}[B_1], \ldots, \hat{\sigma}^{nd}[B_n]\right)$$

was proved in [8].

(2) For any $m, n \in \mathbb{N}^+$, for any $B_1, \ldots, B_n \in P(W_\tau(X_m)$ and $G \in P\left(\mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))\right)$. We can prove by the definition of a formula G of tree languages.

(i) If $G = \{s \approx t\}$, then
$\hat{\sigma}^{nd}\left[\hat{R}^n_m(\{s \approx t\}, B_1, \ldots, B_n)\right]$
$= \hat{\sigma}^{nd}\left[\{r_1 \approx r_2 \mid r_1 \in \hat{S}^n_m(\{s\}, B_1, \ldots, B_n) \text{ and } r_2 \in \hat{S}^n_m(\{t\}, B_1, \ldots, B_n)\}\right]$
$= \left\{u \approx v \mid u \in \hat{\sigma}^{nd}\left[\hat{S}^n_m(\{s\}, B_1, \ldots, B_n)\right] \text{ and } v \in \hat{\sigma}^{nd}\left[\hat{S}^n_m(\{t\}, B_1, \ldots, B_n)\right]\right\}$
$= \{u \approx v \mid u \in \hat{S}^n_m\left(\hat{\sigma}^{nd}[\{s\}], \hat{\sigma}^{nd}[B_1], \ldots, \hat{\sigma}^{nd}[B_n]\right)$
and $v \in \hat{S}^n_m\left(\hat{\sigma}^{nd}[\{t\}], \hat{\sigma}^{nd}[B_1], \ldots, \hat{\sigma}^{nd}[B_n]\right)\}$
$= \hat{R}^n_m\left(\{x \approx y \mid x \in \hat{\sigma}^{nd}[\{s\}] \text{ and } y \in \hat{\sigma}^{nd}[\{t\}]\}, \hat{\sigma}^{nd}[B_1], \ldots, \hat{\sigma}^{nd}[B_n]\right)$
$= \hat{R}^n_m\left(\hat{\sigma}^{nd}[\{s \approx t\}], \hat{\sigma}^{nd}[B_1], \ldots, \hat{\sigma}^{nd}[B_n]\right).$

(ii) If $G = \{\gamma_j(t_1, \ldots, t_{n_j})\}$, then
$\hat{\sigma}^{nd}\left[\hat{R}^n_m\left(\{\gamma_j(t_1, \ldots, t_{n_j})\}, B_1, \ldots, B_n\right)\right]$
$= \hat{\sigma}^{nd}\left[\{\gamma_j(v_1, \ldots, v_{n_j}) \mid v_k \in \hat{S}^n_m(\{t_k\}, B_1, \ldots, B_n); 1 \le k \le n_j\}\right]$
$= \hat{R}^{n_j}_n\left(\sigma(\gamma_j), \hat{\sigma}^{nd}\left[\hat{S}^n_m(\{t_1\}, B_1, \ldots, B_n)\right], \ldots, \hat{\sigma}^{nd}\left[\hat{S}^n_m(\{t_{n_j}\}, B_1, \ldots, B_n)\right]\right)$
$= \hat{R}^{n_j}_n(\sigma(\gamma_j), \hat{S}^n_m\left(\hat{\sigma}^{nd}[\{t_1\}], \hat{\sigma}^{nd}[B_1], \ldots, \hat{\sigma}^{nd}[B_n]\right), \ldots,$
$\hat{S}^n_m\left(\hat{\sigma}^{nd}[\{t_{n_j}\}], \hat{\sigma}^{nd}[B_1], \ldots, \hat{\sigma}^{nd}[B_n]\right))$
$= \hat{R}^n_m\left(\hat{R}^{n_j}_m\left(\sigma(\gamma_j), \hat{\sigma}^{nd}[\{t_1\}], \ldots, \hat{\sigma}^{nd}[\{t_{n_j}\}]\right), \hat{\sigma}^{nd}[B_1], \ldots, \hat{\sigma}^{nd}[B_n]\right)$
$= \hat{R}^n_m\left(\hat{\sigma}^{nd}[\{\gamma_j(t_1, \ldots, t_{n_j})\}], \hat{\sigma}^{nd}[B_1], \ldots, \hat{\sigma}^{nd}[B_n]\right).$

(iii) If $G = \{\neg F\}$, then
$\hat{\sigma}^{nd}\left[\hat{R}^n_m(\{\neg F\}, B_1, \ldots, B_n)\right]$
$= \hat{\sigma}^{nd}\left[\left\{\neg Q \mid Q \in \hat{R}^n_m(\{F\}, B_1, \ldots, B_n)\right\}\right]$
$= \left\{\neg M \mid M \in \hat{\sigma}^{nd}[\hat{R}^n_m(\{F\}, B_1, \ldots, B_n)]\right\}$
$= \left\{\neg M \mid M \in \hat{R}^n_m\left(\hat{\sigma}^{nd}[\{F\}], \hat{\sigma}^{nd}[B_1], \ldots, \hat{\sigma}^{nd}[B_n]\right)\right\}$
$= \hat{R}^n_m\left(\hat{\sigma}^{nd}[\{\neg F\}], \hat{\sigma}^{nd}[B_1], \ldots, \hat{\sigma}^{nd}[B_n]\right).$

(iv) If $G = \{F_1 \vee F_2\}$, then
$\hat{\sigma}^{nd}\left[\hat{R}^n_m(\{F_1 \vee F_2\}, B_1, \ldots, B_n)\right]$
$= \hat{\sigma}^{nd}\left[\{Q_1 \vee Q_2 \mid Q_j \in \hat{R}^n_m(\{F_j\}, B_1, \ldots, B_n); j = 1, 2\}\right]$
$= \{M_1 \vee M_2 \mid M_j \in \hat{\sigma}^{nd}\left[\hat{R}^n_m(\{F_j\}, B_1, \ldots, B_n)\right]; j = 1, 2\}$
$= \{M_1 \vee M_2 \mid M_j \in \hat{R}^n_m\left(\hat{\sigma}^{nd}[\{F_j\}], \hat{\sigma}^{nd}[B_1], \ldots, \hat{\sigma}^{nd}[B_n]\right); j = 1, 2\}$
$= \hat{R}^n_m\left(\hat{\sigma}^{nd}[\{F_1 \vee F_2\}], \hat{\sigma}^{nd}[B_1], \ldots, \hat{\sigma}^{nd}[B_n]\right).$

(v) If $G = \{\exists x_i(F)\}$, then
$\hat{\sigma}^{nd}\left[\hat{R}^n_m(\{\exists x_i(F)\}, B_1, \ldots, B_n)\right]$
$= \hat{\sigma}^{nd}\left[\{\exists x_i(Q) \mid Q \in \hat{R}^n_m(\{F\}, B_1, \ldots, B_n)\}\right]$
$= \{\exists x_i(M) \mid M \in \hat{\sigma}^{nd}\left[\hat{R}^n_m(\{F\}, B_1, \ldots, B_n)\right]\}$
$= \{\exists x_i(M) \mid M \in \hat{R}^n_m\left(\hat{\sigma}^{nd}[\{F\}], \hat{\sigma}^{nd}[B_1], \ldots, \hat{\sigma}^{nd}[B_n]\right)\}$
$= \hat{R}^n_m\left(\hat{\sigma}^{nd}[\{\exists x_i(F)\}], \hat{\sigma}^{nd}[B_1], \ldots, \hat{\sigma}^{nd}[B_n]\right).$

(vi) If A is an arbitrary non-singleton subset of $\mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))$ and $A \neq \emptyset$, then

$$\begin{aligned}\hat{\sigma}^{nd}\left[\hat{R}^n_m(A, B_1, \ldots, B_n)\right] &= \hat{\sigma}^{nd}\left[\bigcup_{a\in A}\hat{R}^n_m(\{a\}, B_1, \ldots, B_n)\right]\\ &= \bigcup_{a\in A}\left\{\hat{\sigma}^{nd}\left[\hat{R}^n_m(\{a\}, B_1, \ldots, B_n)\right]\right\}\\ &= \hat{R}^n_m\left(\hat{\sigma}^{nd}[A], \hat{\sigma}^{nd}[B_1], \ldots, \hat{\sigma}^{nd}[B_n]\right).\end{aligned}$$ ∎

We define a binary operation on the set $Hyp(\tau,\tau')$ of all non-deterministic hypersubstitutions for algebraic systems of type (τ,τ'),

$$\circ_{nd} : Hyp^{nd}(\tau,\tau') \times Hyp^{nd}(\tau,\tau') \longrightarrow Hyp^{nd}(\tau,\tau')$$

by $\sigma_1^{nd} \circ_{nd} \sigma_2^{nd} := \hat{\sigma}_1^{nd} \circ \sigma_2^{nd}$.

An important property of this product $\circ_{nd}$ is the fact that the extension of a product of two nd-hypersubstitutions is equal to the usual composition of the extensions of these both nd-hypersubstitutions (as in the usual case). Then we have

Lemma 4.5. *For any $\sigma_1^{nd}, \sigma_2^{nd} \in Hyp^{nd}(\tau,\tau')$, $\left(\sigma_1^{nd} \circ_{nd} \sigma_2^{nd}\right)\hat{} = \hat{\sigma}_1^{nd} \circ \hat{\sigma}_2^{nd}$.*

Proof. For any $G \in P(W_\tau(X_n))$. The equation $\left(\sigma_1^{nd} \circ_{nd} \sigma_2^{nd}\right)\hat{}[G] = \hat{\sigma}_1^{nd} \circ \hat{\sigma}_2^{nd}[G]$ was proved in [8]. We will show that

$$\left(\sigma_1^{nd} \circ_{nd} \sigma_2^{nd}\right)\hat{}[G] = \hat{\sigma}_1^{nd} \circ \hat{\sigma}_2^{nd}[G]$$

for any $G \in P\left(\mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))\right)$.

(i) If $G = \{s \approx t\}$, then
$\left(\sigma_1^{nd} \circ_{nd} \sigma_2^{nd}\right)\hat{}[\{s \approx t\}]$
$= \{u \approx v \mid u \in \left(\sigma_1^{nd} \circ_{nd} \sigma_2^{nd}\right)\hat{}[\{s\}] \text{ and } v \in \left(\sigma_1^{nd} \circ_{nd} \sigma_2^{nd}\right)\hat{}[\{t\}]\}$
$= \{u \approx v \mid u \in \hat{\sigma}_1^{nd} \circ \hat{\sigma}_2^{nd}[\{s\}] \text{ and } v \in \hat{\sigma}_1^{nd} \circ \hat{\sigma}_2^{nd}[\{t\}]\}$
$= \{u \approx v \mid u \in \hat{\sigma}_1^{nd}\left(\hat{\sigma}_2^{nd}[\{s\}]\right) \text{ and } v \in \hat{\sigma}_1^{nd}\left(\hat{\sigma}_2^{nd}[\{t\}]\right)\}$
$= \hat{\sigma}_1^{nd}\left[\{x \approx y \mid x \in \hat{\sigma}_2^{nd}[\{s\}] \text{ and } y \in \hat{\sigma}_2^{nd}[\{t\}]\}\right]$
$= \hat{\sigma}_1^{nd}\left[\hat{\sigma}_2^{nd}[\{s \approx t\}]\right]$
$= \hat{\sigma}_1^{nd} \circ \hat{\sigma}_2^{nd}[\{s \approx t\}].$

(ii) If $G = \{\gamma_j(t_1, \ldots, t_{n_j})\}$, then
$\left(\sigma_1^{nd} \circ_{nd} \sigma_2^{nd}\right)\hat{}\left[\{\gamma_j(t_1, \ldots, t_{n_j})\}\right]$
$= \hat{R}^{n_j}_n\left(\left(\sigma_1^{nd} \circ_{nd} \sigma_2^{nd}\right)(\gamma_j), \left(\sigma_1^{nd} \circ_{nd} \sigma_2^{nd}\right)\hat{}[\{t_1\}], \ldots, \left(\sigma_1^{nd} \circ_{nd} \sigma_2^{nd}\right)\hat{}[\{t_{n_j}\}]\right)$
$= \hat{R}^{n_j}_n\left(\left(\hat{\sigma}_1^{nd} \circ \sigma_2^{nd}\right)(\gamma_j), \left(\hat{\sigma}_1^{nd} \circ \hat{\sigma}_2^{nd}\right)[\{t_1\}], \ldots, \left(\hat{\sigma}_1^{nd} \circ \hat{\sigma}_2^{nd}\right)[\{t_{n_j}\}]\right)$
$= \hat{R}^{n_j}_n\left(\hat{\sigma}_1^{nd}\left[\sigma_2^{nd}(\gamma_j)\right], \hat{\sigma}_1^{nd}\left(\hat{\sigma}_2^{nd}[\{t_1\}]\right), \ldots, \hat{\sigma}_1^{nd}\left(\hat{\sigma}_2^{nd}[\{t_{n_j}\}]\right)\right)$
$= \hat{\sigma}_1^{nd}\left[\hat{R}^{n_j}_n\left(\sigma_2^{nd}(\gamma_j), \hat{\sigma}_2^{nd}[\{t_1\}], \ldots, \hat{\sigma}_2^{nd}[\{t_{n_j}\}]\right)\right]$

$$= \hat{\sigma}_1^{nd}\left[\hat{\sigma}_2^{nd}\left[\{\gamma_j(t_1,\ldots,t_{n_j})\}\right]\right]$$
$$= \left(\hat{\sigma}_1^{nd}\circ\hat{\sigma}_2^{nd}\right)\left[\{\gamma_j(t_1,\ldots,t_{n_j})\}\right].$$

(iii) If $G = \{\neg F\}$, then

$$\begin{aligned}\left(\sigma_1^{nd}\circ_{nd}\sigma_2^{nd}\right)\hat{}[\{\neg F\}] &= \{\neg Q \mid Q\in\left(\sigma_1^{nd}\circ_{nd}\sigma_2^{nd}\right)\hat{}[\{F\}]\}\\ &= \{\neg Q \mid Q\in\left(\hat{\sigma}_1^{nd}\circ\hat{\sigma}_2^{nd}\right)[\{F\}]\}\\ &= \{\neg Q \mid Q\in\hat{\sigma}_1^{nd}\left[\hat{\sigma}_2^{nd}[\{F\}]\right]\}\\ &= \hat{\sigma}_1^{nd}\left[\{\neg M \mid M\in\hat{\sigma}_2^{nd}[\{F\}]\}\right]\\ &= \hat{\sigma}_1^{nd}\left[\hat{\sigma}_2^{nd}[\{F\}]\right]\\ &= \left(\hat{\sigma}_1^{nd}\circ\hat{\sigma}_2^{nd}\right)[\{F\}].\end{aligned}$$

(iv) If $G = \{F_1\vee F_2\}$, then

$$\begin{aligned}\left(\sigma_1^{nd}\circ_{nd}\sigma_2^{nd}\right)\hat{}[\{F_1\vee F_2\}] &= \{Q_1\vee Q_2 \mid Q_j\in\left(\sigma_1^{nd}\circ_{nd}\sigma_2^{nd}\right)\hat{}[F_j]\ \ ; j=1,2\}\\ &= \{Q_1\vee Q_2 \mid Q_j\in\left(\hat{\sigma}_1^{nd}\circ\hat{\sigma}_2^{nd}\right)[F_j]\ \ j=1,2\}\\ &= \{Q_1\vee Q_2 \mid Q_j\in\hat{\sigma}_1^{nd}\left[\hat{\sigma}_2^{nd}[F_j]\right]\ \ ; j=1,2\}\\ &= \hat{\sigma}_1^{nd}\left[\{M_1\vee M_2 \mid M_j\in\hat{\sigma}_2^{nd}[F_j]\ \ ; j=1,2\}\right]\\ &= \hat{\sigma}_1^{nd}\left[\hat{\sigma}_2^{nd}[\{F_1\vee F_2\}]\right]\\ &= \left(\hat{\sigma}_1^{nd}\circ\hat{\sigma}_2^{nd}\right)[\{F_1\vee F_2\}].\end{aligned}$$

(v) If $G = \{\exists x_i(F)\}$, then

$$\begin{aligned}\left(\sigma_1^{nd}\circ_{nd}\sigma_2^{nd}\right)\hat{}[\{\exists x_i(F)\}] &= \{\exists x_i(Q) \mid Q\in\left(\sigma_1^{nd}\circ_{nd}\sigma_2^{nd}\right)\hat{}[F]\}\\ &= \{\exists x_i(Q) \mid Q\in\left(\hat{\sigma}_1^{nd}\circ\hat{\sigma}_2^{nd}\right)[F]\}\\ &= \{\exists x_i(Q) \mid Q\in\hat{\sigma}_1^{nd}\left[\hat{\sigma}_2^{nd}[F]\right]\}\\ &= \hat{\sigma}_1^{nd}\left[\{\exists x_i(Q) \mid Q\in\hat{\sigma}_2^{nd}[F]\}\right]\\ &= \hat{\sigma}_1^{nd}\left[\hat{\sigma}_2^{nd}[\{\exists x_i(F)\}]\right]\\ &= \left(\hat{\sigma}_1^{nd}\circ\hat{\sigma}_2^{nd}\right)[\{\exists x_i(F)\}].\end{aligned}$$

(vi) If A is an arbitrary non-singleton subset of $\mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))$ and $A\neq\emptyset$, then

$$\begin{aligned}\left(\sigma_1^{nd}\circ_{nd}\sigma_2^{nd}\right)\hat{}[A] &= \bigcup_{a\in A}\left(\sigma_1^{nd}\circ_{nd}\sigma_2^{nd}\right)\hat{}[\{a\}]\\ &= \bigcup_{a\in A}\left(\hat{\sigma}_1^{nd}\circ\hat{\sigma}_2^{nd}\right)[\{a\}]\\ &= \bigcup_{a\in A}\hat{\sigma}_1^{nd}\left[\hat{\sigma}_2^{nd}[\{a\}]\right]\\ &= \hat{\sigma}_1^{nd}\left[\bigcup_{a\in A}\hat{\sigma}_2^{nd}[\{a\}]\right]\\ &= \hat{\sigma}_1^{nd}\left[\hat{\sigma}_2^{nd}\left[\bigcup_{a\in A}[\{a\}]\right]\right]\\ &= \left(\hat{\sigma}_1^{nd}\circ\hat{\sigma}_2^{nd}\right)[A].\end{aligned}$$

∎

Let σ_{id}^{nd} be the nd-hypersubstitution for algebraic systems which is defined by $\sigma_{id}^{nd}(f_i) := \{f_i(x_1,\ldots,x_{n_i})\}$ for all $i\in I$ and $\sigma_{id}^{nd}(\gamma_j) := \{\gamma_j(x_1,\ldots,x_{n_j})\}$ for all $j\in J$. Then we have:

Lemma 4.6. *For any $G\in P(W_\tau(X_n))\cup P\left(\mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))\right)$. Then $\hat{\sigma}_{id}^{nd}[G]=G$.*

Proof. For any $G\in P(W_\tau(X_n))$. The equation $\hat{\sigma}_{id}^{nd}[G]=G$ was proved in [8]. We will show that $\hat{\sigma}_{id}^{nd}[G]=G$ for any $G\in P\left(\mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))\right)$.

(i) If $G=\{s\approx t\}$, then

$$\hat{\sigma}_{id}^{nd}[\{s\approx t\}] = \{u\approx v \mid u\in\hat{\sigma}_{id}^{nd}[\{s\}] \text{ and } v\in\hat{\sigma}_{id}^{nd}[\{t\}]\}$$

$$= \{u \approx v \mid u \in \{s\} \text{ and } v \in \{t\}\}$$
$$= \{s \approx t\}.$$

(ii) If $G = \{\gamma_j(t_1, \ldots, t_{n_j})\}$, then

$$\begin{aligned}\hat{\sigma}_{id}^{nd}[\{\gamma_j(t_1, \ldots, t_{n_j})\}] &= \hat{R}_n^{n_j}\left(\sigma_{id}^{nd}(\gamma_j), \hat{\sigma}_{id}^{nd}[\{t_1\}], \ldots, \hat{\sigma}_{id}^{nd}[\{t_{n_j}\}]\right)\\ &= \hat{R}_n^{n_j}\left(\gamma_j(\{x_1\}, \ldots, \{x_{n_j}\}), \{t_1\}, \ldots, \{t_{n_j}\}\right)\\ &= \{\gamma_j(t_1, \ldots, t_{n_j})\}.\end{aligned}$$

(iii) If $G = \{\neg F\}$, then

$$\hat{\sigma}_{id}^{nd}[\{\neg F\}] = \{\neg Q \mid Q \in \hat{\sigma}_{id}^{nd}[\{F\}]\} = \{\neg Q \mid Q \in \{F\}\} = \{\neg F\}.$$

(iv) If $G = \{F_1 \vee F_2\}$, then

$$\begin{aligned}\hat{\sigma}_{id}^{nd}[\{F_1 \vee F_2\}] &= \{Q_1 \vee Q_2 \mid Q_1 \in \hat{\sigma}_{id}^{nd}[\{F_1\}] \text{ and } Q_2 \in \hat{\sigma}_{id}^{nd}[\{F_2\}]\}\\ &= \{Q_1 \vee Q_2 \mid Q_1 \in \{F_1\} \text{ and } Q_2 \in \{F_2\}\}\\ &= \{F_1 \vee F_2\}.\end{aligned}$$

(v) If $G = \{\exists x_i(F)\}$, then

$$\hat{\sigma}_{id}^{nd}[\{\exists x_i(F)\}] = \{\exists x_i(Q) \mid Q \in \hat{\sigma}_{id}^{nd}[\{F\}]\} = \{\exists x_i(Q) \mid Q \in \{F\}\} = \{\exists x_i(F)\}.$$

(vi) If A is an arbitrary non-singleton subset of $\mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))$ and $A \neq \emptyset$, then

$$\hat{\sigma}_{id}^{nd}[A] = \bigcup_{a \in A} \hat{\sigma}_{id}^{nd}[\{a\}] = \bigcup_{a \in A} \{a\} = A.$$ ■

Hence the nd-hypersubstitution σ_{id}^{nd} is the identity element in $Hyp^{nd}(\tau, \tau')$ with respect to $\circ_{nd}$. Consequently, we can show that $Hyp^{nd}(\tau, \tau')$ forms a monoid.

Theorem 4.7. $\underline{\mathcal{H}yp^{nd}(\tau, \tau')} = \left(Hyp^{nd}(\tau, \tau'); \circ_{nd}, \sigma_{id}^{nd}\right)$ *is a monoid.*

Proof. By Lemma 4.5 and the fact that the usaul composition $\circ$ is associative, it can be shown that $\circ_{nd}$ is an associative binary operation on $Hyp^{nd}(\tau, \tau')$. In fact, for any $\sigma_1^{nd}, \sigma_2^{nd}, \sigma_3^{nd} \in Hyp^{nd}(\tau, \tau')$, we have

$$\begin{aligned}\left(\sigma_1^{nd} \circ_{nd} \sigma_2^{nd}\right) \circ_{nd} \sigma_3^{nd} &= \left(\sigma_1^{nd} \circ_{nd} \sigma_2^{nd}\right)^{\wedge} \circ \sigma_3^{nd}\\ &= \left(\hat{\sigma}_1^{nd} \circ \hat{\sigma}_2^{nd}\right) \circ \sigma_3^{nd}\\ &= \hat{\sigma}_1^{nd} \circ \left(\hat{\sigma}_2^{nd} \circ \sigma_3^{nd}\right)\\ &= \sigma_1^{nd} \circ_{nd} \left(\sigma_2^{nd} \circ_{nd} \sigma_3^{nd}\right).\end{aligned}$$

Let $\sigma^{nd} \in Hyp^{nd}(\tau, \tau')$. Then $\left(\sigma_{id}^{nd} \circ_{nd} \sigma^{nd}\right)(f_i) = \hat{\sigma}_{id}^{nd}\left[\sigma^{nd}(f_i)\right] = \sigma^{nd}(f_i)$ and

$$\begin{aligned}\left(\sigma^{nd} \circ_{nd} \sigma_{id}^{nd}\right)(f_i) &= \hat{\sigma}^{nd}\left[\sigma_{id}^{nd}(f_i)\right]\\ &= \hat{\sigma}^{nd}\left[\{f_i(x_1, \ldots, x_{n_i})\}\right]\\ &= \hat{S}_n^{n_i}\left(\sigma^{nd}(f_i), \hat{\sigma}^{nd}[\{x_1\}], \ldots, \hat{\sigma}^{nd}[\{x_{n_i}\}]\right)\\ &= \hat{S}_n^{n_i}\left(\sigma^{nd}(f_i), \{x_1\}, \ldots, \{x_{n_i}\}\right)\\ &= \sigma^{nd}(f_i).\end{aligned}$$

Also $\left(\sigma_{id}^{nd} \circ_{nd} \sigma^{nd}\right)(\gamma_j) = \hat{\sigma}_{id}^{nd}\left[\sigma^{nd}(\gamma_j)\right] = \sigma^{nd}(\gamma_j)$ and

$$\begin{aligned}\left(\sigma^{nd} \circ_{nd} \sigma_{id}^{nd}\right)(\gamma_j) &= \hat{\sigma}^{nd}\left[\sigma_{id}^{nd}(\gamma_j)\right]\\ &= \hat{\sigma}^{nd}\left[\{\gamma_j(x_1, \ldots, x_{n_j})\}\right]\\ &= \hat{R}_n^{n_j}\left(\sigma^{nd}(\gamma_j), \hat{\sigma}^{nd}[\{x_1\}], \ldots, \hat{\sigma}^{nd}[\{x_{n_j}\}]\right)\\ &= \hat{R}_n^{n_i}\left(\sigma^{nd}(\gamma_j), \{x_1\}, \ldots, \{x_{n_j}\}\right)\\ &= \sigma^{nd}(\gamma_j).\end{aligned}$$

Thus we have σ_{id}^{nd} is an identity element. Therefore $\mathcal{H}yp^{nd}(\tau, \tau') = \left(Hyp^{nd}(\tau, \tau'); \circ_{nd}, \sigma_{id}^{nd}\right)$ is a monoid. ■

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