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On the Bounds of the Domination Numbers of Glued Graphs

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Abstract We obtain a lower bound and an upper bound of the domination numbers of glued graphs, based on those of the original graphs. We also investigate the domination numbers of glued graphs whose clones are paths and the original graphs are in some specific families of bipartite graphs, namely bipartite fan graphs and firecrackers $Fc_{n,3}$. Consequently, we use these results to show that our obtained bounds are sharp.

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1. Introduction

Glue operator is a binary operation on graphs defined by Uiyyasathian to solve the maximal-clique partition problem [1]. Let G_1 and G_2 be non-empty connected graphs with disjoint vertex sets, and $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$ are non-empty connected subgraphs that are isomorphic to each other under an isomorphism ϕ. The *glued graph of G_1 and G_2 at H_1 and H_2 with respect to ϕ*, which is denoted by $G_1 \underset{H_1 \cong_\phi H_2}{\lhd\rhd} G_2$, is the graph resulted from identifying H_1 and H_2 under the isomorphism ϕ.

To give an explicit definition on gluing, we start from another binary operation on graph: graph subtraction. Let G be a graph and H be a subgraph of G. The *subtraction of H from G*, denoted by $G - H$, is the graph obtained from G by deleting all vertices in $V(H)$ and edges incident with the deleted vertices. In this study, we allow the existence of the empty graph of 0 vertices, which is the graph with no vertices and no edges, denoted by $G_\varnothing$. Under these definitions, we can see that $G - H$ is also a subgraph of G and it becomes $G_\varnothing$ when H is a spanning subgraph of G.

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Using the definition of graph subtraction, $G = G_1 \underset{H_1 \cong_\phi H_2}{\lhd\rhd} G_2$ is the graph whose vertex set is

$$V(G) = (V(G_1) - V(H_1)) \cup (V(G_2) - V(H_2)) \cup \{(v, \phi(v)) : v \in V(H_1)\},$$

and edge set is

$$\begin{aligned} E(G) = {} & E(G_1 - H_1) \cup E(G_2 - H_2) \\ & \cup \{\{u, (v, \phi(v))\} : \{u, v\} \in E(G_1)\} \cup \{\{(v, \phi(v)), w\} : \{\phi(v), w\} \in E(G_2)\} \\ & \cup \{\{(u, \phi(u)), (v, \phi(v))\} : \{u, v\} \in E(G_1) \text{ or } \{\phi(u), \phi(v)\} \in E(G_2)\}. \end{aligned}$$

We say that G_1 and G_2 are the *original graphs*. The subgraphs H_1 and H_2, as well as the subgraph H of $G_1 \underset{H_1 \cong_\phi H_2}{\lhd\rhd} G_2$ that is isomorphic to them, are referred as *clones*.

Example 1.1. Let G_1, G_2, H_1 and H_2 be the graphs on the left side of FIGURE 1. The glued graph of G_1 and G_2 at H_1 and H_2 with respect to

$$\phi = \{(v_3, u_1), (v_4, u_2), (v_5, u_3), (v_6, u_4)\}$$

is the graph on the right side of FIGURE 1. The edges of the clone H is shown by the thick lines.

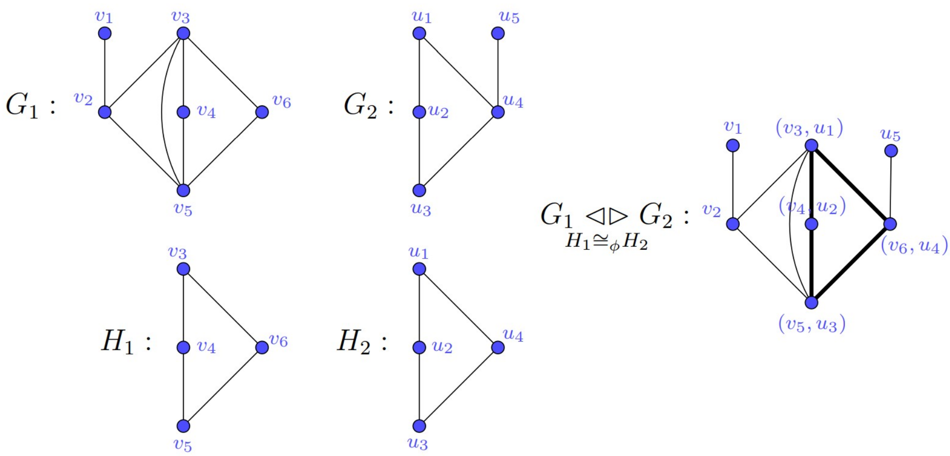

FIGURE 1. An example of glued graphs.

Several properties of glued graph has been studied through the years. In 2006, Promsakon and Uiyyasathian investigated the chromatic numbers of glued graphs [2]. They also published the work on edge-chromatic numbers [3]. A research on perfection of glued graphs was done by Uiyyasathian and Saduakdee [4], while the conditions for glued graphs being Eulerian were considered by Boonthong et al. [5].

Among various graph parameters, domination number is one of the most studied topic (See e.g. [6]). It was introduced by Berge [7] in 1958. Given a graph G, a *dominating set* of G is a set of vertices such that for any vertex v in $V(G)$, there is an edge connecting v to a vertex in the set. The smallest size of dominating sets is the *domination number* of the graph, denoted by $\gamma(G)$. Since we also consider $G_\varnothing$, the smallest value of the domination numbers is 0 which is the domination number of $G_\varnothing$. The concept of dominating sets leads to numerous applications in computer sciences, communication, transportation, as well as physical and digital security.

Some works on the domination numbers of glued graphs have been investigated in particular types of original graphs. Ruangnai and Panma focused on gluing the paths P_m and P_n [8], while Boonmee et al. concentrated on gluing the cycles C_m and C_n [9]. To the best of our knowledge, the bounds on the domination numbers of arbitrary glued graphs have not been studied yet.

There are two types of bipartite graphs that play important roles as examples to show that our achieved bounds are sharp. The first one is the family of *bipartite fan graphs*. It was introduced by Daoud in his research on edge odd graceful labeling [10] as *half gear graphs*. A *fan graph* $F_{m,n}$ is the graph whose vertex set is $V(F_{m,n}) = V(\overline{K_m}) \cup V(P_n)$ and $E(F_{m,n}) = E(P_n) \cup \{\{u,v\} : u \in V(\overline{K_m}), v \in V(P_n)\}$. A bipartite fan graph BF_{2n-1} is resulted from inserting a vertex on each edge of the path P_n of a fan graph $F_{1,n}$. Since a half gear graph is obtained from a fan graph in the same way as a *gear graph*, which is also called a *bipartite wheel graph*, is obtained from a *wheel graph*, we name it a bipartite fan graph instead of a half gear graph to emphasize the relationship between this type of graphs and the fan graphs. Examples of fan graphs and bipartite fan graphs are shown in FIGURE 2.

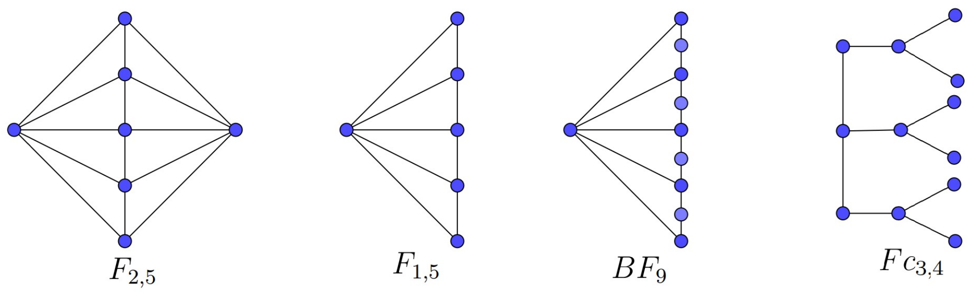

FIGURE 2. Fan graphs $F_{2,5}, F_{1,5}$, bipartite fan graph BF_9 and firecracker $Fc_{3,4}$.

The other family of graphs that we are interested in is the firecrackers, which was defined by Chen et al. [11]. An (n,k)-firecracker, denoted by $Fc_{n,k}$, is constructed from n copies of stars $S_k = K_{1,k-1}$ and a path P_n. Let $S_k^1, S_k^2, \ldots, S_k^n$ be the n copies of stars S_k and $P_n = (u_1, u_2, \ldots, u_n)$. We obtain $Fc_{n,k}$ from identifying one leaf from the star S_k^i and the vertex u_i in the path P_n, for $i = 1, 2, \ldots, n$. An example of firecrackers is shown as FIGURE 2.

In this work, we propose a lower bound and an upper bound on the domination numbers of glued graphs with any original graphs and clones. Besides, to show that our bounds are sharp, we consider the domination numbers of the graphs that we get from gluing two bipartite fan graphs BF_{2n-1} using clones P_{2n-1}, and gluing two firecrackers $Fc_{n,3}$ using clones P_n.

2. Lower Bounds on the Domination Numbers

We offer a lower bound of domination numbers of glued graphs which is the minimum of three values. Furthermore, we find the domination numbers of bipartite fan graphs BF_{2n-1}, and apply this result to show that our proposed bound is sharp.

Theorem 2.1. *Let* $G = G_1 \underset{H_1 \cong_\phi H_2}{\lhd\rhd} G_2$ *be the glued graph of* G_1 *and* G_2 *at clones* H_1 *and* H_2 *with respect to isomorphism* ϕ*. Then*

$$\min\{2, \gamma(G_1), \gamma(G_2)\} \leq \gamma(G).$$

Proof. If $\gamma(G) \geq 2$, we have $\min\{2, \gamma(G_1), \gamma(G_2)\} \leq 2 \leq \gamma(G)$ as required.

Now, assume that $\gamma(G) = 1$. Then there exists a vertex $u \in V(G)$ such that $\{u\}$ is a dominating set of G. In otherwords, for any other vertices $v \in V(G)$, the edge $\{u, v\}$ is in $E(G)$.

<u>Case 1</u> $u \in (V(G_1) - V(H_1)) \cup (V(G_2) - V(H_2))$. Without loss of generality, assume that $u \in V(G_1) - V(H_1)$.

We claim that $\{u\}$ is also a dominating set of G_1. Let $v \in V(G_1)$. If $v \in V(G_1) - V(H_1)$, we have $\{u, v\} \in E(G_1 - H_1) \subseteq E(G_1)$. Otherwise, $v \in V(H_1)$. Since $\{u\}$ is a dominating set of G, vertex u is adjacent to $(v, \phi(v)) \in V(G)$. Thus, the edge $\{u, v\} \in E(G_1)$. Consequently, we get the claim. As a result, $\min\{2, \gamma(G_1), \gamma(G_2)\} \leq \gamma(G_1) = 1 = \gamma(G)$.

<u>Case 2</u> $u = (v, \phi(v))$ for some $v \in V(H_1)$. Claim that $\{v\}$ is a dominating set of G_1. Let $w \in V(G_1)$. In case that $w \in V(G_1) - V(H_1)$, vertex w is adjacent to $u = (v, \phi(v)) \in V(G)$. Hence, $\{v, w\} \in E(G_1)$. If $w \in V(H_1)$, vertices $(w, \phi(w))$ and $u = (v, \phi(v))$ are connected by an edge in G. Therefore, $\{v, w\} \in E(G_1)$. From both cases, we can conclude that $\{v\}$ is a dominating set of G_1. Thus, $\min\{2, \gamma(G_1), \gamma(G_2)\} \leq \gamma(G_1) = 1 = \gamma(G)$. ■

The following result comes directly from Theorem 2.1.

Corollary 2.2. *Let* $G = G_1 \underset{H_1 \cong_\phi H_2}{\lhd\rhd} G_2$ *be the glued graph of* G_1 *and* G_2 *at clones* H_1 *and* H_2 *with respect to isomorphism* ϕ*. If* $\gamma(G) = 1$*, then* $\gamma(G_1) = 1$ *or* $\gamma(G_2) = 1$*.*

However, the converse of the corollary does not hold. In fact, even if we assume a stronger condition that $\gamma(G_1) = \gamma(G_2) = 1$, it does not suffice to conclude that $\gamma(G) = 1$. We can see from the next example.

Example 2.3. Consider a fan graph $F_{2,n}$ where $n \geq 4$. Let $V(\overline{K_2}) = \{u, v\}$ and $V(P_n) = \{w_1, w_2, \ldots, w_n\}$. The fan graph $F_{2,n}$ can be viewed as the glued graph of two fan graphs in the form of $F_{1,n}$. Namely, $F_{2,n} = G_1 \underset{H_1 \cong_\phi H_2}{\lhd\rhd} G_2$ where

- $G_1 = F_{1,n}$ whose vertex set is $\{u, u_1, u_2, \ldots, u_n\}$ and edge set is $\{\{u_i, u_{i+1}\} : i = 1, 2, \ldots, n-1\} \cup \{\{u, u_i\} : i = 1, 2, \ldots, n\}$,
- $G_2 = F_{1,n}$ whose vertex set is $\{v, v_1, v_2, \ldots, v_n\}$ and edge set is $\{\{v_i, v_{i+1}\} : i = 1, 2, \ldots, n-1\} \cup \{\{v, v_i\} : i = 1, 2, \ldots, n\}$,
- $H_1 = P_n$ whose vertex set is $\{u_1, u_2, \ldots, u_n\}$ and edge set is $\{\{u_i, u_{i+1}\} : i = 1, 2, \ldots, n-1\}$,
- $H_2 = P_n$ whose vertex set is $\{v_1, v_2, \ldots, v_n\}$ and edge set is $\{\{v_i, v_{i+1}\} : i = 1, 2, \ldots, n-1\}$,
- ϕ is defined by $\phi(u_i) = v_i$ for $i = 1, 2, \ldots, n$, and
- $w_i := (u_i, \phi(u_i))$ for $i = 1, 2, \ldots, n$.

It is obvious that $\{u\}, \{v\}$ and $\{u, v\}$ are dominating sets of smallest size of $G_1 = F_{1,n}, G_2 = F_{1,n}$ and $F_{2,n} = G_1 \underset{H_1 \cong_\phi H_2}{\lhd\rhd} G_2$, respectively. Hence, the domination numbers of both original graphs are equal to 1, while the domination number of the glued graph is equal to $2 \neq 1$.

Next, we explore the domination numbers of the bipartite fan graphs BF_{2n-1}.

Theorem 2.4. *Let $n \geq 2$ be a positive integer. The domination number of a bipartite fan graph BF_{2n-1} is*

$$\gamma(BF_{2n-1}) = \begin{cases} \frac{n+1}{2} & \text{if } n \equiv 1 \pmod 2, \\ \frac{n+2}{2} & \text{if } n \equiv 0 \pmod 2. \end{cases}$$

Proof. Let $G = BF_{2n-1}$ be the bipartite fan graph originated from the fan graph $F_{1,n}$ with path $P_n = (u_1, u_2, \ldots, u_n)$. Let $v_1, v_2, \ldots, v_{n-1}$ be the vertices inserted to the edges of $F_{1,n}$ where for each $i = 1, 2, \ldots, n$, vertex v_i is inserted to the edge $\{u_i, u_{i+1}\}$.

Then the vertex set of graph G is $V(G) = \{u, u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_{n-1}\}$ where u is the vertex of degree n, and the edge set of G is

$$\begin{aligned} E(G) = \{&\{u, u_i\} : i = 1, 2, \ldots, n\} \\ &\cup \{\{u_i, v_i\} : i = 1, 2, \ldots, n-1\} \\ &\cup \{\{v_i, u_{i+1}\} : i = 1, 2, \ldots, n-1\}. \end{aligned}$$

Case 1 $n = 2k + 1$ for some $k \in \mathbb{Z}$. Since $n \geq 2$, it implies that $k \geq 1$. Consider the set $D = \{u, u_2, u_4, \ldots, u_{2k}\}$. Claim that D is a dominating set of G. Since $u \in D$, vertices $u_1, u_2, \ldots, u_n$ are connected to a vertex in D, namely vertex u. Let $i \in \{1, 2, \ldots, n-1 = 2k\}$. Then vertex v_i is adjacent to vertex $u_i \in D$ if i is even, and to vertex $u_{i+1} \in D$ if i is odd. Thus, we get the claim.

Let D' be a dominating set of G. Suppose that $u \notin D'$. Then D' is also a dominating set of $P_{2n-1} = P_{4k+1}$. It is easy to see that when $m \geq 3$, $\gamma(P_m) = \lceil \frac{m}{3} \rceil$ (See e.g. [12]). Consequently,

$$|D'| \geq \gamma(P_{4k+1}) = \lceil \frac{4k+1}{3} \rceil = k + \lceil \frac{k+1}{3} \rceil \geq k + 1 = |D|.$$

Now we consider the case when $u \in D'$. For $i = 1, 3, ..., 2k-1$, let $S_i = \{u_i, v_i, u_{i+1}, v_{i+1}\}$. If there is a set S_i where $S_i \cap D' = \varnothing$, vertex v_i would not be adjacent to any vertices in D' since its only neighbours are vertices u_i and u_{i+1}. Hence, for each set S_i, there is at least one vertex in S_i that is in D'. Since all S_i's are pairwisely disjoint, $|D'| \geq 1 + k = |D|$.

Therefore, we get

$$\gamma(BF_{2n-1}) = k + 1 = \frac{n+1}{2}.$$

Case 2 $n = 2k$ for some $k \in \mathbb{Z}$. If $k = 1$, then BF_{2n-1} is just C_4 with domination number 2. Assume that $k \geq 2$. Let $D = \{u, u_2, u_4, \ldots, u_{2k}\}$. We claim that D is a dominating set of G. The vertices $u_1, u_2, \ldots, u_n$ are adjacent to vertex $u \in D$. For $i \in \{1, 2, \ldots, n-1 = 2k-1\}$, vertex v_i is adjacent to $u_i \in D$ if i is even, and to $u_{i+1} \in D$ if i is odd. Hence, we get the claim.

Let D' be a dominating set of G. First, we assume that $u \notin D'$. We can see that D' is also a dominating set of $P_{2n-1} = P_{4k-1}$. Since $k \geq 2$, we have

$$|D'| \geq \gamma(P_{4k-1}) = \lceil \frac{4k-1}{3} \rceil = k + \lceil \frac{k-1}{3} \rceil \geq k + 1 = |D|.$$

When $u \in D'$, we denote $S_i = \{u_i, v_i, u_{i+1}, v_{i+1}\}$ for $i = 1, 3, \ldots, 2k-3$, and $S_{2k-1} = \{u_{2k-1}, v_{2k-1}, u_{2k}\}$. Suppose that there is a set S_i where S_i and D' are disjoint. Then vertex v_i is not adjacent to any vertices in D' because it is only connected to vertices u_i and u_{i+1}. Therefore, for each set S_i, there exists a vertex in $S_i \cap D'$. By the fact that these S_i's are pairwisely disjoint, $|D'| \geq 1 + k = |D|$.

As a result,

$$\gamma(BF_{2n-1}) = k + 1 = \frac{n+2}{2}.$$

∎

We use this result to construct a glued graph with domination number equal to 2 from two original graphs whose domination number is an arbitrary positive integer $m \geq 2$.

Lemma 2.5. *Let $m \geq 2$ be a positive integer number. There exists a glued graph of domination number 2 whose two original graphs are of domination number m.*

Proof. Let G_1 and G_2 be bipartite fan graphs BF_{4m-3} whose vertex set are

$$V(G_1) = \{u, u_1, u_2, \ldots, u_{2m-1}, v_1, v_2, \ldots, v_{2m-2}\} \text{ and}$$
$$V(G_2) = \{s, s_1, s_2, \ldots, s_{2m-1}, t_1, t_2, \ldots, t_{2m-2}\}$$

as defined in the proof of Theorem 2.4. Let H_1 and H_2 be paths P_{4m-3} where

$$H_1 = (u_1, v_1, u_2, v_2 \ldots, u_{2m-2}, v_{2m-2}, u_{2m-1}) \text{ and}$$
$$H_2 = (s, s_1, t_1, s_2, t_2 \ldots, t_{2m-3}, s_{2m-2}, t_{2m-2}).$$

Define isomorphism ϕ from $V(G_1)$ to $V(G_2)$ as

$$\phi(w) = \begin{cases} s & \text{if } w = u_1 \\ s_i & \text{if } w = v_i, i = 1, 2, \ldots, 2m-2 \\ t_{i-1} & \text{if } w = u_i, i = 2, 3, \ldots, 2m-1. \end{cases}$$

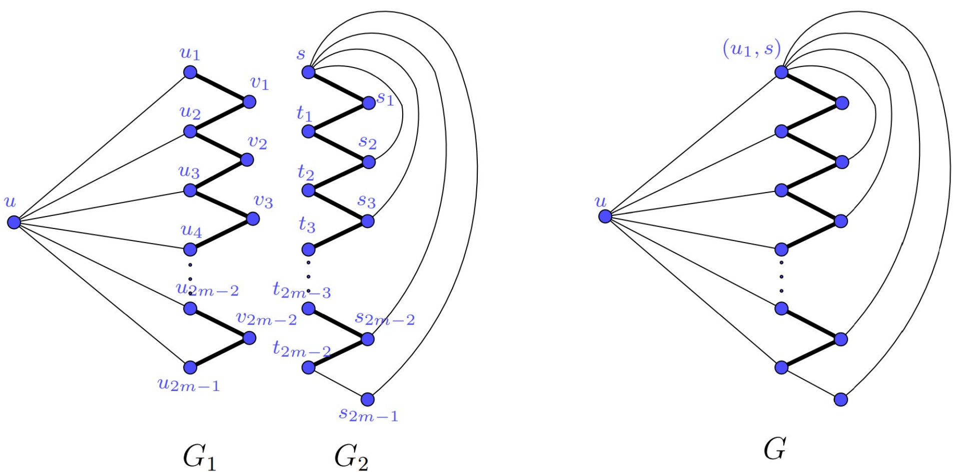

FIGURE 3. The original graphs G_1 and G_2 and the glued graph G.

From Theorem 2.4, since $2m-1$ is odd, $\gamma(G_1) = \gamma(G_2) = \frac{2m-1+1}{2} = m$. Let $G = G_1 \underset{H_1 \cong_\phi H_2}{\lhd\rhd} G_2$. (See FIGURE 3.) Since G does not have any vertex that is adjacent to all other vertices, $\gamma(G) \geq 2$. We claim that $D = \{u, (u_1, s)\}$ is a dominating set of G.

The vertices in $V(G)$ other than u and (u_1, s) are in $\{(v, \phi(v)) : v \in V(H_1)\}$. Let $v \in V(H_1)$.

<u>Case 1</u> $v = v_i, i = 1, 2, \ldots, 2m-2$. We have $(v, \phi(v)) = (v_i, s_i)$. Since $\{s_i, s\} \in E(G_2)$, there is vertex $(u_1, s) \in D$ where $\{(v_i, s_i), (u_1, s)\} \in E(G)$.

<u>Case 2</u> $v = u_i, i = 2, 3, \ldots, 2m-1$. Vertex $(v, \phi(v)) = (u_i, t_{i-1})$ is adjacent to $u \in D$ because $\{u, u_i\} \in E(G_1)$.

Therefore, we get the claim and the glued graph we consider has domination number 2 while the original graphs have domination number m. ∎

Now, we are ready to show that our lower bound is sharp.

Theorem 2.6. *The lower bound from Theorem 2.1 is sharp.*

Proof. Let m be a positive integer. From Lemma 2.5, we get an example of a glue graph with domination number 2 whose original graphs have domination number m.

We can see from Theorem 2.1 that for a glued graph $G = G_1 \underset{H_1 \cong_\phi H_2}{\lhd\rhd} G_2$, its domination number $\gamma(G)$ can be equal to the domination of its original graph only when $\gamma(G_1)$ or $\gamma(G_2)$ is at most 2. Therefore, it remains to show that there is a glued graph with domination number 1 and one of its original graph also has domination number 1.

Let G_1 be a fan graph $F_{1,3m}$ and G_2 be a path P_{3m}. Then $\gamma(G_1) = 1$ and $\gamma(G_2) = \lceil \frac{3m}{3} \rceil = m$ [12]. Let the clones be P_{3m}, which means $H_2 = G_2$. It is clear that $G = G_1 \underset{H_1 \cong_\phi H_2}{\lhd\rhd} G_2$ is isomorphic to G_1. Hence, $\gamma(G) = \gamma(G_1) = 1$, and we are done. ∎

3. Upper Bounds on the Domination Numbers

We start this section from giving an upper bound of the domination numbers of arbitrary glued graphs. Then we consider a glued graph, whose original graphs are firecrackers $Fc_{n,3}$ and the clones are paths P_n, as an example of a glued graph whose domination number is equal to our upper bound.

For many graph parameters such as chromatic number or clique number, if H is a subgraph of G, that parameter of H is less than or equal to that of G. However, it is not true for the domination numbers. The domination number of H can be less than, the same as or greater than the domination number of G, as we illustrate in the next example.

Example 3.1. Consider a star $G = K_{1,n}$ where $n \geq 2$. Let H be a P_2 subgraph of G. Then $G - H$ is an empty graph of $n-1$ vertices. It is clear that $\gamma(G) = 1$, while $\gamma(G - H) = 1 = \gamma(G)$ when $n = 2$ and $\gamma(G - H) = n - 1 > 1 = \gamma(G)$ when $n \geq 3$.

On the other hand, let G be a graph with non-empty vertex set and H be a spanning subgraph of G. Thus, $G - H = G_\varnothing$, and hence, $\gamma(G - H) = \gamma(G_\varnothing) = 0 < \gamma(G)$.

Consequently, when G is an original graph and H is a clone, we cannot tell whether $\gamma(G)$ or $\gamma(G - H)$ is smaller. Thus, we offer an upper bound which is the minimum of three values relating to $\gamma(G_1), \gamma(G_2), \gamma(G_1 - H_1)$ and $\gamma(G_2 - H_2)$.

Theorem 3.2. *Let* $G = G_1 \underset{H_1 \cong_\phi H_2}{\lhd\rhd} G_2$ *be the glued graph of* G_1 *and* G_2 *at clones* H_1 *and* H_2 *with respect to isomorphism* ϕ*. Then*

$$\gamma(G) \leq \min\{\gamma(G_1) + \gamma(G_2), \gamma(G_1) + \gamma(G_2 - H_2), \gamma(G_2) + \gamma(G_1 - H_1)\}.$$

Proof. Let D_1 be a dominating set of G_1 and D_2 be a dominating set of G_2. Then $|D_1| = \gamma(G_1)$ and $|D_2| = \gamma(G_2)$. By symmetry, we can show that our upper bound is valid by constructing dominating sets of size $\gamma(G_1) + \gamma(G_2)$ and $\gamma(G_1) + \gamma(G_2 - H_2)$.

We start from a dominating set of size $\gamma(G_1) + \gamma(G_2)$. Let

$$\begin{aligned} D = &(D_1 - V(H_1)) \cup (D_2 - V(H_2)) \\ &\cup \{(u, \phi(u)) : u \in D_1 \cap V(H_1)\} \cup \{(u, \phi(u)) : \phi(u) \in D_2 \cap V(H_2)\}. \end{aligned}$$

We can see that D is the union of four disjoint sets. Therefore,

$$\begin{aligned} |D| &= (|D_1| - |D_1 \cap V(H_1)|) + (|D_2| - |D_2 \cap V(H_2)|) \\ &\quad + |D_1 \cap V(H_1)| + |D_2 \cap V(H_2)| \\ &= |D_1| + |D_2| \\ &= \gamma(G_1) + \gamma(G_2). \end{aligned}$$

Let $v \in V(G)$. Claim that there exists a vertex in D that is adjacent to v.

<u>Case 1</u> $v \in (V(G_1) - V(H_1)) \cup (V(G_2) - V(H_2))$. Without loss of generality, assume that $v \in V(G_1) - V(H_1)$. Then v is also in $V(G_1)$, and hence, there exists $u \in D_1$ such that $\{u, v\} \in E(G_1)$.

If $u \in D_1 - V(H_1) \subseteq D$, then $\{u, v\} \in E(G_1 - H_1) \subseteq E(G)$ and we are done. Otherwise, $u \in V(H_1)$. Since $\{u, v\} \in E(G_1)$, we have $\{(u, \phi(u)), v\} \in E(G)$. Thus, we obtain vertex $(u, \phi(u)) \in \{(u, \phi(u)) : u \in D_1\} \subseteq D$ that is adjacent to v.

<u>Case 2</u> $v = (w, \phi(w))$ for some $w \in V(H_1)$. Since we know that D_1 is a dominating set of G_1, there exists $u \in D_1$ such that $\{u, w\} \in E(G_1)$.

When $u \in D_1 - V(H_1) \subseteq D$, we have $\{u, v\} = \{u, (w, \phi(w))\} \in E(G)$. In the case that $u \in V(H_1)$, we get $(u, \phi(u)) \in \{(u, \phi(u)) : u \in D_1\} \subseteq D$ is a vertex in D where $\{(u, \phi(u)), v\} = \{(u, \phi(u)), (w, \phi(w))\} \in E(G)$.

The last step is to find a dominating set of size $\gamma(G_1) + \gamma(G_2 - H_2)$. Let D_2' be a dominating set of $G_2 - H_2$, that is $|D_2'| = \gamma(G_2 - H_2)$. We define

$$D' = (D_1 - V(H_1)) \cup \{(u, \phi(u)) : u \in D_1 \cap V(H_1)\} \cup D_2'.$$

Since $D_2' \subseteq V(G_2)$, the set D' is the union of three disjoint sets. Therefore,

$$\begin{aligned} |D| &= (|D_1| - |D_1 \cap V(H_1)|) + |D_1 \cap V(H_1)| + |D_2'| \\ &= |D_1| + |D_2'| \\ &= \gamma(G_1) + \gamma(G_2 - H_2). \end{aligned}$$

To prove that D' is also a dominating set of G, let $v \in V(G)$ and claim that v is adjacent to a vertex in D.

<u>Case 1</u> $v \in V(G_1) - V(H_1)$. Since $v \in V(G_1)$, there exists $u \in D_1$ such that $\{u, v\} \in E(G_1)$. If $u \in D_1 - V(H_1) \subseteq D'$, we get $\{u, v\} \in E(G_1 - H_1) \subseteq E(G)$ as required. When $u \in V(H_1)$, the edge $\{(u, \phi(u)), v\}$ must be in $E(G)$. Therefore, v is adjacent to vertex $(u, \phi(u)) \in \{(u, \phi(u)) : u \in D_1\} \subseteq D'$.

<u>Case 2</u> $v = (w, \phi(w))$ for some $w \in V(H_1)$. From the fact that D_1 is a dominating set of G_1, there is a vertex $u \in D_1$ such that $\{u, w\} \in E(G_1)$. In the case that $u \in D_1 - V(H_1) \subseteq$

D', we have $\{u, v\} = \{u, (w, \phi(w))\} \in E(G)$. If $u \in V(H_1)$, then $(u, \phi(u)) \in \{(u, \phi(u)) : u \in D_1\}$ is a vertex in D' where $\{(u, \phi(u)), v\} = \{(u, \phi(u)), (w, \phi(w))\} \in E(G)$.

Case 3 $v \in V(G_2) - V(H_2)$. Thus, v is also a vertex in $G_2 - H_2$, and there must be a vertex $u \in D'_2$ that is adjacent to v in $G_2 - H_2$. Consequently, the edge $\{u, v\} \in E(G_2 - H_2) \subseteq E(G)$.

Then we get the claim, and hence, we obtain the proposed upper bound. ■

To show that the upper bound from Theorem 3.2 is sharp, we consider the gluing of two firecrackers $Fc_{n,3}$ with the clones being paths P_n.

Theorem 3.3. *The upper bound from Theorem 3.2 is sharp.*

Proof. Let G_1 and G_2 be firecrackers $Fc_{n,3}$ with vertex sets

$$V(G_1) = \{u_{i,j} : i = 1, 2, \ldots, n, j = 1, 2, 3\} \text{ and}$$
$$V(G_2) = \{v_{i,j} : i = 1, 2, \ldots, n, j = 1, 2, 3\},$$

and edge sets

$$\begin{aligned} E(G_1) &= \{\{u_{i,1}, u_{i,2}\} : i = 1, 2, \ldots, n\} \\ &\cup \{\{u_{i,2}, u_{i,3}\} : i = 1, 2, \ldots, n\} \\ &\cup \{\{u_{i,3}, u_{i+1,3}\} : i = 1, 2, \ldots, n-1\} \text{ and} \\ E(G_2) &= \{\{v_{i,1}, v_{i,2}\} : i = 1, 2, \ldots, n\} \\ &\cup \{\{v_{i,2}, v_{i,3}\} : i = 1, 2, \ldots, n\} \\ &\cup \{\{v_{i,3}, v_{i+1,3}\} : i = 1, 2, \ldots, n-1\}. \end{aligned}$$

Let H_1 and H_2 be P_n subgraphs of G_1 and G_2 where $H_1 = (u_{1,3}, u_{2,3}, \ldots, u_{n,3})$ and $H_2 = (v_{1,3}, v_{2,3}, \ldots, v_{n,3})$. We define isomorphism ϕ from $V(G_1)$ to $V(G_2)$ as $\phi(u_{i,3}) = v_{i,3}$ for $i = 1, 2, \ldots, n$.

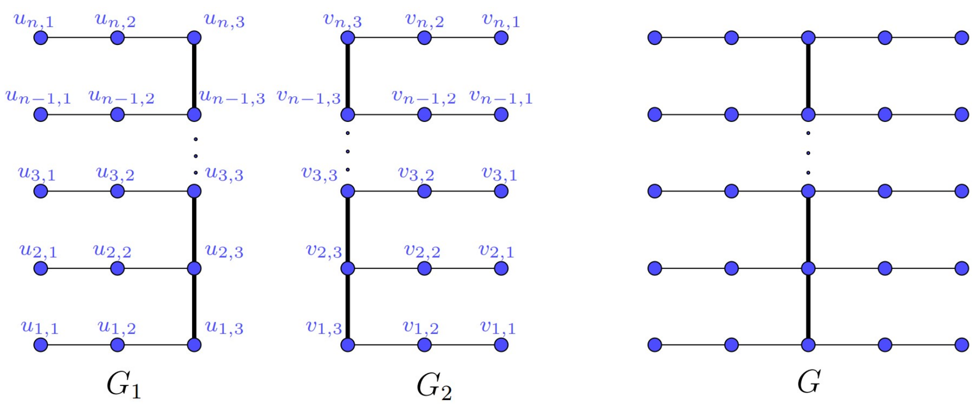

FIGURE 4. The original graphs G_1 and G_2 and the glued graph G.

Consider $G = G_1 \underset{H_1 \cong_\phi H_2}{\triangleleft\triangleright} G_2$. (See FIGURE 4.) Sugumaran and Jayachandran [13] have shown that $\gamma(G_1) = \gamma(G_2) = \gamma(F_{n,3}) = n$. Consider $G_1 - H_1$ and $G_2 - H_2$. These graphs are the union of n paths P_2. Since each path P_2 has domination number $\lceil \frac{2}{3} \rceil = 1$, we get $\gamma(G_1 - H_1) = \gamma(G_2 - H_2) = n$.

Define $D = \{u_{i,2} : i = 1, 2, \ldots, n\} \cup \{v_{i,2} : i = 1, 2, \ldots, n\}$. We have $|D| = 2n = \gamma(G_1) + \gamma(G_2) = \gamma(G_1) + \gamma(G_2 - H_2) = \gamma(G_2) + \gamma(G_1 - H_1)$. Claim that D is a dominating set of G. Consider an arbitrary vertex $w \in V(G) - D$.

<u>Case 1</u> $w = u_{i,1}$ or $w = v_{i,1}$ for some $i \in \{1, 2, \ldots, n\}$. Since $u_{i,2}$ and $v_{i,2}$ are in D, there is a vertex in D that is connected to w.

<u>Case 2</u> $w = (u_{i,3}, v_{i,3})$ for some $i \in \{1, 2, \ldots, n\}$. We know that $u_{i,2}$ and $u_{i,3}$ are adjacent in G_1. Hence, there is an edge between $w = (u_{i,3}, v_{i,3})$ and $u_{i,2} \in D$.

Let D' be a dominating set of G. For each pair of vertices $u_{i,1}$ and $u_{i,2}$, for $i = 1, 2, \ldots, n$, at least one of them must be in D'. Otherwise, $u_{i,1}$ would not be in D' and not be adjacent to any vertices in D' since $u_{i,2}$ is its only neighbour. Similarly, for $i = 1, 2, \ldots, n$, at least one of vertices $v_{i,1}$ and $v_{i,2}$ is in D'. Thus, $|D'| \geq 2n = |D|$.

Therefore, the domination number of G is $2n = \gamma(G_1) + \gamma(G_2) = \gamma(G_1) + \gamma(G_2 - H_2) = \gamma(G_2) + \gamma(G_1 - H_1)$ as required. ∎

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