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Topological properties of two-modular convergence

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Abstract : Let (Ω, Σ, μ) be a σ-finite measure space and assume that E is an ideal of L^0. Let ρ and ρ^* be two modulars defined on E, E_ρ and E_{ρ^*} the modular spaces for ρ and ρ^* respectively. In $E_\rho \cap E_{\rho^*}$ a two-modular convergence can be defined as follows: a sequence (f_n) in $E_\rho \cap E_{\rho^*}$ is said to be two-modular convergent to $f \in E_\rho \cap E_{\rho^*}$ whenever $f_n \to f$ with respect to modular ρ^* and (f_n) is ρ-bounded. We introduce a two-modular topology $\gamma_W(\mathcal{T}_\rho, \mathcal{T}_{\rho^*})$ in $E_\rho \cap E_{\rho^*}$ and show that the convergence in this topology is equivalent to the two-modular convergence. We prove also that the two-modular convergence is equivalent to some modular convergence. The most important fact on this paper is a characterization of linear functionals on the space $L^{\varphi_1} \cap L^{\varphi_2}$, continuous with respect to the two-modular topology $\gamma_W(\mathcal{T}_{m_{\varphi_1}}, \mathcal{T}_{m_{\varphi_2}})$. The functions φ_1 and φ_2 are not assumed to be convex.

Keywords : Modular spaces, modular bases, two-modular convergence, Orlicz spaces, Riesz spaces

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1 Introduction.

Let (Ω, Σ, μ) be a σ-finite complete measure space. We denote by $L^0(\Omega)$ the set of μ-equivalence classes of all real valued Σ -measurable functions defined and a.e. finite on Ω. Then $L^0(\Omega)$ is a super Dedekind complete Riesz space under the ordering $f \leq g$ whenever $f(\omega) \leq g(\omega)$ a.e. on Ω. Recall that a set $E \subset L^0$ is solid whenever from the conditions $|f| \leq |g|$, $g \in E$ and $f \in L^0$ it follows that $f \in E$. If the set E is also linear we call it an ideal of L^0. Assume that E is an ideal of L^0.

1.1 Modular spaces.

1.1.1

A functional $\rho : E \to [0, \infty]$ is called a modular if the following conditions are satisfied:

(i) $\rho(f) = 0 \Leftrightarrow f = 0$.

(ii) $\rho(f) \leq \rho(g)$ if $|f| \leq |g|$.

(iii) $\rho(f + g) = \rho(f) + \rho(g)$ for $|f| \wedge |g| = 0$.

From (ii) and (iii) we obtain the condition

(iv) $\rho(f \vee g) \leq \rho(f) + \rho(g)$.

Since the inequality $\rho(\alpha f + \beta g) \leq \rho(f \vee g)$ is true for all $f, g \in E$ and all scalars α, β such that $|\alpha|^s + |\beta|^s \leq 1$ and $0 < s \leq 1$, then we also obtain $\rho(\alpha f + \beta g) \leq \rho(f) + \rho(g)$.

We have proved that for each $0 < s \leq 1$ modular ρ is an s- modular in sense of [5]. This type of modulars was introduced by Albrycht and Musielak in [1] and in the case $s = 1$ by Musielak and Orlicz in [7].

If $0 < s_1 < s_2 \leq 1$ and ρ is an s_2-modular, then ρ is also an s_1-modular.
The linear space $E_\rho = \{f \in E : \rho(\lambda f) \to 0 \text{ as } \lambda \to 0\}$ is called a modular space for ρ. The formula $|f|_\rho = \inf\{\varepsilon > 0 : \rho(f/\varepsilon^{\frac{1}{s}}) \leq \varepsilon\}$ defines an F-norm in E_ρ, which has the same properties as the norm defined in ([7], 1.21).

1.1.2

The modular $\rho : E \to [0, \infty]$ is called s-convex, ($0 < s \leq 1$) if $\rho(\alpha f + \beta f) \leq |\alpha|^s \rho(f) + |\beta|^s \rho(g)$ for any $f, g \in E$ and any scalars α, β with $|\alpha|^s + |\beta|^s \leq 1$.

A 1-convex modular will be called briefly convex. Every s-convex modular is an s-modular. The formula $||f||_\rho^s = \inf\{\varepsilon > 0 : \rho(f/\varepsilon^{\frac{1}{s}}) \leq 1\}$ defines an s-homogeneous norm in E_ρ (see [1]).

1.1.3

A sequence (f_n) in E_ρ is called ρ-convergent to $f \in E_\rho$, in symbols $f_n \xrightarrow{\rho} f$, if there exists a constant $\lambda > 0$ such that $\rho(\lambda(f_n - f)) \to 0$ as $n \to \infty$ ([1], [7]).

1.1.4

A set Z in E_ρ will be called ρ-bounded if for any sequence (f_n) in Z and any sequence of numbers $\varepsilon_n \to 0$, we have $\varepsilon_n f_n \xrightarrow{\rho} 0$ (see [3]).

If ρ is an s-convex modular, then a set $Z \subset E_\rho$ is ρ-bounded if and only if there exists a constant $r > 0$ such that $||f||_\rho^s \leq r$ for all $f \in Z$.

1.1.5

We say that a modular ρ satisfies the σ-Fatou property if $0 \leq f_n \uparrow f \in E$ implies $\rho(f_n) \uparrow \rho(f)$. We say that a modular ρ satisfies the σ-Lebesgue property if from the conditions $f_n \downarrow 0$ in E and $\rho(\lambda f_1) < \infty$ for some $\lambda > 0$ it follows that $\rho(\lambda f_n) \downarrow 0$. We say that ρ satisfies the σ-Levi property if $0 \leq f_n \uparrow$ in E and the set $\{f_n : n \in \mathbb{N}\}$ is modularly bounded implies that $f_n \uparrow f$ in E for some $f \in E$ (see [9]).

1.1.6

Let ρ_1 be an s_1-modular and ρ_2 an s_2-modular. Let E_0 be a linear subspace of E. We say that ρ_2 is non weaker than ρ_1 on E_0, in symbols $\rho_1 \prec \rho_2$ on E_0, if for every sequence (f_n) in E_0 $f_n \xrightarrow{\rho_2} 0$ implies $f_n \xrightarrow{\rho_1} 0$.

1.1.7

We say that ρ-convergence of sequence (f_n) in E_ρ is generated by a linear topology if there exists a linear topology $\mathcal{T}$ in E_ρ such that $f_n \xrightarrow{\mathcal{T}} 0$ if and only if $f_n \xrightarrow{\rho} 0$ for every sequence (f_n) in E_ρ.

1.1.8

It is said that ρ satisfies the $B2$-condition in E_ρ if $\rho(f_n) \to 0$ implies $\rho(2f_n) \to 0$ for every sequence (f_n) in E_ρ (see [7]).

1.2 Modular bases.

In this section we introduce a notion of an s-modular base in linear space, which is a generalization of the notion of a modular base introduced by Leśniewicz in [5] and [6]. A notion of s-premodular bases in linear lattices was considered by Leśniewicz and Orlicz in [4].

1.2.1

Let U be an arbitrary subset of E. By $bal.U$ we denote the set of all functions $f \in E$ such that $f = ag$, where $|a| \leq 1, g \in U$. If $bal.U = U$, then U is called a balanced set.

1.2.2

Let U be an arbitrary nonempty subset of E,
$\Gamma_s(U) = \{\alpha f + \beta g : f, g \in U, |\alpha|^s + |\beta|^s \leq 1\}$ for $U \subset E$, $0 < s \leq 1$.
A non-void family $\mathcal{B}$ of subsets of E will be called an s-modular base in E if the following conditions are satisfied:

(M1) for every two sets U_1, $U_2 \in \mathcal{B}$ there exists $U \in \mathcal{B}$ such that $\Gamma_s(U) \subset U_1 \cap U_2$,

(M2) every set $U \in \mathcal{B}$ is absorbing in E, i.e. for every $f \in E$ there exists a number $\lambda \neq 0$ such that $\lambda f \in U$.

If $0 < s_1 < s_2 \leq 1$ and a family $\mathcal{B}$ is a s_2-modular base in E, then $\mathcal{B}$ is also a s_1-modular base.

An s-modular base $\mathcal{B} = \{U_n\}$ of absorbing and balanced sets in E such that $\Gamma_s(U_{n+1}) \subset U_n$ for $n \in \mathbb{N}$, is called a sequential s-modular base in E ([5], 1.1). A 1-modular base will be called briefly a modular base.

1.2.3

Let $\mathcal{B}_1$ be an s_1-modular base on E and let $\mathcal{B}_2$ be an s_2-modular base on E. We shall say that $\mathcal{B}_2$ is non-weaker than $\mathcal{B}_1$, in symbols $\mathcal{B}_1 \prec \mathcal{B}_2$ if there exists a number $\alpha \neq 0$ such that for every set $U_1 \in \mathcal{B}_1$ there exists a set $U_2 \in \mathcal{B}_2$ satisfying $\alpha U_2 \subset U_1$.

We say that $\mathcal{B}_1$ and $\mathcal{B}_2$ are equivalent, in symbols $\mathcal{B}_1 \sim \mathcal{B}_2$, if symultaneuosly $\mathcal{B}_1 \prec \mathcal{B}_2$ and $\mathcal{B}_2 \prec \mathcal{B}_1$.

1.2.4

([5], 2.1) We say that the base $\mathcal{B}$ of all neighbourhoods of the origin is a linear-topological base if it satisfies the following conditions:

(LT1) for every set $U_1 \in \mathcal{B}$ there exists a set $U_2 \in \mathcal{B}$ such that $U_2 + U_2 \subset U_1$,

(LT2) for every set $U_1 \in \mathcal{B}$ there exists a set $U_2 \in \mathcal{B}$ such that $bal.U_2 \subset U_1$,

(LT3) for each two sets $U_1, U_2 \in \mathcal{B}$ there exists $U_3 \in \mathcal{B}$ such that $U_3 \subset U_1 \cap U_2$,

(LT4) every set $U \in \mathcal{B}$ is absorbing in E.

If non-void family of sets $\mathcal{B}$ satisfies the conditions (LT1)–(LT4), there is a unique linear topology $\mathcal{T}$ on E, i.e. $(E, \mathcal{T})$ is a linear-topological space.

Theorem 1.1. *If $\mathcal{B}_1 \sim \mathcal{B}_2$ and $\mathcal{B}_1$ is a linear topological base, then $\mathcal{B}_2$ is also a linear topological base (see [5], 2.5).*

1.2.5

Let $\mathcal{B}$ be an s-modular base in E. A sequence (f_n) in E is called convergent to $f \in E$ with respect to $\mathcal{B}$, in symbols $f_n \overset{\mathcal{B}}{\to} f$, if there exists a number $\alpha \neq 0$ such that for every $U \in \mathcal{B}$ there exists a natural number N such that for every $n \geq N$ there holds $\alpha(f_n - f) \in U$ (see [5]).

Theorem 1.2. *([6], 1.3). Let $\mathcal{B}_1$ be an s_1-modular base and let $\mathcal{B}_2$ be a sequential s_2-modular base in E. Relation $\mathcal{B}_1 \prec \mathcal{B}_2$ holds if and only if for every sequence (f_n) in E we have: $f_n \overset{\mathcal{B}_2}{\to} 0$ implies $f_n \overset{\mathcal{B}_1}{\to} 0$.*

Theorem 1.3. *Let $\mathcal{B}$ be an s-modular base in E. Then the family*

$$\mathcal{B}^{\wedge} = \{ \bigcup_{N=1}^{\infty} (\sum_{n=1}^{N} U_n) : \quad (U_n) \textit{ is a sequence of sets in } \mathcal{B}\}$$

is a linear topological base in E. Moreover, the following conditions are satisfied:

(1) $\mathcal{B}^{\wedge} \prec \mathcal{B}$

(2) if $\mathcal{B}_1$ is an arbitrary linear topological base in E such that $\mathcal{B}_1 \prec \mathcal{B}$, then $\mathcal{B}_1 \prec \mathcal{B}^{\wedge}$.

Proof It suffices to note that the inclusion $bal.U \subset \Gamma_s(U)$ holds for any $U \in \mathcal{B}$. The rest follows from ([5], 4.1 and 4.2).

Theorem 1.4. *Let $\mathcal{B}$ be an s-modular base in E. Then the family of sets of the form $\mathcal{B}^{\vee} = \{\alpha U : U \in \mathcal{B}, \alpha \neq 0\}$ is a linear topological base in E. Moreover, the following conditions are satisfied:*

(i) $\mathcal{B} \prec \mathcal{B}^{\vee}$

(ii) if $\mathcal{B}_1$ is an arbitrary linear topological base in E such that $\mathcal{B} \prec \mathcal{B}_1$, then $\mathcal{B}^{\vee} \prec \mathcal{B}_1$.

Proof Denote by $\Delta(A) = \{\alpha f + \beta g : f, g \in A, \ \sup(|\alpha|, |\beta|) \leq 1\}$ for $A \subset E$. It suffices to note that the inclusion $\Delta(2^{-\frac{1}{s}} \lambda U) \subset \lambda \Gamma_s(U)$ holds for any $U \in \mathcal{B}$ and $\lambda > 0$. The rest follows from ([5], 3.1 and 3.4).

Theorem 1.5. *Let $\mathcal{B}$ be an s-modular base in E. Then $\mathcal{B}$ is a linear topological base if and only if $\mathcal{B}^{\wedge} \sim \mathcal{B}^{\vee}$.*

Proof It follows from Theorem 0.1, 0.3 and 0.4 (see [5], 3.5, 4.4).

2 Two-modular topology on modular spaces.

Let ρ be an s-modular on linear space E, E_ρ the modular space of ρ and E^0_ρ a linear subspace of the space E_ρ. Denote $B_\rho(\varepsilon) = \{f \in E : \rho(f) < \varepsilon\}$. Then the family $\mathcal{B}_{\rho,E^0_\rho} = \{B_\rho(\varepsilon) \cap E^0_\rho : \varepsilon > 0\}$ is an s-modular base in E^0_ρ.

Moreover, the family $\mathcal{B}^c_{\rho,E^0_\rho} = \{B_\rho(2^{-n+1}) \cap E^0_\rho : n \in \mathbb{N}\}$ is a sequential s-modular base in E^0_ρ, equivalent to $\mathcal{B}_{\rho,E^0_\rho}$ ([6], 2.2). It is seen that $f_n \xrightarrow{\rho} f$ if and only if $f_n \to f$ with respect to the base $\mathcal{B}_{\rho,E^0_\rho}$ and if and only if $f_n \to f$ with respect to the base $\mathcal{B}^c_{\rho,E^0_\rho}$ for a sequence (f_n) in E^0_ρ and $f \in E^0_\rho$ ([6], 2.3). In view of Theorem 0.3 we get:

Theorem 2.1. *The family*

$$\mathcal{B}^\wedge_{\rho,E^0_\rho} = \{\bigcup_{n=1}^{\infty}(\sum_{i=1}^{n}(B_\rho(\varepsilon_i) \cap E^0_\rho)) : \varepsilon_i > 0\}$$

constitutes a base of neighbourhoods of 0 *for some linear topology in* E^0_ρ, *which we will denote by* $\mathcal{T}^\wedge_{\rho,E^0_\rho}$. *Moreover, the following conditions hold:*

(i) $\mathcal{B}^\wedge_{\rho,E^0_\rho} \prec \mathcal{B}_{\rho,E^0_\rho}$

(ii) if $\mathcal{B}$ *is an arbitrary linear topological base in* E^0_ρ *such that* $\mathcal{B} \prec \mathcal{B}_{\rho,E^0_\rho}$, *then* $\mathcal{B} \prec \mathcal{B}^\wedge_{\rho,E^0_\rho}$.

Theorem 2.2. *The topology* $\mathcal{T}^\wedge_{\rho,E^0_\rho}$ *is the finest of all linear topologies* $\mathcal{T}$ *on* E^0_ρ, *which satisfy the condition* $(*)$: $f_n \xrightarrow{\rho} 0$ *implies* $f_n \xrightarrow{\mathcal{T}} 0$ *for every sequence* (f_n) *in* E^0_ρ.

Proof Let $\mathcal{T}$ be a linear topology on E^0_ρ which satisfies the condition $(*)$. We have $f_n \xrightarrow{\rho} 0$ if and only if $f_n \to 0$ with respect to $\mathcal{B}^c_{\rho,E^0_\rho}$. Hence from Theorem 0.2 we get $\mathcal{B}_\mathcal{T} \prec \mathcal{B}^c_{\rho,E^0_\rho}$, where $\mathcal{B}_\mathcal{T}$ is a base of neighbourhoods of 0 for $\mathcal{T}$. Since $\mathcal{B}^c_{\rho,E^0_\rho} \sim \mathcal{B}_{\rho,E^0_\rho}$, we get $\mathcal{B}_\mathcal{T} \prec \mathcal{B}_{\rho,E^0_\rho}$ and hence by Theorem 1.1 we get that $\mathcal{B}_\mathcal{T} \prec \mathcal{B}^\wedge_{\rho,E^0_\rho}$. Since $\mathcal{B}_T$ and $B^\wedge_{\rho,E^0_\rho}$ are linear-topological bases, we get $\mathcal{T} \prec \mathcal{T}^\wedge_{\rho,E^0_\rho}$.

In the case of $E^0_\rho = E_\rho$ the topology $\mathcal{T}^\wedge_{\rho,E^0_\rho}$ will be denoted by $\mathcal{T}^\wedge_\rho$ and called a lower topology (or the modular topology) for a modular ρ. Moreover, in this case we will denote modular bases by $\mathcal{B}_\rho$ (respectively $\mathcal{B}^\wedge_\rho$) instead of $\mathcal{B}_{\rho,E_\rho}$ (respectively $\mathcal{B}^\wedge_{\rho,E_\rho}$).

Now we recall a definition of two-modular convergence on $E_\rho \cap E_{\rho^*}$ and prove that we can define a modular $\tilde{\rho}$ such that the two-modular convergence is equivalent to the modular convergence with respect to $\tilde{\rho}$.

Definition 2.1. *Let ρ and ρ^* be two modulars on E. We say that a sequence (f_n) in $E_\rho \cap E_{\rho^*}$ is two-modularly convergent* (γ-convergent) to $f \in E_\rho \cap E_{\rho^*}$ if $f_n \xrightarrow{\rho^*} f$ as $n \to \infty$ and (f_n) is ρ-bounded. We denote this by $f_n \xrightarrow{\gamma} f$ (see [8]).

Definition 2.2. *We say that a linear functional F on $E_\rho \cap E_{\rho^*}$ is γ-linear if $f_n \xrightarrow{\rho^*} 0$ and (f_n) is ρ-bounded implies $F(f_n) \to 0$.*

Definition 2.3. *Let ρ and ρ^* be modulars on E. Assume that modular ρ is s-convex. We define a functional $\tilde{\rho}$ by the formula*

$$\tilde{\rho}(f) = \begin{cases} \rho^*(f) & \text{if} \quad \rho(f) \le 1 \\ \infty & \text{if} \quad \rho(f) > 1 \end{cases} .$$

It is obvious that the functional $\tilde{\rho}$ satisfies the conditions (i)–(iii) from 0.1.1, so it is a modular on E.

Theorem 2.3. *If ρ is an s-convex modular then a sequence $(f_n) \subset E_\rho \cap E_{\rho^*}$ is γ-convergent to $f \in E_\rho \cap E_{\rho^*}$ if and only if it is $\tilde{\rho}$-convergent to f.*

Proof Let $f_n \xrightarrow{\tilde{\rho}} f$. Then there exists $\lambda > 0$ such that $\rho^*(\lambda(f_n - f)) = \tilde{\rho}(\lambda(f_n - f)) \to 0$ as $n \to \infty$, so $f_n \xrightarrow{\rho^*} f$. Further, for every $\varepsilon_n \to 0$ we can find $n_0 \in \mathbb{N}$ such that $\frac{|\varepsilon_n|}{\lambda} \le 1$ and $\tilde{\rho}(\lambda(f_n - f)) < \infty$ for $n \ge n_0$. Then $\rho(\lambda(f_n - f)) \le 1$ for every $n \ge n_0$ and $\rho(\varepsilon_n(f_n - f)) \le \frac{|\varepsilon_n|^s}{\lambda^s}\rho(\lambda(f_n - f)) \le \frac{|\varepsilon_n|^s}{\lambda^s} \to 0$. Hence

$$\rho(\tfrac{1}{2}\varepsilon_n f_n) = \rho(\tfrac{1}{2}\varepsilon_n(f_n - f) + \tfrac{1}{2}\varepsilon_n f) \le \rho(\varepsilon_n(f_n - f)) + \rho(\varepsilon_n f) \le \rho(\varepsilon_n(f_n - f)) + |\varepsilon_n|^s \rho(f) \to 0$$

i.e. (f_n) is ρ-bounded. Therefore $f_n \xrightarrow{\gamma} f$.

Assume that $f_n \xrightarrow{\gamma} f$. Since $f_n \xrightarrow{\rho^*} f$, there exists $\lambda_1 > 0$ such that $\rho^*(\lambda_1(f_n - f)) \to 0$. Since $(f_n - f) \subset E_\rho$, then there exists a number $\lambda_2 > 0$ such that $\rho(\lambda_2(f_n - f)) \le 1$. Let $\lambda = \min(\lambda_1, \lambda_2)$. Then we have $\tilde{\rho}(\lambda(f_n - f)) = \rho^*(\lambda(f_n - f)) \to 0$.

Let $B_\rho(\varepsilon) = \{f \in E : \rho(f) \le \varepsilon\}$ and $B_{\rho^*}(\varepsilon) = \{f \in E : \rho^*(f) \le \varepsilon\}$ for $r > 0$ and $\varepsilon > 0$. Denote $\mathcal{B}_\rho = \{B_\rho(\varepsilon) : \varepsilon > 0\}$, $\mathcal{B}_{\rho^*} = \{B_{\rho^*}(\varepsilon) : \varepsilon > 0\}$.

We introduce a linear topology on $E_\rho \cap E_{\rho^*}$ and we will prove that the convergence in this topology is equivalent to the two-modular convergence. Next we prove that introduced two-modular topology is equal to some modular topology (see Theorems 1.4 and 1.5). We also show that this topology is the finest of all linear topologies τ on $E_\rho \cap E_{\rho^*}$ satisfying the condition $\tau|_Z = T_{\rho^*}|_Z$, where $\mathcal{T}_{\rho^*}$ is the topology for which the base of the neighbourhoods of zero is $\mathcal{B}_{\rho^*}$ and Z is an arbitrary ρ-bounded set (see Theorem 1.6).

Theorem 2.4. *Let ρ and ρ^* be two modulars defined on E. Assume that ρ is s-convex and ρ^* satisfies the $B2$ condition. The family $\mathcal{B}_W$ of all sets of the form*

$$(*) \qquad \bigcup_{n=1}^{\infty}\Big(\sum_{i=1}^{n} iB_\rho(r) \cap B_{\rho^*}(\varepsilon_i)\Big),$$

where $\{\varepsilon_i : i \geq 1\}$ is a sequence positive numbers, forms a base of neighbourhoods of 0 for some linear topology on $E_\rho \cap E_{\rho^*}$.

Proof We will show that the conditions (LT1)–(LT4) are satisfied. Let $M(\{B_{\rho^*}(\varepsilon_i)\}, B_\rho(r)) = \bigcup_{n=1}^{\infty}(\sum_{i=1}^{n} iB_\rho(r) \cap B_{\rho^*}(\varepsilon_i))$, where $\{B_{\rho^*}(\varepsilon_i)\}$ is a sequence of the sets of $\mathcal{B}_{\rho^*}$ and $B_\rho(r) \in \mathcal{B}_\rho$ be an arbitrary neighbourhood of 0 in $\mathcal{B}_W$. Let $\lambda \in \mathbb{R}$. Then

$\lambda M(\{B_{\rho^*}(\varepsilon_i)\}, B_\rho(r)) = \{\lambda \sum_{i=1}^{n} f_i : f_i \in B_{\rho^*}(\varepsilon_i), i^{-1}f_i \in B_\rho(r)\} =$
$\{\sum_{i=1}^{n} \lambda f_i : \ \lambda f_i \in \lambda B_{\rho^*}(\varepsilon_i), i^{-1}\lambda f_i \in \lambda B_\rho(r)\} = M(\{\lambda B_{\rho^*}(\varepsilon_i)\}, \lambda B_\rho(r))$.

Hence we obtain that the sets $(*)$ are balanced. For every function $f \in E_\rho \cap E_{\rho^*}$ there exists $\lambda \neq 0$ such that $\lambda f \in B_{\rho^*}(\varepsilon_1) \cap B_\rho(r)$, i.e. the sets $M(\{B_{\rho^*}(\varepsilon_i)\}, B_\rho(r))$ are absorbing. Let $M(\{B_{\rho^*}(\varepsilon_i)\}, B_\rho(r_1))$ and $M(\{B_{\rho^*}(\delta_i)\}, B_\rho(r_2))$ be arbitrary neighbourhoods of 0 in $\mathcal{B}_W$. Since $\mathcal{B}_\rho$ and $\mathcal{B}_{\rho^*}$ are the modular bases, there exist numbers η_i and r_3 such that $B_{\rho^*}(\eta_i) \subset B_{\rho^*}(\varepsilon_i) \cap B_{\rho^*}(\delta_i)$ and $B_\rho(r_3) \subset B_\rho(r_1) \cap B_\rho(r_2)$. Thus we have $M(\{B_{\rho^*}(\eta_i)\}, B_\rho(r_3)) \subset M(\{B_{\rho^*}(\varepsilon_i)\}, B_\rho(r_1)) \cap M(\{B_{\rho^*}(\delta_i)\}, B_\rho(r_2))$.

Since modular ρ is s-convex and ρ^* satisfies the condition $B2$, we can choose $B_{\rho^*}(\alpha_i) \in \mathcal{B}_{\rho^*}$ and $B_\rho(r') \in \mathcal{B}_\rho$ such that $B_{\rho^*}(\alpha_i)+B_{\rho^*}(\alpha_i) \subset B_{\rho^*}(\varepsilon_i)$ and $B_\rho(r')+B_\rho(r') \subset B_\rho(r)$. We prove that $M(\{B_{\rho^*}(\alpha_i)\}, B_\rho(r')) + M(\{B_{\rho^*}(\alpha_i)\}, B_\rho(r')) \subset M(\{B_{\rho^*}(\varepsilon_i)\}, B_\rho(r))$.

In fact, if $f \in M(\{B_{\rho^*}(\alpha_i)\}, B_\rho(r'))+M(\{B_{\rho^*}(\alpha_i)\}, B_\rho(r'))$, then $f = g+h$, where $g = g_1 + \cdots + g_m$, $g_i \in B_{\rho^*}(\alpha_i), i^{-1}g_i \in B_\rho(r'), 1 \leq i \leq m$ and $h = h_1 + \cdots + h_n$, where $h_i \in B_{\rho^*}(\alpha_i), i^{-1}h_i \in B_\rho(r'), 1 \leq i \leq n$. For $m \leq n$ we obtain

$$f = (g_1 + h_1) + (g_2 + h_2) + \cdots + (g_m + h_m) + h_{m+1} + \cdots + h_n.$$

Since $B_{\rho^*}(\alpha_i)+B_{\rho^*}(\alpha_i) \subset B_{\rho^*}(\varepsilon_i)$ and $B_\rho(r')+B_\rho(r') \subset B_\rho(r)$, so $g_i+h_i \in B_{\rho^*}(\varepsilon_i)$ and $i^{-1}(g_i + h_i) \in B_\rho(r)$ for $i = 1, ..., m$ and $h_i \in B_{\rho^*}(\alpha_i), i^{-1}h_i \in B_\rho(r')$ for $i = m+1, ..., n$. From this it follows that
$f \in M(\{B_{\rho^*}(\varepsilon_i)\}, B_\rho(r))$.

We have proved that there is a unique linear topology τ on $E_\rho \cap E_{\rho^*}$ such that $\mathcal{B}_W$ is the base of the neighbourhoods of zero of τ.

Definition 2.4. *The topology defined in Theorem 1.4 is called the two-modular topology on* $E_\rho \cap E_{\rho^*}$ *and it is denoted by* $\gamma_W(\mathcal{T}_\rho, \mathcal{T}_{\rho^*})$ *or shortly* γ_W.

Theorem 2.5. *Let* ρ *and* ρ^* *be two modulars defined on* E. *Assume that* ρ *is* s*-convex and* ρ^* *satisfies the* $B2$ *condition. The base of neighbourhoods of zero in the modular topology* $\mathcal{T}^\wedge_{\tilde\rho}$ *is equivalent to the base of neighbourhoods zero of* $\gamma_W(\mathcal{T}_\rho, \mathcal{T}_{\rho^*})$.

Proof Let $K = B_\rho(1)$. Since ρ is s-convex, for arbitrary set $B_\rho(r) \in \mathcal{B}_\rho$ there exists a natural number n_0 such that $K \subset nB_\rho(r)$ for $n > n_0$, hence

$$\begin{aligned}&W(\{B_{\rho^*}(\varepsilon_{n_0+n}) \cap K\}) =\\ &\textstyle\bigcup_{n=1}^\infty (B_{\rho^*}(\varepsilon_{n_0+1}) \cap K + \cdots + B_{\rho^*}(\varepsilon_{n_0+n}) \cap K) \subset\\ &\textstyle\bigcup_{n=1}^\infty (B_{\rho^*}(\varepsilon_{n_0+1}) \cap (n_0+1)B_\rho(r) + \cdots + B_{\rho^*}(\varepsilon_{n_0+n}) \cap (n_0+n)B_\rho(r)) =\\ &M(\{B_{\rho^*}(\varepsilon_i)\}, B_\rho(r)).\end{aligned}$$

We have showed that for every set $M \in \mathcal{B}_W$ there exists a set $W \in \mathcal{B}^\wedge_{\tilde\rho}$ such that $W \subset M$, i.e. $\mathcal{B}_W \prec \mathcal{B}^\wedge_{\tilde\rho}$.

For every sequence $\{B_{\rho^*}(\varepsilon_i)\} \subset \mathcal{B}_{\rho^*}$ there exists a sequence $\{B_{\rho^*}(\beta_i)\} \subset \mathcal{B}_{\rho^*}$ such that $B_{\rho^*}(\varepsilon_1) = B_{\rho^*}(\beta_1)$ and $(k+1)^{-1}B_{\rho^*}(\beta_{k+1}) \subset B_{\rho^*}(\varepsilon_{n_k+1}) \cap \cdots \cap B_{\rho^*}(\varepsilon_{n_{k+1}})$, where $n_k = \frac{1}{2}k(k+1)$, $k \in \mathbb{N}$. Thus we obtain:

$$\begin{aligned}&M(\{B_{\rho^*}(\beta_k)\}, K) = \textstyle\bigcup_{k=1}^\infty (B_{\rho^*}(\beta_1) \cap K + \cdots + B_{\rho^*}(\beta_{k+1}) \cap (k+1)K) \subset\\ &\textstyle\bigcup_{k=1}^\infty (B_{\rho^*}(\varepsilon_1) \cap K + \cdots + (B_{\rho^*}(\varepsilon_{n_k+1}) \cap K + \cdots + B_{\rho^*}(\varepsilon_{n_{k+1}}) \cap K)) =\\ &W(\{B_{\rho^*}(\varepsilon_i) \cap K\}).\end{aligned}$$

We have proved that for arbitrary $W \in \mathcal{B}^\wedge_{\tilde\rho}$ there exists the set $M \in \mathcal{B}_W$ such that $M \subset W$, i.e. $\mathcal{B}^\wedge_{\tilde\rho} \prec \mathcal{B}_W$.

Since $\mathcal{T}^\wedge_{\tilde\rho} \sim \gamma_W(\mathcal{T}_\rho, \mathcal{T}_{\rho^*})$, from Theorem 2.5 in [5] we obtain that $\mathcal{T}^\wedge_{\tilde\rho}$ is a linear topology on $E_\rho \cap E_{\rho^*}$.

Theorem 2.6. *Let* ρ *and* ρ^* *be two modulars defined on* E. *Assume that* ρ *is* s*-convex and* ρ^* *satisfies the* $B2$ *condition. The two-modular topology* γ_W *is the finest of all linear topologies* τ *on* $E_\rho \cap E_{\rho^*}$ *satisfying the condition* $\tau|_Z = \mathcal{T}_{\rho^*}|_Z$, *where* Z *is arbitrary* ρ*-bounded set.*

Proof Let Z be arbitrary ρ-bounded set. First we show that $\gamma_W|_Z = \mathcal{T}_{\rho^*}|_Z$. Let $\varepsilon > 0$. Since ρ^* satisfies the $B2$ condition, there exists a number $\varepsilon_1 > 0$ such that $B_{\rho^*}(\varepsilon_1) + B_{\rho^*}(\varepsilon_1) \subset B_{\rho^*}(\varepsilon)$. Next, there exists a number $\varepsilon_2 > 0$ such that $B_{\rho^*}(\varepsilon_2) + B_{\rho^*}(\varepsilon_2) \subset B_{\rho^*}(\varepsilon_1)$. By induction we obtain a sequence (ε_n) such that $B_{\rho^*}(\varepsilon_n) + B_{\rho^*}(\varepsilon_n) \subset B_{\rho^*}(\varepsilon_{n-1})$ for all $n \in \mathbb{N}$. Hence we obtain that

$$B_{\rho^*}(\varepsilon_1) + B_{\rho^*}(\varepsilon_2) + \cdots + B_{\rho^*}(\varepsilon_n) \subset$$
$$B_{\rho^*}(\varepsilon_1) + B_{\rho^*}(\varepsilon_2) + \cdots + B_{\rho^*}(\varepsilon_n) + B_{\rho^*}(\varepsilon_n) \subset$$
$$B_{\rho^*}(\varepsilon_1) + B_{\rho^*}(\varepsilon_2) + \cdots + B_{\rho^*}(\varepsilon_{n-1}) + B_{\rho^*}(\varepsilon_{n-1}) \subset \cdots \subset$$
$$B_{\rho^*}(\varepsilon_1) + B_{\rho^*}(\varepsilon_2) + B_{\rho^*}(\varepsilon_2) \subset B_{\rho^*}(\varepsilon_1) + B_{\rho^*}(\varepsilon_1) \subset B_{\rho^*}(\varepsilon)$$

for every n and

$$B_{\rho^*}(\varepsilon_1) \cap B_\rho(r) + B_{\rho^*}(\varepsilon_2) \cap 2B_\rho(r) + \cdots B_{\rho^*}(\varepsilon_n) \cap nB_\rho(r) \subset$$
$$B_{\rho^*}(\varepsilon_1) + B_{\rho^*}(\varepsilon_2) + \cdots + B_{\rho^*}(\varepsilon_n) \subset B_{\rho^*}(\varepsilon).$$

Thus we obtain that $M(\{B_{\rho^*}(\varepsilon_i)\}, B_\rho(r)) \subset B_{\rho^*}(\varepsilon)$ i.e. $\mathcal{T}_{\rho^*} \prec \gamma_W$.

We will prove that $\gamma_W|_Z \prec \mathcal{T}_{\rho^*}|_Z$, i.e. every neighbourhood of the form $(f_0 + M(\{B_{\rho^*}(\varepsilon_i)\}, B_\rho(r))) \cap Z$ contains a neighbourhood of the form $(f_0 + B_{\rho^*}(\varepsilon)) \cap Z$. Since the set $Z - Z$ is ρ-bounded for every $M = M(\{B_{\rho^*}(\varepsilon_i)\}, B_\rho(r))$ there exists $m \in \mathbb{N}$ such that $Z - Z \subset mB_\rho(r)$. Let $\varepsilon_m = \varepsilon$ for $m \in \mathbb{N}$. Then $B_{\rho^*}(\varepsilon) \cap (Z - Z) \subset B_{\rho^*}(\varepsilon_m) \cap mB_\rho(r) \subset M$. Since for every $f_0 \in Z$ we have $(f_0 + B_{\rho^*}(\varepsilon)) \cap Z \subset f_0 + B_{\rho^*}(\varepsilon) \cap (Z - Z)$, so $(f_0 + B_{\rho^*}(\varepsilon)) \cap Z \subset f_0 + B_{\rho^*}(\varepsilon) \cap (Z - Z) \subset (f_0 + M) \cap Z$.

Assume that $\tau|_Z = \mathcal{T}_{\rho^*}|_Z$. Let B' be a ρ-bounded neighbourhood of 0 in τ. Since τ is a linear topology, there exists a sequence $\{B'_n\}$ such that $B'_1 + \cdots + B'_n \subset B'$ for all n. Since the sets $nB_\rho(r)$ are ρ-bounded, $\mathcal{T}_{\rho^*}|_{nB_\rho(r)} = \tau|_{nB_\rho(r)}$. Hence for every n there exists $\mathcal{T}_{\rho^*}$-neighbourhood of zero $B_{\rho^*}(\varepsilon_n)$ such that $B_{\rho^*}(\varepsilon_n) \cap nB_\rho(r) \subset B'_n$. We have

$$B_{\rho^*}(\varepsilon_1) \cap B_\rho(r) + B_{\rho^*}(\varepsilon_2) \cap 2B_\rho(r) + \cdots + B_{\rho^*}(\varepsilon_n) \cap nB_\rho(r) \subset$$
$B'_1 + B'_2 + \cdots + B'_n \subset B'$, i.e. $M \subset B'$.

We have proved that if $\tau|_Z = \mathcal{T}_{\rho^*}|_Z$, then $\tau \prec \gamma_W$.

Now we will show that the convergence in two-modular topology γ_W is equivalent with the two-modular convergence (see Theorem 1.8). First we will prove a fact, which we use in the proof.

Denote:

$$\mathcal{B}_1 = \{B_{\rho^*}(\varepsilon_0) \cap \bigcap_{n=1}^{\infty} ((nB_\rho(r) + B_{\rho^*}(\varepsilon_n)) : \varepsilon_n > 0\}, \tag{1}$$

$$\mathcal{B}_2 = \{B_{\rho^*}(\varepsilon_0) \cap \bigcap_{n=1}^{\infty} ((a_n B_\rho(r) + B_{\rho^*}(\varepsilon_n)) : \varepsilon_n > 0\}, \tag{2}$$

where (a_n) is an arbitrary sequence positive numbers such that $a_n \to \infty$ as $n \to \infty$ and $a_1 \leq 1$.

Theorem 2.7. *Let ρ and ρ^* be two modulars defined on E. Assume that modular ρ is s-convex and modular ρ^* satisfies the B2 condition. The bases $\mathcal{B}_1$ i $\mathcal{B}_2$ are equivalent and the sets of the form* (1) *and* (2) *constitute a base of neighbourhoods of zero of two-modular topology.*

Proof Assume that U is an arbitrary set of the form (1). Let (k_n) be an increasing sequence of natural numbers such that $k_1 = 1$, $k_n \geq a_n$ for all $n \geq 2$ and (δ_n) be a sequence such that the following inclusions hold $B_{\rho^*}(\delta_0) \subset \bigcap_{p=0}^{k_1-1} B_{\rho^*}(\varepsilon_p)$ and $B_{\rho^*}(\delta_n) \subset \bigcap_{p=k_n}^{k_{n+1}-1} B_{\rho^*}(\varepsilon_p)$ for every $n \in \mathbb{N}$. Since $a_n B_\rho(r) + B_{\rho^*}(\delta_n) \subset \bigcap_{p=k_n}^{k_{n+1}-1}(pB_\rho(r)+B_{\rho^*}(\varepsilon_p))$ we get $B_{\rho^*}(\delta_0)\cap\bigcap_{n=1}^{\infty}(a_n B_\rho(r) + B_{\rho^*}(\delta_n)) \subset B_{\rho^*}(\varepsilon_0)\cap \bigcap_{p=1}^{\infty}(pB_\rho(r) + B_{\rho^*}(\varepsilon_p))$, i.e. $\mathcal{B}_1 \prec \mathcal{B}_2$.

We prove that $\mathcal{B}_2 \prec \mathcal{B}_1$. Assume that V is an arbitrary set of the form (2). Let (m_n) be an increasing sequence such that $a_{m_n} \geq n$ for all $n \in \mathbb{N}$ and (δ_n) a sequence of positive numbers such that $B_{\rho^*}(\delta_0) \subset \bigcap_{p=0}^{m_1-1} B_{\rho^*}(\varepsilon_p)$ and $B_{\rho^*}(\delta_n) \subset \bigcap_{p=m_n}^{m_{n+1}-1} B_{\rho^*}(\varepsilon_p)$ for all $n \in \mathbb{N}$. Hence we get $B_{\rho^*}(\delta_n) + nB_\rho(r) \subset \bigcap_{p=m_n}^{m_{n+1}-1}(B_{\rho^*}(\varepsilon_p) + a_p B_\rho(r))$ and from this it follows that
$B_{\rho^*}(\delta_0) \cap \bigcap_{n=1}^{\infty}(nB_\rho(r) + B_{\rho^*}(\delta_n)) \subset B_{\rho^*}(\varepsilon_0) \cap \bigcap_{p=1}^{\infty}(a_p B_\rho(r) + B_{\rho^*}(\varepsilon_p))$.

We will show that the sets of the form (1) and (2) constitute the base of neighbourhoods of zero of two-modular topology $\gamma_W(\mathcal{T}_\rho, \mathcal{T}_{\rho^*})$. First we will prove that every set of the form $B_{\rho^*}(\varepsilon_0)\cap\bigcap_{n=1}^{\infty}\left(n \cdot 2^{\frac{n}{s}} B_\rho(r) + B_{\rho^*}(\varepsilon_n)\right)$ contains some neighbourhood $M(\{B_{\rho^*}(\delta_n)\}, B_\rho(r))$ in the topology γ_W, i.e. $\mathcal{B}_2 \prec \mathcal{B}_W$ for $a_n = n \cdot 2^{\frac{n}{s}}$. Let $B_{\rho^*}(\delta_1)$ be an arbitrary set such that $B_{\rho^*}(\delta_1) + B_{\rho^*}(\delta_1) \subset B_{\rho^*}(\varepsilon_0)$. By the induction we get a sequence $B_{\rho^*}(\delta_n)$ such that $B_{\rho^*}(\delta_n) + B_{\rho^*}(\delta_n) \subset B_{\rho^*}(\delta_{n-1}) \cap B_{\rho^*}(\varepsilon_{n-1})$ for all $n > 1$. Thus we have

$B_{\rho^*}(\delta_1) + B_{\rho^*}(\delta_2) + \cdots + B_{\rho^*}(\delta_n) \subset B_{\rho^*}(\delta_1) + B_{\rho^*}(\delta_2) + \cdots + B_{\rho^*}(\delta_n) + B_{\rho^*}(\delta_n) \subset B_{\rho^*}(\delta_1) + B_{\rho^*}(\delta_2) + \cdots + B_{\rho^*}(\delta_{n-1}) + B_{\rho^*}(\delta_{n-1}) \cap B_{\rho^*}(\varepsilon_{n-1}) \subset B_{\rho^*}(\delta_1) + B_{\rho^*}(\delta_2) + \cdots + B_{\rho^*}(\delta_{n-1}) + B_{\rho^*}(\delta_{n-1}) \subset \cdots \subset B_{\rho^*}(\delta_1) + B_{\rho^*}(\delta_1) \subset B_{\rho^*}(\varepsilon_0)$.

Hence for every $p \in \mathbb{N}$ we have $B_{\rho^*}(\delta_n) + B_{\rho^*}(\delta_{n+1}) + \cdots + B_{\rho^*}(\delta_{n+p}) \subset B_{\rho^*}(\delta_n) + B_{\rho^*}(\delta_n) \subset B_{\rho^*}(\delta_{n-1}) \cap B_{\rho^*}(\varepsilon_{n-1}) \subset B_{\rho^*}(\varepsilon_{n-1})$ and

$$
\begin{aligned}
&M(\{B_{\rho^*}(\delta_n)\}, B_\rho(r)) = \\
&= \textstyle\bigcup_{p=1}^{\infty}(B_{\rho^*}(\delta_1) \cap B_\rho(r) + B_{\rho^*}(\delta_2) \cap 2B_\rho(r) + \cdots \\
&+ B_{\rho^*}(\delta_{n-1}) \cap (n-1)B_\rho(r) + B_{\rho^*}(\delta_n) \cap nB_\rho(r) + \\
&B_{\rho^*}(\delta_{n+1}) \cap (n+1)B_\rho(r) + \cdots + B_{\rho^*}(\delta_{n+p}) \cap (n+p)B_\rho(r)) \subset \\
&\textstyle\bigcup_{p=1}^{\infty}(B_\rho(r) + 2B_\rho(r) + \cdots + (n-1)B_\rho(r) + B_{\rho^*}(\delta_n) + \\
&B_{\rho^*}(\delta_{n+1}) + \cdots + B_{\rho^*}(\delta_{n+p})) \subset n \cdot 2^{\frac{n}{s}} B_\rho(r) + B_{\rho^*}(\varepsilon_{n-1})
\end{aligned}
$$

for every $n > 1$, so $M(\{B_{\rho^*}(\delta_n)\}, B_\rho(r)) \subset \bigcap_{n=1}^{\infty}(n \cdot 2^{\frac{n}{s}} B_\rho(r) + B_{\rho^*}(\varepsilon_{n-1}))$, i.e. $\mathcal{B}_2 \prec \mathcal{B}_W$.

Now we prove that every neighbourhood of zero $M(\{B_{\rho^*}(\varepsilon_n)\}, B_\rho(r))$ contains a set of the form $\bigcap_{n=1}^{\infty}(nB_\rho(r) + B_{\rho^*}(\delta_n))$, i.e. $\mathcal{B}_W \prec \mathcal{B}_1$. Let (m_n) be a sequence of natural numbers such that $m_n \geq n \cdot 2^{\frac{1}{s}}$ for $n \in \mathbb{N}$. Since modular ρ^* satisfies the $B2$ condition, there exists a sequence (δ_n) such that $B_{\rho^*}(\delta_0) + B_{\rho^*}(\delta_0) \subset B_{\rho^*}(\varepsilon_{m_1})$, $B_{\rho^*}(\delta_{p-1}) + B_{\rho^*}(\delta_{p-1}) \subset B_{\rho^*}(\varepsilon_{m_p})$, $B_{\rho^*}(\delta_p) \subset B_{\rho^*}(\delta_{p-1})$ for $p \in \mathbb{N}$. Let $f \in B_{\rho^*}(\delta_0) \cap \bigcap_{n=1}^{\infty}(B_{\rho^*}(\delta_n) + nB_\rho(r))$. Then $f \in B_{\rho^*}(\delta_0)$ and for every $n \in \mathbb{N}$ we have $f = g_n + h_n$, where $g_n \in nB_\rho(r), h_n \in B_{\rho^*}(\delta_n)$. Let $f_1 = g_1$, $f_n = g_n - g_{n-1}$ for $n > 1$. Hence for every $n \in \mathbb{N}$ we have

$$f_1 + f_2 + \cdots + f_n + h_n = g_1 + (g_2 - g_1) + \cdots$$
$$+(g_n - g_{n-1}) + h_n = g_n + h_n = f.$$

Since $f = g_{n-1} + h_{n-1}$, then $h_{n-1} = f - g_{n-1} = f_n + h_n$, so

$$f_n = h_n - h_{n-1} \in B_{\rho^*}(\delta_{n-1}) + B_{\rho^*}(\delta_n).$$

On the other hand $f_n = g_n - g_{n-1} \in nB_\rho(r) + (n-1)B_\rho(r) \subset m_n B_\rho(r)$. From the definition of the sets $B_{\rho^*}(\delta_n)$ it follows that $B_{\rho^*}(\delta_{n-1}) + B_{\rho^*}(\delta_n) \subset B_{\rho^*}(\delta_{n-1}) + B_{\rho^*}(\delta_{n-1}) \subset B_{\rho^*}(\varepsilon_{m_n})$, so $f_n \in B_{\rho^*}(\varepsilon_{m_n}) \cap m_n B_\rho(r)$.

Since $f \in E_\rho$ and modular ρ is s-convex, there exists $k_0 \in \mathbb{N}$ such that $f \in k_0 B_\rho(r)$. From the equality $h_n = f - g_n$ we get $h_n \in (k_0 + n)B_\rho(r)$, where k_0 is a natural number such that $f \in k_0 B_\rho(r)$. If $n_0 > \dfrac{k_0 - 2^{\frac{1}{s}}}{2^{\frac{1}{s}} - 1}$, then $m_{n_0+1} \geq (n_0 + 1) \cdot 2^{\frac{1}{s}} > k_0 + n_0$. Hence $h_{n_0} \in (k_0 + n_0)B_\rho(r) \subset m_{n_0+1} B_\rho(r)$.

On the other hand $h_{n_0} \in B_{\rho^*}(\delta_{n_0}) \subset B_{\rho^*}(\delta_{n_0}) + B_{\rho^*}(\delta_{n_0}) \subset B_{\rho^*}(\varepsilon_{m_{n_0+1}})$. Hence we have $h_{n_0} \in B_{\rho^*}(\varepsilon_{m_{n_0+1}}) \cap m_{n_0+1} B_\rho(r)$ and $f = f_1 + f_2 + \cdots + f_{n_0} + h_{n_0} \in B_{\rho^*}(\varepsilon_{m_1}) \cap m_1 B_\rho(r) + B_{\rho^*}(\varepsilon_{m_2}) \cap m_2 B_\rho(r) + \cdots + B_{\rho^*}(\varepsilon_{m_{n_0}}) \cap m_{n_0} B_\rho(r) + B_{\rho^*}(\varepsilon_{m_{n_0+1}}) \cap m_{n_0+1} B_\rho(r) \subset M(\{B_{\rho^*}(\varepsilon_n)\}, B_\rho(r))$. Since $f = g_n + h_n \in B_{\rho^*}(\delta_n) + nB_\rho(r)$ for every $n \in \mathbb{N}$, so $\bigcap_{n=1}^{\infty}(B_{\rho^*}(\delta_n) + nB_\rho(r)) \subset M(\{B_{\rho^*}(\varepsilon_n)\}, B_\rho(r))$.

Since $\mathcal{B}_W \prec \mathcal{B}_1$, $\mathcal{B}_2 \prec \mathcal{B}_W$ i $\mathcal{B}_1 \sim \mathcal{B}_2$, so the sets of the form (1) and (2) constitute the base of the neighbourhoods of zero of the two-modular topology γ_W.

From Theorem 1.8 in [9] we know that if ρ satisfies the $B2$ condition, then $f_n \xrightarrow{\rho} f$ if and only if $f_n \to f$ with respect to the topology $\mathcal{T}_\rho^\wedge$ and if and only if $f_n \to f$ with respect to the topology $\mathcal{T}_\rho$.

Theorem 2.8. *Let ρ and ρ^* be two modulars defined on E. Assume that ρ is s-convex, ρ^* satisfies the $B2$ condition and the sets $nB_\rho(r)$ are closed in topology $\mathcal{T}_{\rho^*}$ for all $n \in \mathbb{N}$. For an arbitrary sequence $(f_n) \subset E_\rho \cap E_{\rho^*}$ and $f \in E_\rho \cap E_{\rho^*}$, we have $f_n \xrightarrow{\rho^*} f$ and (f_n) is ρ-bounded if and only if $f_n \to f$ with respect to the two-modular topology $\gamma_W(\mathcal{T}_\rho, \mathcal{T}_{\rho^*})$.*

Proof Let $f_n \to f$ with respect to the two-modular topology γ_W. Since $\mathcal{T}_{\rho^*} \prec \gamma_W$ (see proof of Theorem 1.6), we get $f_n \to f$ with respect to the topology $\mathcal{T}_{\rho^*}$. Since ρ^* satisfies the $B2$ condition, from Theorem 1.8 in [9] we get that $f_n \xrightarrow{\rho^*} f$. It suffices to prove that the sequence (f_n) is ρ-bounded. Assume that the sequence (f_n) is not ρ-bounded. Since the two-modular topology is linear, we can assume that $f = 0$, i.e. $f_n \to 0$ with respect to γ_W. If the sequence (f_n) is not ρ-bounded, there exists a sequence (k_n) such that $f_{k_n} \notin nB_\rho(r)$. Since all the sets $nB_\rho(r)$ are closed in topology $\mathcal{T}_{\rho^*}$, for every $n \in \mathbb{N}$ there exists a number $\varepsilon_n > 0$

such that $f_{k_n} \notin nB_\rho(r) + B_{\rho^*}(\varepsilon_n)$. Hence the set $U = \bigcap_{n=1}^{\infty} (nB_\rho(r) + B_{\rho^*}(\varepsilon_n))$ does not contain any element f_{k_n}. Since the set U is a neighbourhood of zero in two-modular topology, we obtain contradiction with the fact that $f_n \to 0$ with respect to γ_W. Thus the sequence (f_n) is ρ-bounded.

Now we assume that $f_n \xrightarrow{\gamma} f$, i.e. $f_n \xrightarrow{\rho^*} f$ and (f_n) is ρ-bounded. Let $Z = \{f_n : n \in \mathbb{N}\}$. Since ρ^* satisfies the $B2$ condition, from Theorem 1.8 in [9] we get $f_n \to f$ with respect to the topology $\mathcal{T}_{\rho^*}$. Since Z is ρ-bounded, $\mathcal{T}_{\rho^*}|_Z = \gamma_W|_Z$, hence $f_n \to f$ with respect γ_W.

Theorem 2.9. *Let ρ and ρ^* be two modulars defined on E. Assume that ρ is s-convex, ρ^* satisfies the $B2$ condition and the sets $nB_\rho(r)$ are closed in topology $\mathcal{T}_{\rho^*}$ for all $n \in \mathbb{N}$. For a linear functional F on $E_\rho \cap E_{\rho^*}$ the following conditions are equivalent:*

(i) F is continuous in the modular topology $\mathcal{T}_{\tilde{\rho}}^{\wedge}$.

(ii) F is γ-linear.

(iii) F is continuous in the two-modular topology γ_W.

(iv) For every $i > 0$ the restriction $F|_{iB_\rho(r)}$ is continuous in $\mathcal{T}_{\rho^*}|_{iB_\rho(r)}$.

Proof $(i) \Leftrightarrow (iii)$. This equivalence follows from Theorem 1.5.
$(iii) \Rightarrow (iv)$. Since $\gamma_W|_{iB_\rho(r)} = \mathcal{T}_{\rho^*}|_{iB_\rho(r)}$, the functional $F|_{iB_\rho(r)}$ is continuous in $\mathcal{T}_{\rho^*}|_{iB_\rho(r)}$.
$(iv) \Rightarrow (iii)$. We will show that F is continuous at 0. Let $\varepsilon > 0$. Since the functional $F|_{iB_\rho(r)} : (iB_\rho(r), \mathcal{T}_{\rho^*}|_{iB_\rho(r)})$ is continuous, for arbitrary $i > 0$ there exists $\varepsilon_i > 0$ such that $F(B_{\rho^*}(\varepsilon_i) \cap iB_\rho(r)) \subset (-\frac{\varepsilon}{2^{i+1}}, \frac{\varepsilon}{2^{i+1}})$. Hence $F(\sum_{i=1}^n iB_\rho(r) \cap B_{\rho^*}(\varepsilon_i)) \subset \sum_{i=1}^n (-\frac{\varepsilon}{2^{i+1}}, \frac{\varepsilon}{2^{i+1}}) \subset (-\varepsilon, \varepsilon)$,

and we obtain that

$$F(\bigcup_{n=1}^{\infty} \sum_{i=1}^n iB_\rho(r) \cap B_{\rho^*}(\varepsilon_i)) \subset \bigcup_{n=1}^{\infty} (F(\sum_{i=1}^n (iB_\rho(r) \cap B_{\rho^*}(\varepsilon_i)))) \subset (-\varepsilon, \varepsilon),$$

so F is continuous with respect to γ_W.

$(ii) \Rightarrow (iii)$. If the functional F is γ-linear, for every $i > 0$ the restriction $F|_{iB_\rho(r)}$ is continuous with respect to $\mathcal{T}_{\rho^*}|_{iB_\rho(r)}$. Hence F is continuous with respect to γ_W.

$(iii) \Rightarrow (ii)$. Let $f_n \xrightarrow{\gamma} 0$. ¿From Theorem 1.3 $f_n \xrightarrow{\tilde{\rho}} 0$ and from Theorem 1.2 $f_n \xrightarrow{\mathcal{T}_{\tilde{\rho}}^{\wedge}} 0$. Since $\gamma_W = \mathcal{T}_{\tilde{\rho}}^{\wedge}$ (Theorem 1.5) and F is continuous with respect to the two-modular topology γ_W, then $F(f_n) \to 0$.

3 Applications to Orlicz spaces.

In this section we will apply theorems 1.8 and 1.9 to the theory of Orlicz spaces. We will show that these theorems are true only in case when Orlicz functions take the value $+\infty$ for some $u > 0$, i.e. when $L^\varphi \subset L^\infty$.

Assume that (Ω, Σ, μ) is a σ-finite atomless measure space. An Orlicz function $\varphi : [0, \infty) \to [0, \infty)$ is a non-decreasing, left-continuous, continuous at 0 function, which is not identically equal to 0. Assume that $\lim_{u\to\infty} \inf \varphi(u)/u > 0$. The Orlicz function φ determines a functional
$m_\varphi : L^0 \to [0, \infty]$ by the formula $m_\varphi(f) = \int_\Omega \varphi(|f(\omega)|)d\mu$.

The Orlicz space L^φ generated by φ is the ideal of L^0 defined by:

$$L^\varphi = \{f \in L^0 : m_\varphi(\lambda f) < \infty \text{ for some } \lambda > 0\}$$

and the space E^φ of finite elements is defined by:

$$E^\varphi = \{f \in L^0 : m_\varphi(\lambda f) < \infty \text{ for every } \lambda > 0\}.$$

The functional m_φ restricted to L^φ is a semimodular, i.e. it satisfies the following conditions:

1) $m_\varphi(\lambda f) = 0$ for all $\lambda > 0$ if and only if $f = 0$
2) $m_\varphi(\alpha f + \beta g) \le m_\varphi(f) + m_\varphi(g)$ for the scalars α and β such that $\alpha, \beta \ge 0$ and $\alpha + \beta = 1$
3) $m_\varphi(\alpha f) = m_\varphi(f)$ for $|\alpha| = 1$
4) $|f| \le |g|$ implies $m_\varphi(f) \le m_\varphi(g)$.

Recall that the Orlicz function φ satisfies the Δ_2- condition if there exists a number $K > 0$ such that $\varphi(2u) \le K\varphi(u)$ for all $u \ge 0$.

It is known that the modular m_φ satisfies the σ-Levi and the σ-Fatou properties and if the function φ satisfies the Δ_2-condition, then m_φ satisfies also the σ-Lebegue property.

Let φ_1 i φ_2 be the Orlicz functions. Assume that φ_1 is s-convex and φ_2 satisfies the Δ_2 condition. Then the modular m_{φ_1} is s-convex and m_{φ_2} satisfies the B2 condition. We know that the functional m defined by the formula

$$m(f) = \begin{cases} m_{\varphi_2}(f) & \text{when} \quad m_{\varphi_1}(f) \le 1 \\ \infty & \text{when} \quad m_{\varphi_1}(f) > 1 \end{cases}$$

is a modular on L^0. We have created a modular space (L, m). It turns out that there does not exist an Orlicz function φ (with the assumption that φ_1 and φ_2

have only the finite values) such that $m = m_\varphi$. The space L is not an Orlicz space except a trivial case when the measure μ takes only the values 0 and ∞.

Let φ_1 and φ_2 be the Orlicz functions, m_{φ_1} and m_{φ_2} modulars generated by them. In the case when the measure μ takes only the values 0 and ∞, the only integrable function is $f = 0$ and the Orlicz space $L^\varphi = \{0\}$ for every φ. In this case we get that the condition $m_{\varphi_1}(f) \leq 1$ is satisfied only for $f = 0$ and then also $m_{\varphi_2}(f) = 0$, so $m(f) = m_\varphi(f) = 0$ for every Orlicz function φ. When $f \neq 0$, then $\infty = \int_\Omega \varphi_1(f)d\mu > 1$, so $m(\lambda f) = m_\varphi(\lambda f) = \infty$ for every $\lambda > 0$, so $f \notin L^\varphi$.

We shall show that if μ has the positive values there does not exist the function φ such that $m(f) = m_\varphi(f)$. Assume that such function exists and take an arbitrary $\lambda > 0$. Consider the expression $m_{\varphi_1}(\lambda f) = \int_\Omega \varphi_1(\lambda|f|)d\mu$ as $f \neq 0$. Taking $\lambda \to \infty$, from the property of Orlicz function φ_1 we have $\varphi_1(\lambda|f|) \to \infty$ on the set of positive measure. From Beppo-Levi Theorem $\int_\Omega \varphi_1(\lambda|f|)d\mu \to \infty$, so this integral is bigger than 1 for sufficiently large λ i.e. $\lambda > \lambda_0$. From the definition of m_φ we get that $m_\varphi(\lambda f) = \int_\Omega \varphi(\lambda|f|)d\mu = \infty$ for $\lambda > \lambda_0$. Hence $f \notin E^\varphi$, i.e. $E^\varphi = \{0\}$. On the other hand take the set $A \in \Sigma$ such that $0 < \mu(A) < \infty$ and denote by f a characteristic function of the set A. This is a non-zero function and for every $\lambda > 0$ we have $m_\varphi(\lambda f) = \int_\Omega \varphi(\lambda|f|)d\mu = \int_A \varphi(\lambda)d\mu = \mu(A)\varphi(\lambda) < \infty$. We get that $f = \chi_A \in E^\varphi$, so we have obtained a contradiction.

It turns out that there exist functions φ_1 and φ_2 such that $m = m_\varphi$ (see Example 2.1) In this case let $r > 0$ be fixed real number and (ε_i) a sequence of positive numbers. Denote $B_{m_\varphi}(\varepsilon) = \{f \in L^0 : m_\varphi(f) \leq \varepsilon\}$. Then from Theorem 1.4 the family $\bigcup_{n=1}^\infty(\sum_{i=1}^n iB_{m_{\varphi_1}}(r) \cap B_{m_{\varphi_2}}(\varepsilon_i))$ constitutes a base of the neighbourhoods 0 for topology γ_W on $L^{\varphi_1} \cap L^{\varphi_2}$. From Theorem 1.5 we obtain that the modular topology $\mathcal{T}^\wedge_{m_\varphi}$ for which the base of the neighbourhoods of 0 are sets of the form $\bigcup_{n=1}^\infty(\sum_{i=1}^n B_{m_\varphi}(\varepsilon_i))$ is equal to the topology γ_W.

We will prove that for all $n \in \mathbb{N}$ the sets $nB_{m_{\varphi_1}}(r)$ are closed in topology $\mathcal{T}_{m_{\varphi_2}}$. Since modular m_{φ_1} is s-convex and modular m_{φ_2} satisfies the $B2$ condition, it suffices to show that the set $B_{m_{\varphi_1}}(r)$ is m_{φ_2}-closed. Let $f \in L^{\varphi_1} \cap L^{\varphi_2}$. Assume that $m_{\varphi_1}(f_n) \leq r$ for all $n \in \mathbb{N}$ and $f_n \xrightarrow{m_{\varphi_2}} f$. Since $m_{\varphi_2}(f_n - f) \to 0$, $f_n \to f$ a.e. and $|f_n| \to |f|$ a.e. Denote $g_n = \inf\limits_{n \leq m} |f_m|$. Then $0 \leq g_n \uparrow |f|$ and $g_n \leq |f_n|$ for all $n \in \mathbb{N}$. Since modular m_{φ_1} satisfies the σ-Fatou property, we get $m_{\varphi_1}(f) = \lim\limits_{n\to\infty} m_{\varphi_1}(g_n) \leq \lim \inf\limits_{n\to\infty} m_{\varphi_1}(f_n) \leq r$, i.e. $f \in B_{m_{\varphi_1}(r)}$.

Hence from Theorems 1.8 and 1.9 we obtain the following corollaries:

Corollary 3.1. *Let φ_1 and φ_2 be two Orlicz functions. Assume that φ_1 is s-convex and φ_2 satisfies the Δ_2 condition. For an arbitrary sequence $(f_n) \subset L^{\varphi_1} \cap L^{\varphi_2}$ and $f \in L^{\varphi_1} \cap L^{\varphi_2}$, we have: $f_n \xrightarrow{m_{\varphi_2}} f$ and (f_n) is m_{φ_1}-bounded if and only if $f_n \to f$ with respect to two-modular topology $\gamma_W(\mathcal{T}_{m_{\varphi_1}}, \mathcal{T}_{m_{\varphi_2}})$*

Corollary 3.2. *Let φ_1 and φ_2 be two Orlicz functions. Assume that φ_1 is s-convex and φ_2 satisfies the Δ_2 condition. For a linear functional F on $L^{\varphi_1} \cap L^{\varphi_2}$ the following conditions are equivalent:*

(i) F is continuous in the modular topology $\mathcal{T}^{\wedge}_{m_\varphi}$.

(ii) F is γ-linear.

(iii) F is continuous in the two-modular topology γ_W.

(iv) For every $i > 0$ the restriction $F|_{iB_{\varphi_1}(r)}$ is continuous in $\mathcal{T}_{m_{\varphi_2}}|_{iB_{\varphi_1}(r)}$.

Example 2.1. Let $\Omega = \mathbb{R}$, μ be the Lebesgue measure on $\mathbb{R}$. Assume that φ_2 is an arbitrary Orlicz function satisfying the Δ_2 condition for all $u \geq 0$ and the function φ_1 is defined by the formula

$$\varphi_1(u) = \begin{cases} 0 & \text{for } u \in [0,1] \\ \infty & \text{for } u > 1. \end{cases}$$

Then the modular m is defined by the formula

$$m(f) = \begin{cases} m_{\varphi_2}(f) & \text{for } ||f||_\infty \leq 1 \\ \infty & \text{for } ||f||_\infty > 1. \end{cases}$$

We will prove that $m = m_\varphi$, where the function φ is defined by the formula

$$\varphi(u) = \begin{cases} \varphi_2(u) & \text{for } u \in [0,1] \\ \infty & \text{for } u > 1. \end{cases}$$

Let $||f||_\infty \leq 1$, i.e. $|f(\omega)| \leq 1$ for a.e. $\omega \in \mathbb{R}$. Thus $m(f) = m_{\varphi_2}(f)$ from the definition of m and $m_\varphi(f) = \int_\Omega \varphi(|f(\omega)|)d\omega = \int_\Omega \varphi_2(|f(\omega)|)d\omega = m_{\varphi_2}(f) = m(f)$. If $||f||_\infty > 1$, then $m(f) = \infty = m_\varphi(f)$.

It turns out that the function φ_2 is not assumed to be convex. In this case the Orlicz space L^{φ_2} is not locally convex.

Define the Orlicz function $\varphi : [0,\infty) \to [0,\infty)$ by the formula

$$\varphi(u) = \begin{cases} u^2 & \text{for } u \in [0,1] \\ (u-2n)^2 + n & \text{for } u \in [2n, 2n+1), n \in \mathbb{N} \\ n & \text{for } u \in [2n-1, 2n), n \in \mathbb{N}. \end{cases}$$

This function is not equivalent to any s-convex Orlicz function, but it satisfies the Δ_2 condition for all $u \geq 0$ and $\lim_{u\to\infty} \inf \frac{\varphi(u)}{u} > 0$.

Let φ_1 be a function from Example 2.1 and φ_2 the function defined above. Then there exist nonzero linear and continuous functionals on the space $L^{\varphi_1} \cap L^{\varphi_2}$ and we obtain the equivalence of the conditions $(i) - (iv)$ in Corollary 2.2.

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