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Semiabelian Hyperoperations with Applications to Menger Hyperalgebras

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Abstract Functions of many variables and their algebras are generally called multiplace functions and Menger algebras, respectively. The purpose of this paper is to present a particular kind of n-ary operations which is called semiabelian n-ary operations where n is a positive integer. Menger algebras of such n-ary operations, defined on some sets, are constructed and some of their algebraic properties are proposed. Particularly, we really provide the necessary and sufficient conditions under which an abstract Menger algebra can be isomorphically embedded into the algebras of semiabelian n-ary operations. Additionally, based on the theory of hypercompositional algebra, the structure of semiabelian n-ary hyperoperations are further defined. This leads us to give an abstract characterization and representation of Menger hyperalgebras satisfying a certain identity.

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1. Introduction and Preliminaries

It is widely accepted that Menger algebras always play a vital role in different branches of mathematics, espectially in the theory of multiple valued logic and multiplace functions. The study of Menger algebras traces back to the mathematical work of K. Menger [1], while the first attempt to present an interaction of Menger algebras and the theory of multiplace functions is due to W. A. Dudek and V. S. Trokhimenko [2–4]. In recent years, a number of scientific publications of V. S. Trokhimenko in the theory of multiplace functions were memorial collected in [5]. Actually, a *Menger algebra of rank n* for a fixed

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positive integer n is a pair of a nonempty set G with an $(n+1)$-ary operation $\circ$ satisfying the following identity, called superassociative law:

$$\circ(\circ(x, y_1, \ldots, y_n), z_1, \ldots, z_n) = \circ(x, \circ(y_1, z_1, \ldots, z_n), \ldots, \circ(y_n, z_1, \ldots, z_n)),$$

for all $x, y_1, \ldots, y_n, z_1, \ldots, z_n \in G$. Obviously, if $n = 1$, it is a semigroup. Menger algebras of different types have been studied by various authors for a long time, see for example [6–13]. Fundamental properties of Menger algebras were examined in [14]. We now propose a basic example of Menger algebras. The set $\mathbb{R}^+$ of all positive real numbers with the operation $\circ : (\mathbb{R}^+)^{n+1} \to \mathbb{R}^+$, defined by $\circ(x_0, \ldots, x_n) = x_0 \sqrt[n]{x_1 \cdots x_n}$, forms a Menger algebra.

The most natural example of Menger algebras is a Menger algebra of all full n-ary functions or a Menger algebra of all n-ary operations, which has been widely used in mulitivalued calculus. In fact, some primary tools are essentially recalled as the following. Let A^n be the n-th Cartesian product of a nonempty set A. Any mapping from A^n to A is called a *full n-ary function* or an *n-ary operation.* The set of all such mappings is denoted by $T(A^n, A)$. One can consider the *Menger's superposition* on the set $T(A^n, A)$, i.e., an $(n+1)$-ary operation $\mathcal{O} : T(A^n, A)^{n+1} \to T(A^n, A)$ defined by

$$\mathcal{O}(f, g_1, \ldots, g_n)(a_1, \ldots, a_n) = f(g_1(a_1, \ldots, a_n), \ldots, g_n(a_1, \ldots, a_n)), \tag{1.1}$$

where $f, g_1, \ldots, g_n \in T(A^n, A), a_1, \ldots, a_n \in A$. A Menger algebra of all full n-ary functions or a Menger algebra of all n-ary operations is a pair of the set $T(A^n, A)$ of all full n-ary functions defined on A and the Menger composition of full n-ary functions satisfying the superassociative law.

In a structural point of view, both of semigroups and groups can be isomorphically represented by functions of one variables. Analogously, some types of Menger algebras are also investigated in the same direction. Namely, it turned out that some types of Menger algebras of rank n can be represented by n-ary operations. The construction of many kinds of arbitrary n-ary operations was given by W. A. Dudek and V. S. Trokhimenko in the last few years. For more details, one can refer the reader to [15]. A connection of polynomial functions and n-ary semigroups was investigated in [16].

Recently, the authors were firstly introduced the concept of Menger hyperalgebras, which is an algebraic hypercompositional extension of original Menger algebras. Recall from [17] that a *Menger hyperalgebra of rank n* is a pair of $(n+1)$-ary hypergroupoid (G, f) where $f : G^{n+1} \to P^*(G)$ which is called an *$(n+1)$-ary hyperoperation* satisfying superassociative law:

$$f(f(x, y_1, \ldots, y_n), z_1, \ldots, z_n) = f(x, f(y_1, z_1, \ldots, z_n), \ldots, f(y_n, z_1, \ldots, z_n))$$

for every $x, y_1, \ldots, y_n, z_1, \ldots, z_n \in G$. We can say that every Menger algebra can be constructed a Menger hyperalgebra. In general, in case of $n = 1$, Menger hyperalgebras can be considered as a generalization of semihypergroups, see [18].

Some fundamental examples of Menger hyperalgebras are collected. Let H be the unit interval $[0, 1]$. For every $x_0, x_1 \ldots, x_n \in H$, we define

$$f(x_0, x_1 \ldots, x_n) = [0, \tfrac{x_0 x_1 \cdot \ldots \cdot x_n}{n+1}].$$

Then (H, f) is a Menger hyperalgebra. For more details on hyperstructure, one can refer the reader to [19–25].

In 2016, a precise idea of an n-ary opearation which is called semiabelian was firstly mentioned by N. A. Shchuchkin [26]. It is represented by an n-ary operation ζ on a nonempty set A satisfying the identity

$$\zeta(a_1, a_2, \dots, a_{n-1}, a_n) = \zeta(a_n, a_2, \dots, a_{n-1}, a_1)$$

for every $a_1, \dots, a_n \in A$. If ζ satisfies $\zeta(a_1, \dots, a_n) = \zeta(a_{\sigma(1)}, \dots, a_{\sigma(n)})$ for any permutation $\sigma \in S_n$, then it said to be abelian or commutative.

The purpose of this paper is to apply the notions of semiabelian n-ary opearation for studying in an algebraic viewpoint, specifically, to propose two novel algebraic structures which are called a Menger algebra of semiabelian n-ary opearation and a Menger algebra of semiabelian n-ary hyperopearation. Further more, the necessary and sufficient conditions under which a Menger algebra of rank n satisfying some certain equation can be isomorphically embeded by a Menger algebra of cyclic operation and a Menger algebra of semiabelian n-ary opearation are presented in the Section 2. After we concentrate on algebraic structure of Menger algebras, we continue our investigation of a representaion of Menger hyperalgebras in Section 3 by presenting a structure of semiabelian n-ary hyperopearations. We attempt to study a characterization of abstract Menger hyperalgebras by semiabelian n-ary hyperopearations. We complete this paper in Section 4 by discussing some of our results and giving some further topics for the interested readers.

2. Menger algebras represented by semiabelian n-ary operations

In this section, we aim to construct a Menger algebra of semiabelian n-ary operations and to present a characterization of any abstract Menger algebras. The answer to the challenging question "When is a Menger algebra of rank $n \geq 2$ isomorphically embedded into a Menger algebra of all semiabelian n-ary operations defined on some set?" is absolutely given in this section.

Let $n \geq 2$ and A be a nonempty set. An n-ary operation $\gamma : A^n \to A$ is said to be *semiabelian* if for all $a_1, \dots, a_n \in A$, it satisfies the identity

$$\gamma(a_1, a_2, \dots, a_{n-1}, a_n) = \gamma(a_n, a_2, \dots, a_{n-1}, a_1).$$

The symbol $SA(A^n, A)$ denotes the set of all semiabelian n-ary operations.

There are several possibilities to provide examples of semiabelian n-ary operation. Let us present in the following example.

Example 2.1. Let $\mathbb{Z}$ be the set of all integers. An n-ary operation f on $\mathbb{Z}$ is defined by

$$f(a_1, \dots, a_n) = \max\{a_1, \dots, a_n\}$$

where $\max A$ denotes the maximum of elements in A. Then f is semiabelian.

The fact that the set $SA(A^n, A)$ of all semiabelian n-ary operations closed with respect to the Menger superposition can be proved by the following lemma.

Lemma 2.2. *The set of all semiabelian n-ary operations forms a Menger algebra with respect to the Menger's composition $\mathcal{O}$.*

Proof. Let $\gamma, \beta_1, \dots, \beta_n \in SA(A^n, A)$. In order to show that $\mathcal{O}(\gamma, \beta_1, \dots, \beta_n)$ is also a semiabelian n-ary operation, we assume that $a_1, a_2, \dots, a_{n-1}, a_n$ are elements in A. Then

$$\begin{aligned}
&\mathcal{O}(\gamma, \beta_1, \dots, \beta_n)(a_1, a_2, \dots, a_{n-1}, a_n) \\
&= \gamma(\beta_1(a_1, a_2, \dots, a_{n-1}, a_n), \dots, \beta_n(a_1, a_2, \dots, a_{n-1}, a_n)) \\
&= \gamma(\beta_1(a_n, a_2, \dots, a_{n-1}, a_1), \dots, \beta_n(a_n, a_2, \dots, a_{n-1}, a_1)) \\
&= \mathcal{O}(\gamma, \beta_1, \dots, \beta_n)(a_n, a_2, \dots, a_{n-1}, a_1).
\end{aligned}$$

The proof is completely finished. ∎

Lemma 2.3. *In* $(SA(A^n, A), \mathcal{O})$*, for all* $\gamma, \beta_1, \ldots, \beta_n \in SA(A^n, A)$*, the following identity is satisfied*

$$\mathcal{O}(\gamma, \beta_1, \beta_2, \ldots, \beta_{n-1}, \beta_n) = \mathcal{O}(\gamma, \beta_n, \beta_2, \ldots, \beta_{n-1}, \beta_1).$$

Proof. Let $a_1, \ldots, a_n \in A$. Then we have

$$\begin{aligned}
&\mathcal{O}(\gamma, \beta_1, \beta_2, \ldots, \beta_{n-1}, \beta_n)(a_1, \ldots, a_n)\\
&= \gamma(\beta_1(a_1, \ldots, a_n), \beta_2(a_1, \ldots, a_n), \ldots, \beta_{n-1}(a_1, \ldots, a_n), \beta_n(a_1, \ldots, a_n))\\
&= \gamma(\beta_n(a_1, \ldots, a_n), \beta_2(a_1, \ldots, a_n), \ldots, \beta_{n-1}(a_1, \ldots, a_n), \beta_1(a_1, \ldots, a_n))\\
&= \mathcal{O}(\gamma, \beta_n, \beta_2, \ldots, \beta_{n-1}, \beta_1)(a_1, \ldots, a_n).
\end{aligned}$$

∎

The following theorem presents the necessary and sufficient conditions for embedding of an abstract Menger algebra to a Menger algebra of semiabelian n-ary operations.

Theorem 2.4. *A Menger algebra* $(G, \circ)$ *of rank* $n \geq 2$ *is isomorphically embedded into the Menger algebra of all semiabelian* n*-ary operations defined on some set if and only if* $(G, \circ)$ *satisfies the identity*

$$\circ(a, b_1, b_2, \ldots, b_{n-1}, b_n) = \circ(a, b_n, b_2, \ldots, b_{n-1}, b_1).$$

Proof. Assume that φ is a monomorphism from a Menger algebra G to $SA((G')^n, G')$ for some set G'. Let $a, b_1, \ldots, b_n$ be arbitrary elements in G. Then the homomorphism property of φ and Lemma 2.3 directly imply that $\varphi(\circ(a, b_1, b_2, \ldots, b_{n-1}, b_n)) = \mathcal{O}(\varphi(a), \varphi(b_1), \varphi(b_2), \ldots, \varphi(b_{n-1}), \varphi(b_n)) = \mathcal{O}(\varphi(a), \varphi(b_n), \varphi(b_2), \ldots, \varphi(b_{n-1}), \varphi(b_1)) = \varphi(\circ(a, b_n, b_2, \ldots, b_{n-1}, b_1))$. This means that the equality $\varphi(\circ(a, b_1, b_2, \ldots, b_{n-1}, b_n)) = \varphi(\circ(a, b_n, b_2, \ldots, b_{n-1}, b_1))$ is valid. It follows directly from the injectivity of a mapping φ that $\circ(a, b_1, b_2, \ldots, b_{n-1}, b_n) = \circ(a, b_n, b_2, \ldots, b_{n-1}, b_1)$.

For the converse, let e, c be different two elements not belonging in G. Consider $G' = G \cup \{e, c\}$. For each element $g \in G'$, we define an n-ary operation on G' by setting

$$\lambda_g(a_1, \ldots, a_n) = \begin{cases} \circ(g, a_1, \ldots, a_n) & \text{if } a_i \in G \text{ for all } 1 \leq i \leq n,\\ g & \text{if } a_i = e \text{ for all } 1 \leq i \leq n,\\ c & \text{otherwise.} \end{cases}$$

First, we prove that the n-ary operation λ_g defined above is semiabelian. For this, let $a_1, \ldots, a_n \in G'$. If $a_1, \ldots, a_n \in G$, then according to the assumption, we have $\lambda_g(a_1, a_2, \ldots, a_{n-1}, a_n) = \circ(g, a_1, a_2, \ldots, a_{n-1}, a_n) = \circ(g, a_n, a_2, \ldots, a_{n-1}, a_1) = \lambda_g(a_n, a_2, \ldots, a_{n-1}, a_1)$. If $a_1 = \cdots = a_n = e$, then by defining λ_g, we get $\lambda_g(a_1, a_2, \ldots, a_{n-1}, a_n) = \lambda_g(e, \ldots, e) = g = \lambda_g(a_n, a_2, \ldots, a_{n-1}, a_1)$. In all other cases, $\lambda_g(a_1, a_2, \ldots, a_{n-1}, a_n) = c = \lambda_g(a_n, a_2, \ldots, a_{n-1}, a_1)$. So, the n-ary operation λ_g is semiabelian, i.e., $\lambda_g \in SA((G')^n, G')$. Put $\overline{SA((G')^n, G')} := \{\lambda_g \mid g \in G\}$. It follows directly from Lemma 2.2 that the set $\overline{SA((G')^n, G')}$ forms a Menger subalgebra of $S((G')^n, G')$ under the $(n+1)$-ary Menger composition $\mathcal{O}$.

Later, we show that a mapping $\varphi : (G, \circ) \to (\overline{S((G')^n, G')}, \mathcal{O})$ which is defined by

$$\varphi(g) = \lambda_g$$

is an isomorphism. Clearly, a mapping φ is surjective. The injectivity is also satisfied, because from $\varphi(g_1) = \varphi(g_2)$, we have $\lambda_{g_1} = \lambda_{g_2}$. Then $\lambda_{g_1}(a_1, \ldots, a_n) = \lambda_{g_2}(a_1, \ldots, a_n)$ for all $a_1, \ldots, a_n \in G'$. Particularly, $\lambda_{g_1}(e, \ldots, e) = \lambda_{g_2}(e, \ldots, e)$, which implies $g_1 = g_2$. To show that φ have a preserving operation property, we consider in the following cases.

We begin with the case when $a_1, \ldots, a_n \in G$. Then, for $x, y_1, \ldots, y_n \in G$, applying the superassociativity of $(n+1)$-ary operation $\circ$ on G, we have

$$\begin{aligned}
\lambda_{\circ(x,y_1,\ldots,y_n)}(a_1, \ldots, a_n) &= \circ(\circ(x, y_1, \ldots, y_n), a_1, \ldots, a_n) \\
&= \circ(x, \circ(y_1, a_1, \ldots, a_n), \ldots, \circ(y_n, a_1, \ldots, a_n)) \\
&= \lambda_x(\lambda_{y_1}(a_1, \ldots, a_n), \ldots, \lambda_{y_n}(a_1, \ldots, a_n)) \\
&= \mathcal{O}(x, y_1, \ldots, y_n)(a_1, \ldots, a_n).
\end{aligned}$$

If $a_i = e$ for every $1 \leq i \leq n$, by the definition of λ_g, we obtain $\lambda_{\circ(x,y_1,\ldots,y_n)}(e, \ldots, e) = \circ(x, y_1, \ldots, y_n)$. On the other hand, by the definition of Menger composition, we also have $\mathcal{O}(x, y_1, \ldots, y_n)(e, \ldots, e) = \lambda_x(\lambda_{y_1}(e, \ldots, e), \ldots, \lambda_{y_n}(e, \ldots, e) = \lambda_x(y_1, \ldots, y_n) = \circ(x, y_1, \ldots, y_n)$. Thus, $\lambda_{\circ(x,y_1,\ldots,y_n)} = \mathcal{O}(x, y_1, \ldots, y_n)$. Otherwise, by defining an n-ary operation λ_g we obtain immediatly that $\lambda_{\circ(x,y_1,\ldots,y_n)}(a_1, \ldots, a_n) = c = \lambda_x(c, \ldots, c) = \lambda_x(\lambda_{y_1}(a_1, \ldots, a_n), \ldots, \lambda_{y_n}(a_1, \ldots, a_n)) = \mathcal{O}(x, y_1, \ldots, y_n)(a_1, \ldots, a_n)$. Consequently, a mapping φ is an isomorphism from $(G, \circ)$ to $(\overline{SA((G')^n, G')}, \mathcal{O})$. ■

3. An embedding theorem for Menger hyperalgebras via semiabelian n-ary Hyperoperations

We begin this section with recalling some useful definitions of Menger hyperalgebras and n-ary hyperoperations. For an overview of these topics, we refer to [17].

Let A^n be the n-th Cartesian product of a nonempty set A. On the set $T(A^n, P^*(A))$ of all n-ary hyperoperations $f : A^n \to P^*(A)$, one can define the following $(n+1)$-ary operation $\bullet : T(A^n, P^*(A))^{n+1} \to T(A^n, P^*(A))$, called the *Menger superposition* $\bullet$, defined by

$$\bullet(f, g_1, \ldots, g_n)(x_1, \ldots, x_n) = \bigcup_{\substack{y_i \in g_i(x_1, \ldots, x_n) \\ i \in \{1, \ldots, n\}}} f(y_1, \ldots, y_n),$$

where $f, g_1, \ldots, g_n \in T(A^n, P^*(A))$, $x_1, \ldots, x_n \in A$.

We now obtain the primary properties of the Menger composition $\bullet$.

Theorem 3.1. [17] The composition of n-ary hyperoperations is superassociative, i.e., if $f, g_1, \ldots, g_n, h_1, \ldots, h_n$ are n-ary hyperoperations, then

$$\bullet(\bullet(f, g_1, \ldots, g_n), h_1, \ldots, h_n) = \bullet(f, \bullet(g_1, h_1, \ldots, h_n), \ldots, \bullet(g_n, h_1, \ldots, h_n)).$$

By Theorem 3.1, we immediately obtain the following important corollary:

Corollary 3.2. [17] The structure $(T(A^n, P^*(A)), \bullet)$ forms a Menger algebra.

So it forms a Menger algebra $(T(A^n, P^*(A)), \bullet)$, which will be called a *Menger algebra of all n-ary hyperoperations*. By a Menger algebra of n-ary hyperoperations, we mean a Menger subalgebra of $(T(A^n, P^*(A)), \bullet)$.

Formally, an n-ary hyperoperation ν on A is said to be semiabelian if it satisfies

$$\nu(a_1, a_2, \ldots, a_{n-1}, a_n) = \nu(a_n, a_2, \ldots, a_{n-1}, a_1)$$

for all $a_1, \ldots, a_n \in A$. Actually, both of the left and right handside of this identity is a nonempty set. By $SA(A^n, P^*(A))$ we denote the set of all semiabelian n-ary hyperoperations on a set A.

A strong relationship between semiabelian n-ary operations and semiabelian n-ary hyperoperations is naturally presented as follows.

Theorem 3.3. *Any Menger algebra of semiabelian n-ary operations can be embedded into a Menger algebra of semiabelian n-ary hyperoperations.*

Proof. Assume that $\gamma : A^n \to A$ is a semiabelian n-ary operation. Let $\overline{\gamma} : A^n \to P^*(A)$ be an n-ary hyperoperation defined by $\overline{\gamma}(a_1, \ldots, a_n) = \{\gamma(a_1, \ldots, a_n)\}$ for all $a_1, \ldots, a_n \in A$. Since γ is semiabelian, then so $\overline{\gamma}$ is. We now define a mapping $\sigma : SA(A^n, A) \to SA(A^n, P^*(A))$ by

$$\sigma(\gamma) = \overline{\gamma}$$

for all $\gamma \in SA(A^n, A)$. Using the definition of the composition $\bullet$, it is clear that the mapping σ preserves the operations, i.e.,

$$\sigma(\mathcal{O}(\gamma, \beta_1, \ldots, \beta_n)) = \overline{\mathcal{O}(\gamma, \beta_1, \ldots, \beta_n)} = \bullet(\overline{\gamma}, \overline{\beta_1}, \ldots, \overline{\beta_n}) = \bullet(\sigma(\gamma), \sigma(\beta_1), \ldots, \sigma(\beta_n)).$$

In fact, if we let $a_1, \ldots, a_n \in A$ then

$$\begin{aligned}
\overline{\mathcal{O}(\gamma, \beta_1, \ldots, \beta_n)}(a_1, \ldots, a_n) &= \{\mathcal{O}(\gamma, \beta_1, \ldots, \beta_n)(a_1, \ldots, a_n)\} \\
&= \{\gamma(\beta_1(a_1, \ldots, a_n), \ldots, \beta_n(a_1, \ldots, a_n))\} \\
&= \overline{\gamma}(\beta_1(a_1, \ldots, a_n), \ldots, \beta_n(a_1, \ldots, a_n)) \\
&= \bigcup_{\substack{y_i \in \{\beta_i(a_1,\ldots,a_n)\} \\ i \in \{1,\ldots,n\}}} \overline{\gamma}(y_1, \ldots, y_n) \\
&= \bigcup_{\substack{y_i \in \overline{\beta_i}(a_1,\ldots,a_n) \\ i \in \{1,\ldots,n\}}} \overline{\gamma}(y_1, \ldots, y_n) \\
&= \bullet(\overline{\gamma}, \overline{\beta_1}, \ldots, \overline{\beta_1})(a_1, \ldots, a_n).
\end{aligned}$$

Finally, let $\gamma, \beta \in SA(A^n, A)$ and $a_1, \ldots, a_n \in A$ be such that $\overline{\gamma}(a_1, \ldots, a_n) = \overline{\beta}(a_1, \ldots, a_n)$. Then we have that $\{\gamma(a_1, \ldots, a_n)\} = \{\beta(a_1, \ldots, a_n)\}$ and thus $\gamma = \beta$. Hence σ is injective. ■

The fact that a composition on the set of all n-ary hyperoperations is superassociative was already proved in [17]. Then we obtain the following lemmas.

Lemma 3.4. *On the set $SA(A^n, P^*(A))$, the Menger superposition $\bullet$ satisfies the superassociative law, i.e., $SA(A^n, P^*(A))$ forms a Menger algebra under the $(n+1)$-operation $\bullet$.*

Some algebraic properties of semiabelian n-ary hyperoperations are given below.

Lemma 3.5. *Let $\nu, \mu_1, \ldots, \mu_n$ be semiabelian n-ary hyperoperations on A. Then the identity*

$$\bullet(\nu, \mu_1, \mu_2, \ldots, \mu_{n-1}, \mu_n) = \bullet(\nu, \mu_n, \mu_2, \ldots, \mu_{n-1}, \mu_1)$$

holds.

Proof. It is not difficult to prove the statement by using the definition of Menger superposition $\bullet$ and the definition of semiabelian n-ary hyperoperations. ■

The main result of this section concerns a representation of Menger hyperalgebras satisfying some condition. In the following theorem, we give a proof of this fact.

Theorem 3.6. *An abstract Menger hyperalgebra (G, f) can be represented by semiabelian n-ary hyperoperations defined on some set if and only if it satisfies the equation*

$$f(a, b_1, b_2, \ldots, b_{n-1}, b_n) = f(a, b_n, b_2, \ldots, b_{n-1}, b_1).$$

Proof. The necessity follows immediatly from Lemma 3.5, hence we prove only the converse direction. Let (G, f) be a Menger hyperalgebra satisfying $f(a, b_1, b_2, \ldots, b_{n-1}, b_n) = f(a, b_n, b_2, \ldots, b_{n-1}, b_1)$. Let e, c be two different elements which are not belonging to G. Let $\overline{G} = G \cup \{e, c\}$. For every element $g \in G$, we define the n-ary hyperoperation $\eta_g : (\overline{G})^n \to \overline{G}$ by setting

$$\eta_g(x_1, \ldots, x_n) = \begin{cases} f(g, x_1, \ldots, x_n) & \text{if } x_1, \ldots, x_n \in G; \\ \{g\} & \text{if } x_1 = \cdots = x_n = e; \\ \{c\} & \text{otherwise}. \end{cases}$$

Moreover, we extend the n-ary hyperoperation η_g as follows: For any nonempty subset A of G, the hyperoperaion η_A is defined by

$$\eta_A(x_1, \ldots, x_n) = \begin{cases} f(A, x_1, \ldots, x_n) & \text{if } x_1, \ldots, x_n \in G; \\ A & \text{if } x_1 = \cdots = x_n = e; \\ \{c\} & \text{otherwise}. \end{cases}$$

In order to show that η_g is semiabelian, we let $x_1, \ldots, x_n \in \overline{G}$. We first consider when $x_1, \ldots, x_n$ are arbitrary elements in G. By the hypothesis, we have $\eta_g(x_1, x_2, \ldots, x_{n-1}, x_n) = f(g, x_1, x_2, \ldots, x_{n-1}, x_n) = f(g, x_n, x_2, \ldots, x_{n-1}, x_1) = \eta_g(x_n, x_2, \ldots, x_{n-1}, x_1)$. If $x_i = e$ for all $1 \leq i \leq n$, then we obtain $\eta_g(x_1, x_2, \ldots, x_{n-1}, x_n) = \eta_g(e, \ldots, e) = \{g\} = \eta_g(x_n, x_2, \ldots, x_{n-1}, x_1)$. In all oher cases, we also have

$$\eta_g(x_1, x_2, \ldots, x_{n-1}, x_n) = \{c\} = \eta_g(x_n, x_2, \ldots, x_{n-1}, x_1).$$

Hence, the n-ary hyperoperation η_g is semiabelian.

We now prove that the mapping $\varphi : G \to SA((\overline{G})^n, P^*(\overline{G}))$ which is defined by $\varphi(a) = \eta_a$ for all $a \in G$, is a monomorphism. For this, we first show that the equality

$$\eta_{f(a,b_1,\ldots,b_n)} = \bullet(\eta_a, \eta_{b_1}, \ldots, \eta_{b_n})$$

holds for any $a, b_1, \ldots, b_n \in \overline{G}$.

Let $x_1, \ldots, x_n \in G$. Then, by defining the hyperoperation η_a and its extension η_A, we obtain

$$\begin{aligned} \eta_{f(a,b_1,\ldots,b_n)}(x_1, \ldots, x_n) &= f(f(a, b_1, \ldots, b_n), x_1, \ldots, x_n) \\ &= f(a, f(b_1, x_1, \ldots, x_n), \ldots, f(b_n, x_1, \ldots, x_n)) \\ &= f(a, \eta_{b_1}(x_1, \ldots, x_n), \ldots, \eta_{b_n}(x_1, \ldots, x_n)) \\ &= \bigcup_{\substack{y_i \in \eta_{b_i}(x_1,\ldots,x_n) \\ i \in \{1,\ldots,n\}}} f(a, y_1, \ldots, y_n) \\ &= \bigcup_{\substack{y_i \in \eta_{b_i}(x_1,\ldots,x_n) \\ i \in \{1,\ldots,n\}}} \eta_a(y_1, \ldots, y_n) \\ &= \bullet(\eta_a, \eta_{b_1}, \ldots, \eta_{b_n})(x_1, \ldots, x_n). \end{aligned}$$

For each $1 \leq i \leq n$, if an element x_i equals to e, then according to the definition of η_A, we have

$$\begin{aligned}
\eta_{f(a,b_1,\ldots,b_n)}(x_1,\ldots,x_n) &= \eta_{f(a,b_1,\ldots,b_n)}(e,\ldots,e)\\
&= f(a,b_1,\ldots,b_n)\\
&= \eta_a(b_1,\ldots,b_n)\\
&= \bigcup_{\substack{y_i\in\{b_i\}\\ i\in\{1,\ldots,n\}}} \eta_a(y_1,\ldots,y_n)\\
&= \bigcup_{\substack{y_i\in\lambda_{b_i}(e,\ldots,e)\\ i\in\{1,\ldots,n\}}} \eta_a(y_1,\ldots,y_n)\\
&= \bullet(\eta_a,\eta_{b_1},\ldots,\eta_{b_n})(e,\ldots,e)\\
&= \bullet(\eta_a,\eta_{b_1},\ldots,\eta_{b_n})(x_1,\ldots,x_n),
\end{aligned}$$

which implies $\eta_{f(a,b_1,\ldots,b_n)}(e,\ldots,e) = \bullet(\eta_a,\eta_{b_1},\ldots,\eta_{b_n})(e,\ldots,e)$.
Finally, let $(z_1,\ldots,z_n) \in (\overline{G})^n \setminus (G^n \cup \{(e,\ldots,e)\})$. This means that we consider in other cases. Then

$$\begin{aligned}
\eta_{f(a,b_1,\ldots,b_n)}(z_1,\ldots,z_n) &= \{c\}\\
&= \eta_a(c,\ldots,c)\\
&= \bigcup_{\substack{y_i\in\{c\}\\ i\in\{1,\ldots,n\}}} \eta_a(y_1,\ldots,y_n)\\
&= \bigcup_{\substack{y_i\in\lambda_{b_i}(z_1,\ldots,z_n)\\ i\in\{1,\ldots,n\}}} \eta_a(y_1,\ldots,y_n)\\
&= \bullet(\eta_a,\eta_{b_1},\ldots,\eta_{b_n})(z_1,\ldots,z_n).
\end{aligned}$$

Hence $\eta_{f(a,b_1,\ldots,b_n)}(z_1,\ldots,z_n) = \bullet(\eta_a,\eta_{b_1},\ldots,\eta_{b_n})(z_1,\ldots,z_n)$. This completes the proof of the homomorphism property.

To prove the injectivity of φ, let $\eta_a = \eta_b$. Then $\eta_a(e,\ldots,e) = \eta_b(e,\ldots,e)$, and $\{a\} = \{b\}$. Hence $a = b$. The proof is actually finished. ∎

Below we give a concrete example that demonstrates a representation of Menger hyperalgebras in a specific rank.

Example 3.7. Let $H = \{0,1,2\}$ be a semihypergroup, i.e., a Menger hyperalgebra of rank 1 with respect to the following Cayley's table.

$\star$	0	1	2
0	0	$\{0,1\}$	$\{0,2\}$
1	$\{0,1\}$	1	$\{1,2\}$
2	$\{0,2\}$	$\{1,2\}$	2

Clearly, a hyperoperation $\star$ on H satisfies $\star(a,b) = \star(b,a)$ for all $a,b \in H$. We now illustrate that H is isomorphic to some set of a semigroup of unary hyperoperations. For this, we consider a mapping $\eta_0 : H^1 \to P^*(H^1)$ such that $\eta_0(0) = 0\star 0 = 0$, $\eta_0(1) = 0\star 1 = \{0,1\}$ and $\eta_0(2) = 0\star 2 = \{0,2\}$. Thus $\eta_0 = \begin{pmatrix} 0 & 1 & 2\\ 0 & \{0,1\} & \{0,2\}\end{pmatrix}$ and then $0 \mapsto \eta_0 = \begin{pmatrix} 0 & 1 & 2\\ 0 & \{0,1\} & \{0,2\}\end{pmatrix}$. Furthermore, we consider a mapping $\eta_1 : H^1 \to P^*(H^1)$ is defined by $\eta_1(0) = \{0,1\}$, $\eta_1(1) = 1$ and $\eta_1(2) = \{1,2\}$. Thus $1 \mapsto \eta_1 = \begin{pmatrix} 0 & 1 & 2\\ \{0,1\} & 1 & \{1,2\}\end{pmatrix}$. Similar to the element 1 in H, we obtain $2 \mapsto \eta_2 = \begin{pmatrix} 0 & 1 & 2\\ \{0,2\} & \{1,2\} & 2\end{pmatrix}$. By Theorem

3.6, we conclude that $(H, \star) \cong (\{\eta_0, \eta_1, \eta_2\}, \bullet)$ where $\bullet$ can be described by the following table.

$\bullet$	η_0	η_1	η_2
η_0	η_0	$\eta_{\{0,1\}}$	$\eta_{\{0,2\}}$
η_1	$\eta_{\{0,1\}}$	η_1	$\eta_{\{1,2\}}$
η_2	$\eta_{\{0,2\}}$	$\eta_{\{1,2\}}$	η_2

In addition, the table for these representations is just like the original table with elements in H, renamed by a unary hyperoperation η.

4. Conclusions and future work

In the paper we have introduced two algebraic structures of the n-ary operations and n-ary hyperoperations which are called semiabelian. We remark here that our concepts can be regarded as a natural similarity with others kind of n-ary operation in case of $n = 2$. But the result of this paper offered a different perspective proof. For instance, if $n = 2$, then semiabelian binary operations and 1-commutative binary operations [4] are the same concepts. For the significant results, we have particularly proved that every Menger algebra of rank $n \geq 2$ is isomorphically embeded into a Menger algebra of all semiabelian n-ary operations defined on some set if it satisfies certain identity. In Section 3, we also attempted to extend our study from classical algebras to hypercompositional algebras, whence, the main results of this paper are novel contributions for solving the challenging problem of subclasses of both n-ary operations and n-ary hyperoperations.

In the future work, we would like to extend this idea to the study of commutative n-ary operations and its many-sorted operations.

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References

[1] K. Menger, The algebra of functions: past, present, future. Rendicionti di mathematica, 20 (1961), 409-430.
[2] W. A. Dudek, V. S. Trokhimenko, Menger Algebras of associative and self-distributive n-ary operations, Quasigroups Relat. Syst., 26 (2018), 45-52.
[3] W. A. Dudek, V. S. Trokhimenko, Menger algebras of idempotent n-ary operations, Stud. Sci. Math. Hung., 55(2) (2019), 260-269.
[4] W. A. Dudek, V. S. Trokhimenko, Menger algebras of k-commutative n-place functions, Georgian Math. J., 28(3) (2021), 355-361.
[5] W. A. Dudek, In memoriam: Valentin S. Trokhimenko, Quasigroups Relat. Syst., 28 (2020), 171-176.
[6] K. Denecke, Partial clones, Asian-Eur. J. Math., 13(8) (2020), 2050161.
[7] W. A. Dudek, V. S. Trokhimenko, On σ-commutativity in Menger algebras of n-place functions, Comm. Algebra, 45(10) (2017), 4557–4568.
[8] K. A. Kearnes, A. Szendrei, Clones of algebras with parallelogram terms, Internat. J. Algebra Comput., 22 (2012), 1250005.

[9] T. Kumduang, S. Leeratanavalee, Semigroups of terms, tree languages, Menger algebra of n-ary functions and their embedding theorems, Symmetry, 13(4) (2021), 558.

[10] Yu. M. Movsisyan, E. Nazari, A Cayley theorem for the multiplicative semigroup of a field, J. Algebra Appl., 11(2) (2012), 1250042.

[11] F. Kh. Muradov, On the semigroups of quasi-open transformations, J. Semigroup Theory Appl., 5 (2015), 1-11.

[12] K. Wattanatripop, T. Changphas, The Menger algebra of terms induced by order-decreasing transformations, Comm. Algebra, 49(7) (2021), 3114-3123.

[13] K. Wattanatripop, T. Kumduang, T. Changphas, S. Leeratanavalee, Power Menger algebras of terms induced by order-decreasing transformations and superpositions, Int. J. Math. Comput. Sci., 16(4) (2021), 1697-1707.

[14] T. Kumduang, S. Leeratanavalee, Left translations and isomorphism theorems for Menger algebras of rank n, Kyungpook Math. J., 61(2) (2021), 223-237.

[15] W. A. Dudek, V. S. Trokhimenko, Algebras of multiplace functions, De Gruyter, Berlin, 2012.

[16] J. Marichal, P. Mathone, A description of n-ary semigroups polynomial-derived from integral domains, Semigroup Forum, 83 (2011), 241–249.

[17] T. Kumduang, S. Leeratanavalee, Menger hyperalgebras and their representations, Comm. Algebra, 49(4) (2020), 1513-1533.

[18] B. Davvaz, Semihypergroup theory, Elsevier, Sci, Publication, London, 2016.

[19] I. Cristea, M. Stefanescu, Hypergroups and n-ary relations, Eur. J. Comb., 31(3) (2010), 780– 789.

[20] V. Leoreanu-Fotea, B. Davvaz, n-hypergroups and binary relations, Eur. J. Comb. 29 (2008), 1207-1218.

[21] D. Heidari, I. Cristea, On factorizable semihypergroups, Mathematics, 2020, 8(7), 1064

[22] A. Nikkhah, B. Davvaz, Hypergroups associated with hypergraphs, Discrete Math. Algorithms Appl., 13(3) (2021), 2150018.

[23] M. Norouzi, I. Cristea, Hyperrings with n-ary composition hyperoperation, J. Algebra Appl., 17(2) (2018), 1850022.

[24] A. C. Sonea, New aspects in polygroup theory, An. Ştiinţ. Univ. "Ovidius" Constanţa, Ser. Mat., 28(3) (2020), 241–254.

[25] H. Bordbar, I. Cristea, Regular parameter elements and regular local hyperrings, Mathematics, 2021, 9(3), 243.

[26] N. A. Shchuchkin, The structure of finite semiabelian n-ary groups, Chebyshevskii Sb., 17(1) (2016), 254–269.