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Solving Split Equality Fixed Point Problem for Quasi-Phi-Nonexpansive Mappings

Charles Ejike Chidume[1] **and Abubakar Adamu**[1,2*]

[1] *Department of Pure and Applied Mathematics, Mathematics Institute (Emerging Regional Centre of Excellence (ERCE) of the European Mathematical Society (EMS)), African University of Science and Technology, PMB 681, Abuja 900107, Nigeria*

[2] *Center of Excellence in Theoretical and Computational Science (TaCS-CoE), Science Laboratory Building, King Mongkuts University of Technology Thonburi (KMUTT), 126 Pracha-Uthit Road, Bang Mod, Thung Khru, Bangkok 10140, Thailand*
e-mail : aadamu@aust.edu.ng

Abstract An iterative algorithm is constructed to approximate solutions of split equality fixed point problem (SEFPP) for quasi-ϕ-nonexpansive mappings in real Banach spaces more general than Hilbert spaces. Weak convergence of the sequence generated by the algorithm is proved. The theorem proved complements recent important results to provide algorithms for approximating solutions of SEFPP. Furthermore, strong convergence of the sequence generated by the algorithm is proved under the assumption that the operators are semi-compact. Moreover, applications of the theorem to split equality problem and split variational inclusion problem are presented. Finally, numerical examples are presented to illustrate the strong convergence of the sequence generated by our algorithm.

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1. Introduction

Let K be a nonempty closed and convex subset of a normed space, E. Let $T : K \to K$ be a map. A point $x^* \in K$ is called a *fixed point of* T if $Tx^* = x^*$. We shall denote the set of fixed points of any map T by $F(T)$.

Let H_1, H_2 and H_3 be real Hilbert spaces, $T : H_1 \to H_1$ and $S : H_2 \to H_2$ be nonlinear mappings such that $F(T) \neq \emptyset$ and $F(S) \neq \emptyset$. Let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be bounded linear maps. The *split equality fixed point problem* (SEFPP) studied by Moudafi [1] and a host of other authors is the following:

$$\text{find} \quad x^* \in F(T),\ y^* \in F(S) \quad \text{such that} \quad Ax^* = By^*.$$

*Corresponding author.

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We shall denote the set of solutions of the SEFPP by Ω. The SEFPP has applications in several important fields such as in decomposition method for partial differential equation, game theory, intensity modulated radiation therapy, and in many other fields (see, e.g., [2], [3] and the references contained in them). Consequently, the problem has attracted the attention of several researchers, especially within the past 15 years, or so (see, e.g., [4–7] and the references therein). We remark here that if $H_2 = H_3$ and $B = I$, where I is the identity map on H_2, then the SEFPP reduces to the *split common fixed point problem* (SCFPP) introduced by Censor and Segal [8] which is known to have applications in several real life problems (see, e.g., [9] and the references therein).

Zhao [10] introduced the following iterative algorithm for approximating a solution of SCFPP in real Hilbert spaces where T and S are *quasi-nonexpansive mappings*:

$$\begin{cases} x_0 \in H_1, \ y_0 \in H_2, \\ x_{n+1} = \alpha_n u_n + (1-\alpha_n)Tu_n, \quad u_n = x_n - \gamma_n A^*(Ax_n - By_n), \\ y_{n+1} = \beta_n v_n + (1-\beta_n)Sv_n, \quad v_n = y_n + \gamma_n B^*(Ax_n - By_n), \end{cases} \tag{1.1}$$

where A^* and B^* are adjoints of A and B, respectively, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$, $\{\gamma_n\}$ is a sequence of positive numbers satisfying appropriate conditions. Under the assumption that $(I-T)$ and $(I-S)$ are *demiclosed at zero*, Zhao proved that the sequence generated by (1.1) converges weakly to a solution of the SCFPP.

Chidume *et al.* [11], studied the following algorithm for approximating a solution of the SEFPP in real Hilbert spaces where T and S are *demi-contractive mappings.*

$$\begin{cases} x_1 \in H_1, \ y_1 \in H_2, \\ x_{n+1} = (1-\alpha)\big(x_n - \gamma A^*(Ax_n - By_n)\big) + \alpha S\big(x_n - \gamma A^*(Ax_n - By_n)\big), \\ y_{n+1} = (1-\alpha)\big(y_n - \gamma B^*(Ax_n - By_n)\big) + \alpha T\big(y_n - \gamma B^*(Ax_n - By_n)\big), \quad n \geq 1. \end{cases} \tag{1.2}$$

Under the assumption that $(I-T)$ and $(I-S)$ are demiclosed at zero, Chidume *et al.* [11] proved that the sequence generated by (1.2) converges weakly to a solution of the SEFPP.

In 2014, Wu *et al.* [12] studied the split equality problem (SEP) and multiple sets split equality problems for quasi-nonexpansive multi-valued mappings. In the same year, Chang and Agarwal [13] proved a strong convergence theorem for general split equality problems for *quasi-nonexpansive mappings.* For more on iterative algorithms for solving SEFPP in *real Hilbert spaces*, the reader may consult the following references: [12–16], and the references contained in them.

It is well-known that most mathematical problems that arise in real life lie in Banach spaces more general than Hilbert space: Hazewinkel, Series Editor, *Mathematics and its Applications*, rightly stated this fact when he wrote:

> " ... many, and probably most mathematical objects and models do not naturally live in a Hilbert space" [17], pg. viii.

In 2018, Zhaoli *et al.* [18], studied the split feasibility and fixed point problem in *2-uniformly convex and 2-uniformly smooth real Banach spaces*. They considered the following algorithm:

$$\begin{cases} z_n = J_1^{-1}\big(J_1x_n + \gamma A^* J_2(P_Q - I)Ax_n\big), \\ y_n = J_1^{-1}\big((1-\alpha_n)J_1z_n + \alpha_n J_1 S z_n\big), \\ C_{n+1} = \{x \in C_n : \phi(x,y_n) \le \phi(x,x_n);\ \phi(x,z_n) \le \phi(x,x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}x_1, \quad n \ge 1, \end{cases} \tag{1.3}$$

where S is a closed quasi-ϕ-nonexpansive map, P_Q is the metric projection of E_2 onto Q, $\Pi_{C_{n+1}}$ is the generalized projection of x_1 onto C_{n+1}, $\{\beta_n\} \subset [\delta, 1)$, $\delta > 0$ and γ is a constant satisfying $0 < \gamma < \frac{1}{\|A\|^2\kappa^2}$, $\kappa > 0$ is best smoothness constant of the underlying space. They proved that the sequence generated by (1.3) converges strongly to a solution of the SEFPP.

In 2019, Chidume *et al.* [19] studied an iterative algorithm for solving SEFPP for quasi-ϕ-nonexpansive mappings in a 2-uniformly convex and smooth real Banach space. They proved that the sequence generated by their algorithm converges weakly to a solution of the SEFPP.

Remark 1.1. We note, however, that if a real normed space is 2-uniformly convex and 2-uniformly smooth, it is necessarily a real *Hilbert space*.

Remark 1.2. While 2-uniformly convex and smooth real Banach spaces are more general Banach spaces than Hilbert spaces (they include, for example, l_p, for $1 < p \le 2$), *they do not include* the real Banach spaces: l_p **for** $2 < p < \infty$.

Motivated by Remark 1.2, it is our purpose in this paper to introduce a new iterative algorithm for studying the SEFPP for quasi-ϕ-nonexpansive mappings in real Banach spaces that will include *all* l_p, for $1 < p < \infty$.

2. Preliminaries

A real Banach space E is called an *Opial space* (see, e.g., Opial [20]) or is said to satisfy an *Opial condition* if for any sequence $\{x_n\}$ in E such that $\{x_n\}$ converges weakly to some $x \in E$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \tag{2.1}$$

holds for $y \ne x$. It is well known that every real Hilbert space is an Opial spaces (see, e.g., Opial [20]). Furthermore, l_p spaces, $1 < p < \infty$, are Opial spaces but L_p spaces $1 < p < \infty$, $p \ne 2$ are not.

Remark 2.1. Gosse and Lami-Dozo [21] have shown that for any normed space E, the existence of a weakly continuous duality map implies that E is an Opial space (i.e., E satisfies condition (2.1)) but the converse implication does not hold.

Let E be a strictly convex and smooth real Banach space. For $p > 1$, define $J_p : E \to 2^{E^*}$ by

$$J_p(x) = \{u^* \in E^* : \langle x, u^* \rangle = \|x\|\|u^*\|,\ \|u^*\| = \|x\|^{p-1}\}.$$

J_p is called the *generalized duality map on E.* If $p = 2$, J_2 is called the *normalized duality map* and is denoted by J. In a real Hilbert space H, J is the identity map on H. It is easy to see from the definition that

$$J_p(x) = \|x\|^{p-2}Jx, \quad \text{and} \quad \langle x, J_px\rangle = \|x\|^p, \ \forall x \in E.$$

It is well-known that if E is smooth, then J is single-valued and if E is strictly convex, J is one-to-one, and J is surjective if E is reflexive.

Let E be a reflexive, strictly convex and smooth real Banach space with dual space E^*. For $p \geq 2$, Chidume [22] define the following functionals: $\phi_p : E \times E \to \mathbb{R}^+$ by

$$\phi_p(x, y) := \|x\|^p - p\langle x, J_py\rangle + (p-1)\|y\|\|^p, \ \forall x, y \in E.$$

$V_p : E \times E^* \to \mathbb{R}^+$ by

$$V_p(x, x^*) := \|x\|^p - p\langle x, x^*\rangle + (p-1)\|x^*\|^{\frac{p}{p-1}}, \ \forall x \in E, \ x^* \in E^*.$$

It is clear from these definitions that

$$V_p(x, x^*) = \phi_p(x, J_p^{-1}x^*), \ \forall x \in E, \ x^* \in E^*. \tag{2.2}$$

Remark 2.2. We observe that ϕ_p is the Bregman distance for the strictly convex functional $f(x) = \|x\|^p, \ p > 1$. Thus, $\phi_p(x, y) \geq 0, \ \forall \ x, y \in E$. Furthermore, it is easy to see that $\phi_p(x, x) = 0, \ \forall \ x \in E$. Moreover, if $p = 2$, we shall denote $\phi_2(x, y)$ simply as $\phi(x, y)$, so that

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \ \forall x, y \in E. \tag{2.3}$$

The functional ϕ was first introduced by Alber and has been studied extensively by many authors (see, for example, [23], [24], [25], [26], [27], [28], [29], [30], [31], and the references therein). It is easy to see from the definition of ϕ that, in a real Hilbert space H, equation (2.3) reduces to $\phi(x, y) = \|x - y\|^2, \forall x, y \in H$.

Definition 2.3. Let C be a nonempty, closed and convex subset of a real normed space, E. A mapping $T : C \to C$ is said to be *quasi-ϕ-nonexpansive* if $F(T) \neq \emptyset$ and

$$\phi(p, Tx) \leq \phi(p, x), \ \forall \ p \in F(T), \ x \in C.$$

Definition 2.4. A mapping $T : C \to C$ is said to be semi-compact if for any bounded sequence $\{x_n\}$ in C with $x_n - Tx_n \to 0$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to some $x \in C$.

Definition 2.5. Let E be a real normed space with dimension $E \geq 2$. The *modulus of convexity* of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) := \left\{1 - \left\|\frac{u+v}{2}\right\| : \|u\| = \|v\| = 1; \epsilon = \|u - v\|\right\}.$$

Let $p > 1$ be a real number and $\delta_E : (0, 2] \to [0, 1]$ be the modulus of convexity of E. Then a normed space E is said to be *p-uniformly convex* if there exists a constant $c > 0$ such that

$$\delta_E(\epsilon) \geq c\epsilon^p.$$

It is well known that L_p, l_p and the Sobolev spaces $W_p^m(\Omega)$, $1 < p < \infty$, are all p-uniformly convex and that the following estimates hold:

$$\delta_{l_p}(\epsilon) = \delta_{L_p}(\epsilon) = \delta_{W_m^p(\Omega)}(\epsilon) = \begin{cases} \frac{p-1}{8}\epsilon^2 + o(\epsilon^2) > \frac{p-1}{8}\epsilon^2, \ 1 < p < 2; \\ 1 - \left[1 - (\frac{\epsilon}{2})^p\right]^{\frac{1}{p}} > \frac{1}{p}\left(\frac{\epsilon}{2}\right)^p, \ p \geq 2; \end{cases}$$

(see, e.g., [32, 33]).

Lemma 2.6 ([34]). *For $p > 1$, let E be a p-uniformly convex real Banach space. Then, there exists a constant $c_p > 0$ such that*

$$\|\lambda x + (1-\lambda)y\|^p \leq \lambda\|x\|^p + (1-\lambda)\|y\|^p - c_p w_p(\lambda)\|x-y\|^p, \tag{2.4}$$

for all $\lambda \in [0,1]$, $x, y \in E$, where $w_p(\lambda) = \lambda^p(1-\lambda) + \lambda(1-\lambda)^p$.

Lemma 2.7 ([34]). *Let E be a p-uniformly convex real Banach space. Then, there exists a constant $d_p > 0$ such that for all $x, y \in E$,*

$$\|x + y\|^p \geq \|x\|^p + p\langle y, j_p(x)\rangle + d_p\|y\|^p. \tag{2.5}$$

Lemma 2.8 ([35]). *Let D be a nonempty, closed and convex subset of a reflexive strictly convex and smooth Banach space X. Then,*

$$\phi(u, \Pi_D y) + \phi(\Pi_D y, y) \leq \phi(u, y), \ \forall u \in D, \ y \in X.$$

Lemma 2.9 ([36]). *Let E be a real reflexive, strictly convex and smooth Banach space, $A : E \to 2^E$ be a maximal monotone operator with $A^{-1}0 \neq \emptyset$, then for any $x \in E$, $y \in A^{-1}0$ and $r > 0$, we have*

$$\phi(y, Q_r^A x) + \phi(Q_r^A x, x) \leq \phi(y, x),$$

where $Q_r^A : E \to E$ is defined by $Q_r^A x := (J + rA)^{-1}Jx$.

Lemma 2.10. *Let E be a reflexive, strictly convex and smooth real Banach space. Then, for $p > 1$,*

$$V_p(u, u^*) + p\langle J_p^{-1}u^* - u, v^*\rangle \leq V_p(u, u^* + v^*), \ \forall u \in E, \ u^*, v^* \in E^*. \tag{2.6}$$

Proof. We compute as follows: Using the definition of V_p, expand $V_p(u, u^*) + p\langle J_p^{-1}u^* - u, v^*\rangle$ to establishing the lemma. ∎

Lemma 2.11. *Let E be a reflexive, strictly convex and smooth real Banach space. Then, for $p > 1$,*

$$\phi_p(x, J_p^{-1}(\lambda J_p u + (1-\lambda)J_p v)) \leq \lambda\phi_p(x, u) + (1-\lambda)\phi_p(x, v), \ \forall x, u, v \in E. \tag{2.7}$$

Proof. Use the definition of ϕ_p and Lemma 2.6 to establish the lemma. ∎

Lemma 2.12. *Let E be a p-uniformly convex and smooth real Banach space with dual space E^*. For $p > 1$, let $J_p : E \to E^*$ be the generalized duality map. Then,*

$$\|J_p^{-1}x - J_p^{-1}y\| \leq \kappa_p\|x - y\|^{\frac{1}{p-1}}, \ \forall x, y \in E, \tag{2.8}$$

where $\kappa_p = \left(\frac{1}{c_2}\right)^{\frac{1}{p-1}}$, for some constant $c_2 > 0$.

Proof. For E, the following inequality holds:

$$\langle x - y, J_p x - J_p y\rangle \geq c_2\|x - y\|^p, \ \forall x, y \in E, \tag{2.9}$$

for some constant $c_2 > 0$. Inequality (2.8) follows from inequality (2.9), establishing the Lemma. ∎

2.1. Analytical Representations of Generalized Duality Maps in L_p, l_p, and W_m^p, Spaces, $1 < p < \infty$

Using the analytic representation of the *normalized* duality maps in L_p, l_p, and W_m^p, $1 < p < \infty$ (see e.g., Lindenstrauss and Tzafriri [32]) and the relation $J_p(x) = \|x\|^{p-2}J(x)$, we obtain the analytical representations of *generalized* duality maps in these spaces as follows:

$$
\begin{aligned}
Jz &= y \in l_q,\ \ y = \{|z_1|^{p-2}z_1, |z_2|^{p-2}z_2, ...\},\ \ z = \{z_1, z_2, ...\},\\
J^{-1}z &= y \in l_p,\ \ y = \{|z_1|^{q-2}z_1, |z_2|^{q-2}z_2, ...\},\ \ z = \{z_1, z_2, ...\},\\
Jz &= \|z\|_{L_p}^{2-p}|z(s)|^{p-2}z(s) \in L_q(G),\ \ s \in G,\\
J^{-1}z &= |z(s)|^{q-2}z(s) \in L_p(G),\ \ s \in G,\ \text{ and}\\
Jz &= \sum_{|\alpha|\le m}(-1)^{|\alpha|}D^{\alpha}(|D^{\alpha}z(s)|^{p-2}D^{\alpha}z(s)) \in W_{-m}^{q}(G), m > 0, s \in G,
\end{aligned}
$$

3. Main Result

3.1. A Weak Convergence Theorem

In Theorem 3.2 below, for $p > 1$,

(i) E_1 and E_2 are p-uniformly convex and uniformly smooth real Banach spaces which satisfy Opial condition, and E_3 is a smooth real Banach space, with dual spaces, E_1^*, E_2^* and E_3^*, respectively.
(ii) $T : E_1 \to E_1$ and $S : E_2 \to E_2$ are quasi-ϕ-nonexpansive maps.
(iii) $A : E_1 \to E_3$ and $B : E_2 \to E_3$ are bounded linear maps with adjoints A^* and B^*, respectively.
(iv) J_{pE_i} denotes the *generalized* duality map on E_i, for $i = 1, 2, 3$; $J_{pE_i}^{-1}$ denotes the *generalized* duality map on E_i^*, for $i = 1, 2, 3$.

Algorithm 3.1.

$$
\begin{cases}
x_1 \in E_1,\ y_1 \in E_2,\ z_n = J_{pE_3}(Ax_n - By_n),\\
x_{n+1} = J_{pE_1}^{-1}\big(\alpha_n J_{pE_1}u_n + (1-\alpha_n)J_{pE_1}Tu_n\big),\\
u_n = J_{pE_1}^{-1}\big(J_{pE_1}x_n - \gamma A^*z_n\big),\\
y_{n+1} = J_{pE_2}^{-1}\big(\alpha_n J_{pE_2}v_n + (1-\alpha_n)J_{pE_2}Su_n\big),\\
v_n = J_{pE_2}^{-1}\big(J_{pE_2}y_n + \gamma B^*z_n\big),
\end{cases}
\tag{3.1}
$$

where $\alpha_n \in (0,1)$ and $0 < \gamma < \left[\frac{1}{\kappa_p\left(\|A\|^{\frac{p}{p-1}} + \|B\|^{\frac{p}{p-1}}\right)}\right]^{p-1}$.

We now prove the following theorem:

Theorem 3.2. *Let $\{(x_n, y_n)\}$ be a sequence generated in $E_1 \times E_2$ by algorithm 3.1. Assume that $(I - T)$ and $(I - S)$ are demiclosed at zero and that $\Omega := \{(x, y) \in F(T) \times F(S) : Ax = By\} \neq \emptyset$. Then, $\{(x_n, y_n)\}$ converges weakly to some $(x^*, y^*) \in \Omega$.*

Proof. We first show that $\{x_n\}$ and $\{y_n\}$ are bounded. Let $(x,y)\in\Omega$. Then, using Lemma 2.11, we obtain

$$\begin{aligned}\phi_{pE_1}(x,x_{n+1}) &= \phi_{pE_1}(x, J_{pE_1}^{-1}(\alpha_n J_{pE_1}u_n + (1-\alpha_n)J_{pE_1}Tu_n))\\ &\le \alpha_n\phi_{pE_1}(x,u_n) + (1-\alpha_n)\phi_{pE_1}(x,Tu_n)\\ &\le \phi_{pE_1}(x,u_n).\end{aligned} \tag{3.2}$$

Furthermore,

$$\begin{aligned}\phi_{pE_1}(x,u_n) &= \phi_{pE_1}(x, J_{pE_1}^{-1}(J_{pE_1}x_n - \gamma A^* z_n))\\ &= V_{pE_1}(x, J_{pE_1}x_n - \gamma A^* z_n)\\ &\le V_{pE_1}(x, J_{pE_1}x_n) - p\gamma\langle J_{pE_1}^{-1}(J_{pE_1}x_n - \gamma A^* z_n) - x, A^* z_n\rangle\\ &= \phi_{pE_1}(x,x_n) - p\gamma\langle Au_n - Ax, z_n\rangle.\end{aligned} \tag{3.3}$$

Substituting (3.3) in (3.2), we obtain

$$\phi_{pE_1}(x,x_{n+1}) \le \phi_{pE_1}(x,x_n) - p\gamma\langle Au_n - Ax, z_n\rangle. \tag{3.4}$$

Following a similar argument, we obtain the following:

$$\phi_{pE_2}(y,y_{n+1}) \le \phi_{pE_2}(y,y_n) - p\gamma\langle By - Bv_n, z_n\rangle. \tag{3.5}$$

Adding inequalities (3.4) and (3.5), we obtain that

$$\begin{aligned}\phi_{pE_1}(x,x_{n+1}) + \phi_{pE_2}(y,y_{n+1}) \le \phi_{pE_1}(x,x_n) &+ \phi_{pE_2}(y,y_n)\\ &- p\gamma\langle Au_n - Bv_n, z_n\rangle.\end{aligned} \tag{3.6}$$

Using Lemma 2.12 and the fact that $z_n = J_{pE_3}(Ax_n - By_n)$, we compute as follows:

$$\begin{aligned}-p\gamma\langle Au_n - Bv_n, z_n\rangle &= -p\gamma\|Ax_n - By_n\|^p - p\gamma\langle Au_n - Bv_n, z_n\rangle + p\gamma\langle Ax_n - By_n, z_n\rangle\\ &= -p\gamma\|Ax_n - By_n\|^p + p\gamma\langle A(x_n - u_n), z_n\rangle + p\gamma\langle B(v_n - y_n), z_n\rangle\\ &\le -p\gamma\|Ax_n - By_n\|^p + p\gamma\Big(\|A\|\cdot\|x_n - J_{pE_1}^{-1}(J_{pE_1}x_n - \gamma A^* z_n)\|\\ &\quad + \|B\|\cdot\|y_n - J_{pE_2}^{-1}(J_{pE_2}y_n + \gamma B^* z_n)\|\Big)\cdot\|z_n\|\\ &\le -p\gamma\|Ax_n - By_n\|^p + p\gamma^{\frac{p}{p-1}}\kappa_p\Big(\|A\|\cdot\|A^* z_n\|^{\frac{1}{p-1}}\\ &\quad + \|B\|\cdot\|B^* z_n\|^{\frac{1}{p-1}}\Big)\cdot\|z_n\|.\end{aligned} \tag{3.7}$$

But $\quad \|A\|\cdot\|A^* z_n\|^{\frac{1}{p-1}}\|z_n\| \le \|A\|^{\frac{p}{p-1}}\|Ax_n - By_n\|^p, \quad$ and,

$$\|B\|\cdot\|B^* z_n\|^{\frac{1}{p-1}}\|z_n\| \le \|B\|^{\frac{p}{p-1}}\|Ax_n - By_n\|^p.$$

Substituting these inequalities in inequality (3.7), we obtain:

$$-p\gamma\langle Au_n - Bv_n, z_n\rangle \le -p\gamma\|Ax_n - By_n\|^p + p\gamma^{\frac{p}{p-1}}\kappa_p\Big(\|A\|^{\frac{p}{p-1}} + \|B\|^{\frac{p}{p-1}}\Big)\|Ax_n - By_n\|^p.$$

Substituting this inequality in inequality (3.6), we obtain:

$$\begin{aligned}\phi_{pE_1}(x,x_{n+1}) + \phi_{pE_2}(y,y_{n+1}) &\le \phi_{pE_1}(x,x_n) + \phi_{pE_2}(y,y_n)\\ &\quad - p\gamma\Big[1 - \gamma^{\frac{1}{p-1}}\kappa_p\big(\|A\|^{\frac{p}{p-1}} + \|B\|^{\frac{p}{p-1}}\big)\Big]\|Ax_n - By_n\|^p\\ &\le \phi_{pE_1}(x,x_n) + \phi_{pE_2}(y,y_n).\end{aligned} \tag{3.8}$$

Define $\Lambda_n(x,y) := \phi_p(x,x_n) + \phi_p(y,y_n)$. Then, from inequality (3.8), we obtain that $\{\Lambda_n(x,y)\}$ is convergent. This implies that $\{x_n\}$ and $\{y_n\}$ are bounded and, consequently, $\{u_n\}$ and $\{v_n\}$ are bounded. Furthermore, $\|Ax_n - By_n\| \to 0$, as $n \to \infty$. Hence,

$$\|x_n - u_n\| = \|J_{pE_1}^{-1}(J_{pE_1}x_n - \gamma A^* z_n) - x_n\| \leq \kappa_p\big(\gamma\|A\|\big)^{\frac{1}{p-1}}\|Ax_n - By_n\| \to 0, \quad \text{as} \quad n \to \infty.$$

Similarly, $\|v_n - y_n\| \to 0$, as $n \to \infty$.

Using the definition of ϕ_p, Lemma 2.6 and the quasi-ϕ-nonexpansiveness of T, we obtain

$$\begin{aligned}\phi_p(x,x_{n+1}) &= \phi_p(x, J_{pE_1}^{-1}\big(\alpha_n J_{pE_1}u_n + (1-\alpha_n)J_{pE_1}Tu_n\big))\\ &\leq \|x\|^p - p\langle x, \alpha_n J_{pE_1}u_n\rangle - p\langle x, (1-\alpha_n)J_{pE_1}Tu_n\rangle + \alpha_n\|J_{pE_1}u_n\|^p\\ &\quad + (1-\alpha_n)\|J_{pE_1}Tu_n\|^p - c_p w_p(\alpha_n)\|J_{pE_1}u_n - J_{pE_1}Tu_n\|^p\\ &= \alpha_n\phi_p(x,u_n) + (1-\alpha_n)\phi_p(x,Tu_n) - c_p w_p(\alpha_n)\|J_{pE_1}u_n - J_{pE_1}Tu_n\|^p,\end{aligned}$$

so that

$$\phi_p(x,x_{n+1}) \leq \phi_p(x,u_n) - c_p w_p(\alpha_n)\|J_{pE_1}u_n - J_{pE_1}Tu_n\|^p.$$

From inequality (3.3), we obtain that:

$$\phi_p(x,x_{n+1}) \leq \phi_p(x,x_n) - p\gamma\langle Au_n - Ax, z_n\rangle - c_p w_p(\alpha_n)\|J_{pE_1}u_n - J_{pE_1}Tu_n\|^p. \quad (3.9)$$

Following a similar argument, we obtain that:

$$\phi_p(y,y_{n+1}) \leq \phi_p(y,y_n) - p\gamma\langle By - Bv_n, z_n\rangle - c_p w_p(\alpha_n)\|J_{pE_2}v_n - J_{pE_1}Sv_n\|^p. \quad (3.10)$$

Adding (3.9) and (3.10), and using the fact that $Ax = By$, and the condition on γ, we obtain that

$$\begin{aligned}\Lambda_{n+1}(x,y) \leq \Lambda_n(x,y) - c_p w_p(\alpha_n)\Big[&\|J_{pE_1}u_n - J_{pE_1}Tu_n\|^p\\ &+ \|J_{pE_2}v_n - J_{pE_1}Sv_n\|^p\Big].\end{aligned} \quad (3.11)$$

Since $\lim\limits_{n\to\infty}\Lambda_n(x,y)$ exists, it follows from inequality (3.11) that:

$$\lim_{n\to\infty}\|J_{pE_1}u_n - J_{pE_1}Tu_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|J_{pE_2}v_n - J_{pE_1}Sv_n\| = 0.$$

By the uniform continuity of $J_{pE_1}^{-1}$ and $J_{pE_2}^{-1}$ on bounded sets, we obtain that $\|u_n - Tu_n\| \to 0$ and $\|v_n - Sv_n\| \to 0$, as $n \to \infty$. Since $\{x_n\}$ and $\{y_n\}$ are bounded, there exist $r_1 > 0$ and $r_2 > 0$ such that $\{x_n\} \subset \overline{B_1}(0,r_1) := \{u \in E_1 : \|u\| \leq r_1\}$ and $\{y_n\} \subset \overline{B_2}(0,r_2) := \{v \in E_2 : \|v\| \leq r_1\}$. Furthermore, there exist subsequences $\{x_{n_k}\}$ and $\{y_{n_k}\}$ of $\{x_n\}$ and $\{y_n\}$ respectively such that $x_{n_k} \rightharpoonup x^*$ and $y_{n_k} \rightharpoonup y^*$, as $k \to \infty$, for some $x^* \in \overline{B_1}$, $y^* \in \overline{B_2}$. Since $\|x_n - u_n\| \to 0$ and $\|y_n - v_n\| \to 0$, as $n \to \infty$, we have, in particular, that $u_{n_k} \rightharpoonup x^*$ and $v_{n_k} \rightharpoonup y^*$, as $k \to \infty$. By demi-closedness of $(I-T)$ and $(I-S)$, we obtain that $x^* \in F(T)$ and $y^* \in F(S)$. Furthermore, by the weak lower semi-continuity of the norm, we obtain that:

$$0 = \lim_{n\to\infty}\|Ax_n - By_n\| = \liminf_{n\to\infty}\|Ax_n - By_n\| \geq \|Ax^* - By^*\|,$$

which implies that $Ax^* = By^*$. Hence, $(x^*, y^*) \in \Omega$.

Let $\{x_{n_j}\}$ be an arbitrary subsequence of $\{x_n\}$ such that $x_{n_j} \rightharpoonup q \in E_1$, as $j \to \infty$. We claim $q = x^*$. Suppose this claim is false. Then $q \neq x^*$. Since E_1 satisfies Opial condition, we have:

$$\liminf_{n\to\infty} \|x_n - x^*\| < \liminf_{n\to\infty} \|x_n - q\| < \liminf_{n\to\infty} \|x_n - x^*\|,$$

and this contradiction yields that $q = x^*$. Hence, $\{x_n\}$ has a unique weak cluster point and so $x_n \rightharpoonup x^*$. A similar argument yields that $y_n \rightharpoonup y^*$. The proof of the theorem is complete. ∎

3.2. A Strong Convergence Theorem

In Theorem 3.4 below, for $p > 1$,

(i) E_1 and E_2 are p-uniformly convex and uniformly smooth real Banach spaces which satisfy Opial condition, and E_3 is a smooth real Banach space, with dual spaces, E_1^*, E_2^* and E_3^*, respectively.

(ii) $T : E_1 \to E_1$ and $S : E_2 \to E_2$ are quasi-ϕ-nonexpansive maps.

(iii) $A : E_1 \to E_3$ and $B : E_2 \to E_3$ are bounded linear maps with adjoints A^* and B^*, respectively.

(iv) J_{pE_i} denotes the *generalized* duality map on E_i, for $i = 1, 2, 3$; $J_{pE_i}^{-1}$ denotes the *generalized* duality map on E_i^*, for $i = 1, 2, 3$.

Algorithm 3.3.

$$\begin{cases} x_1 \in E_1,\ y_1 \in E_2,\ z_n = J_{pE_3}(Ax_n - By_n), \\ x_{n+1} = J_{pE_1}^{-1}\big(\alpha_n J_{pE_1} u_n + (1-\alpha_n) J_{pE_1} T u_n\big), \\ u_n = J_{pE_1}^{-1}\big(J_{pE_1} x_n - \gamma A^* z_n\big), \\ y_{n+1} = J_{pE_2}^{-1}\big(\alpha_n J_{pE_2} v_n + (1-\alpha_n) J_{pE_2} S u_n\big), \\ v_n = J_{pE_2}^{-1}\big(J_{pE_2} y_n + \gamma B^* z_n\big), \end{cases} \tag{3.12}$$

where $\alpha_n \in (0,1)$ and $0 < \gamma < \left[\frac{1}{\kappa_p\left(\|A\|^{\frac{p}{p-1}} + \|B\|^{\frac{p}{p-1}}\right)}\right]^{p-1}$.

We now prove the following theorem:

Theorem 3.4. *Let $\{(x_n, y_n)\}$ be a sequence generated in $E_1 \times E_2$ by algorithm (3.3). Assume that T and S are semi-compact and, $(I - T)$ and $(I - S)$ are demiclosed at zero and that $\Omega := \{(x, y) \in F(T) \times F(S) : Ax = By\} \neq \emptyset$. Then, $\{(x_n, y_n)\}$ converges strongly to some $(x^*, y^*) \in \Omega$.*

Proof. Following the same argument as in the proof of Theorem 3.2, we obtain that:

$$\lim_{n\to\infty} \|u_n - x_n\| = 0 = \lim_{n\to\infty} \|v_n - y_n\| \quad \text{and} \tag{3.13}$$

$$\lim_{n\to\infty} \|u_n - Tu_n\| = 0 = \lim_{n\to\infty} \|v_n - Sv_n\|. \tag{3.14}$$

Thus, $u_n \rightharpoonup x^*$ and $v_n \rightharpoonup y^*$. By semi-compactness of T and S, there exist subsequences $\{u_{n_j}\}$ of $\{u_n\}$ and $\{v_{n_j}\}$ of $\{v_n\}$ such that $u_{n_j} \to x^*$ and $v_{n_j} \to y^*$, as $j \to \infty$. Let $\{u_{n_i}\}$ be any other subsequence of $\{u_n\}$ such that $u_{n_i} \to q$, as $i \to \infty$. Let $w := \liminf_{n\to\infty}(\phi(q, u_n) - \phi(x^*, u_n))$.

$$\phi(q, u_n) - \phi(x^*, u_n) = 2\langle x^* - q, Ju_n\rangle + \|q\|^2 - \|x^*\|^2. \tag{3.15}$$

Since $u_{n_i} \to q$, as $i \to \infty$ and $u_{n_j} \to x^*$, as $j \to \infty$, from (3.15) we have

$$w = 2\langle x^* - q, Jx^* \rangle + \|q\|^2 - \|x^*\|^2$$

and

$$w = 2\langle x^* - q, Jq \rangle + \|q\|^2 - \|x^*\|^2.$$

Thus, $\langle x^* - q, Jx^* - Jq \rangle = 0$. This implies that $x^* = q$, since J is strictly monotone. Therefore, $\{u_n\}$ converges strongly to x^*. Hence, from (3.13), $\{x_n\}$ converges strongly to x^*. Following a similar argument, we obtain that $\{y_n\}$ converges strongly to y^*. This completes the proof. ∎

3.3. Corollaries

The setting for Corollary 3.6 is as follows:

- (i) $E_1 = L_{p_1}$, and $E_2 = L_{p_2}$, $p_1, p_2 \in [2, \infty)$, and E_3 is a smooth real Banach space;
- (ii) $T : E_1 \to E_1$ and $S : E_2 \to E_2$ are quasi-ϕ-nonexpansive maps;
- (iii) $A : E_1 \to E_3$ and $B : E_2 \to E_3$ are bounded linear maps with adjoints A^* and B^*, respectively;
- (iv) J_{p_1}, J_{p_2} and J_{p_3} are the *generalized* duality maps on E_1, E_2 and E_3, respectively.

Algorithm 3.5.

$$\begin{cases} x_1 \in E_1, \; y_1 \in E_2, \; z_n = J_{p_3}(Ax_n - By_n), \\ x_{n+1} = J_{p_1}^{-1}\big(\alpha J_{p_1} u_n + (1-\alpha) J_{p_1} T u_n\big), \\ u_n = J_{p_1}^{-1}\big(J_{p_1} x_n - \gamma A^* z_n\big), \\ y_{n+1} = J_{p_2}^{-1}\big(\alpha J_{p_2} v_n + (1-\alpha) J_{p_2} S u_n\big), \\ v_n = J_{p_2}^{-1}\big(J_{p_2} y_n + \gamma B^* z_n\big), \end{cases} \tag{3.16}$$

where $\alpha \in (0,1)$ and $0 < \gamma < \left[\frac{1}{\kappa_p\left(\|A\|^{\frac{p}{p-1}} + \|B\|^{\frac{p}{p-1}}\right)}\right]^{p-1}$. We now deduce the following corollary:

Corollary 3.6. *Let $\{(x_n, y_n)\}$ be a sequence generated in $E_1 \times E_2$ by algorithm 3.5. Assume that $(I - T)$ and $(I - S)$ are demi-closed at zero and that $\Omega := \{(x, y) \in F(T) \times F(S) : Ax + By\} \neq \emptyset$. Then, $\{(x_n, y_n)\}$ converges weakly to some $(x^*, y^*) \in \Omega$.*

The setting for Corollary 3.8 is as follows:

- (i) $E_1 = L_{p_1}$, and $E_2 = L_{p_2}$, $p_1, p_2 \in (1, 2]$, and E_3 is a smooth real Banach space;
- (ii) $T : E_1 \to E_1$ and $S : E_2 \to E_2$ are quasi-ϕ-nonexpansive maps;
- (iii) $A : E_1 \to E_3$ and $B : E_2 \to E_3$ are bounded linear maps with adjoints A^* and B^*, respectively;
- (iv) J_{p_1}, J_{p_2} and J_{p_3} are the *generalized* duality maps on E_1, E_2 and E_3, respectively.

Algorithm 3.7.

$$\begin{cases} x_1 \in E_1,\ y_1 \in E_2,\ z_n = J_3(Ax_n - By_n), \\ x_{n+1} = J_1^{-1}\big(\alpha J_1 u_n + (1-\alpha)J_1 T u_n\big), \\ u_n = J_1^{-1}\big(J_1 x_n - \gamma A^* z_n\big), \\ y_{n+1} = J_2^{-1}\big(\alpha J_2 v_n + (1-\alpha)J_2 S u_n\big), \\ v_n = J_2^{-1}\big(J_2 y_n + \gamma B^* z_n\big), \end{cases} \tag{3.17}$$

where $\alpha \in (0,1)$ and $0 < \gamma < \left[\frac{1}{\kappa_2\left(\|A\|^2 + \|B\|^2\right)}\right]$. We now deduce the following corollary:

Corollary 3.8. *Let $\{(x_n, y_n)\}$ be a sequence generated in $E_1 \times E_2$ by algorithm 3.7. Assume that $(I - T)$ and $(I - S)$ are demi-closed at zero and that $\Omega := \{(x, y) \in F(T) \times F(S) : Ax + By\} \neq \emptyset$. Then, $\{(x_n, y_n)\}$ converges weakly to some $(x^*, y^*) \in \Omega$.*

Remark 3.9. The condition on γ involves the norms, $\|A\|$, $\|B\|$ of A and B, respectively. This is not a drawback on implementing the algorithm because, *for applying the algorithm*, **one does not need to compute these norms**. The norms can be replaced with two constants associated with the maps A and B, as follows. To assert that A is a bounded linear map, one has to show that:

$$\|Ax\| \leq K\|x\|,\ \forall x \in E,$$

and some constant $K > 0$. This constant $K > 0$ which is an upper bound for $\|A\|$ is generally fairly easy to obtain (since it is not unique) for any bounded linear map. Similarly, to assert that B is a bounded linear map, one has to show that:

$$\|Bx\| \leq L\|x\|,\ \forall x \in E,$$

and some constant $L > 0$. Again, this constant $L > 0$ is an upper bound for $\|B\|$ and is generally fairly easy to obtain for any bounded linear map. It is easy to see from the proof of Theorem 3.2 that the condition

$$0 < \gamma < \left[\frac{1}{\kappa_p\left(\|A\|^{\frac{p}{p-1}} + \|B\|^{\frac{p}{p-1}}\right)}\right]^{p-1},$$

can be replaced with the condition

$$0 < \gamma < \left[\frac{1}{\kappa_p\left(K^{\frac{p}{p-1}} + L^{\frac{p}{p-1}}\right)}\right]^{p-1},$$

where K and L are easily obtained.

4. Applications

In this section, we shall present some applications of Theorem 3.2. In the sequel, we assume that H_1, H_2, and H_3 are real Hilbert spaces, C and Q are nonempty, closed and convex subsets of H_1 and H_2, respectively, and $A : H_1 \to H_3$, $B : H_2 \to H_3$ are bounded linear mappings.

4.1. Split Equality Problem (SEP)

The split equality problem (SEP) is to

$$\text{find } x \in C,\ y \in Q \text{ such that } Ax = By.$$

Several iterative algorithms have been proposed to approximate solutions of SEP in real Hilbert spaces and in Banach spaces more general than Hilbert spaces (see, e.g., [19, 37, 38]).

We shall apply Theorem 3.2 to approximate a solution of the SEP in Banach spaces more general than Hilbert spaces.

In Theorem 4.2 below, for $p > 1$,

(i) E_1 and E_2 are p-uniformly convex and uniformly smooth real Banach spaces which satisfy Opial condition, and E_3 is a smooth real Banach space, with dual spaces, E_1^*, E_2^* and E_3^*, respectively.
(ii) $T : E_1 \to E_1$ and $S : E_2 \to E_2$ are quasi-ϕ-nonexpansive maps.
(iii) $A : E_1 \to E_3$ and $B : E_2 \to E_3$ are bounded linear maps with adjoints A^* and B^*, respectively.
(iv) J_{pE_i} denotes the *generalized* duality map on E_i, for $i = 1, 2, 3$; $J_{pE_i}^{-1}$ denotes the *generalized* duality map on E_i^*, for $i = 1, 2, 3$.

Algorithm 4.1.

$$\begin{cases} x_1 \in E_1,\ y_1 \in E_2,\ z_n = J_{pE_3}(Ax_n - By_n), \\ x_{n+1} = J_{pE_1}^{-1}\big(\alpha_n J_{pE_1} u_n + (1-\alpha_n) J_{pE_1} \Pi_C u_n\big), \\ u_n = J_{pE_1}^{-1}\big(J_{pE_1} x_n - \gamma A^* z_n\big), \\ y_{n+1} = J_{pE_2}^{-1}\big(\alpha_n J_{pE_2} v_n + (1-\alpha_n) J_{pE_2} \Pi_Q u_n\big), \\ v_n = J_{pE_2}^{-1}\big(J_{pE_2} y_n + \gamma B^* z_n\big), \end{cases} \tag{4.1}$$

where $\alpha_n \in (0,1)$ and $0 < \gamma < \left[\frac{1}{\kappa_p\left(\|A\|^{\frac{p}{p-1}} + \|B\|^{\frac{p}{p-1}}\right)}\right]^{p-1}$.

We now prove the following theorem:

Theorem 4.2. *Let $\{(x_n, y_n)\}$ be a sequence generated in $E_1 \times E_2$ by algorithm 4.1. Assume that $\Omega := \{(x,y) \in C \times Q : Ax = By\} \neq \emptyset$. Then, $\{(x_n, y_n)\}$ converges weakly to some $(x^*, y^*) \in \Omega$.*

Proof. Set $T = \Pi_C$ and $S = \Pi_Q$. Clearly, $F(T) = C$ and $F(S) = Q$ are nonempty. Thus, using Lemma 2.8 it is easy to see that T and S are quasi-ϕ-nonexpansive. Hence, the conclusion follows from Theorem 3.2. ∎

4.2. Split Equality Variational Inclusion Problem (SEVIP)

Let $M : H_1 \to 2^{H_1}$ and $N : H_2 \to 2^{H_2}$ be maximal monotone mappings. The split equality variational inclusion problem (SEVIP) is:

$$\text{finding } x \in M^{-1}(0),\ y \in N^{-1}(0) \text{ such that } Ax = By,$$

where $M^{-1}(0) = \{x \in H_1 : 0 \in M(x)\}$ and $N^{-1}(0) = \{x \in H_2 : 0 \in N(x)\}$ are called the set of zeros of M and N, respectively. We shall apply Theorem 3.2 to approximate a solution of a SEVIP in certain Banach spaces. Theorem 4.4 below improves the results in [39] and [19].

In Theorem 4.4 below, for $p > 1$,

(i) E_1 and E_2 are p-uniformly convex and uniformly smooth real Banach spaces which satisfy Opial condition, and E_3 is a smooth real Banach space, with dual spaces, E_1^*, E_2^* and E_3^*, respectively.
(ii) $T : E_1 \to E_1$ and $S : E_2 \to E_2$ are quasi-ϕ-nonexpansive maps.
(iii) $A : E_1 \to E_3$ and $B : E_2 \to E_3$ are bounded linear maps with adjoints A^* and B^*, respectively.
(iv) J_{pE_i} denotes the *generalized* duality map on E_i, for $i = 1, 2, 3$; $J_{pE_i}^{-1}$ denotes the *generalized* duality map on E_i^*, for $i = 1, 2, 3$.

Algorithm 4.3.

$$\begin{cases} x_1 \in E_1,\ y_1 \in E_2,\ z_n = J_{pE_3}(Ax_n - By_n), \\ x_{n+1} = J_{pE_1}^{-1}\big(\alpha_n J_{pE_1} u_n + (1-\alpha_n) J_{pE_1} Q_r^M u_n\big), \\ u_n = J_{pE_1}^{-1}\big(J_{pE_1} x_n - \gamma A^* z_n\big), \\ y_{n+1} = J_{pE_2}^{-1}\big(\alpha_n J_{pE_2} v_n + (1-\alpha_n) J_{pE_2} Q_r^N v_n\big), \\ v_n = J_{pE_2}^{-1}\big(J_{pE_2} y_n + \gamma B^* z_n\big), \end{cases} \tag{4.2}$$

where $Q_r^M := (J_{pE_1} + rM)^{-1} J_{pE_1}$ and $Q_r^N := (J_{pE_2} + rN)^{-1} J_{pE_2}$, $r > 0$, $\alpha_n \in (0,1)$ and $0 < \gamma < \left[\frac{1}{\kappa_p\left(\|A\|^{\frac{p}{p-1}} + \|B\|^{\frac{p}{p-1}}\right)}\right]^{p-1}$.

We now prove the following theorem:

Theorem 4.4. *Let $\{(x_n, y_n)\}$ be a sequence generated in $E_1 \times E_2$ by algorithm 4.3. Assume that $\Omega := \{(x, y) \in M^{-1}(0) \times N^{-1}(0) : Ax = By\} \neq \emptyset$. Then, $\{(x_n, y_n)\}$ converges weakly to some $(x^*, y^*) \in \Omega$.*

Proof. Set $T = Q_r^M$ and $S = Q_r^N$. Clearly, $F(T) = C$ and $F(S) = Q$ are nonempty. Thus, using Lemma 2.9, it is easy to see that T and S are quasi-ϕ-nonexpansive. Hence, the conclusion follows from Theorem 3.2. ∎

5. Numerical Illustrations

In this section, we give numerical examples to illustrate the convergence of sequences generated by our algorithm.

Example 5.1.

In Theorem 3.4, set $E_1 = \mathbb{R}$, $E_2 = \mathbb{R}^2$ and $E_3 = \mathbb{R}^2$. Let $A : E_1 \to E_3$ and $B : E_2 \to E_3$ be defined by

$$Ax := \left(\frac{x}{2}, \frac{x}{3}\right), \qquad B(x, y) := (x + 2y, y),$$

respectively. Then,

$$A^*(u, v) = \frac{u}{2} + \frac{v}{3} \quad \text{and} \quad B^*(u, v) = (u, 2u + v).$$

Let $T : E_1 \to E_1$ and $S : E_2 \to E_2$ be defined by

$$Tx := \frac{x}{2} \quad \text{and} \quad S(u,v) := (u,v).$$

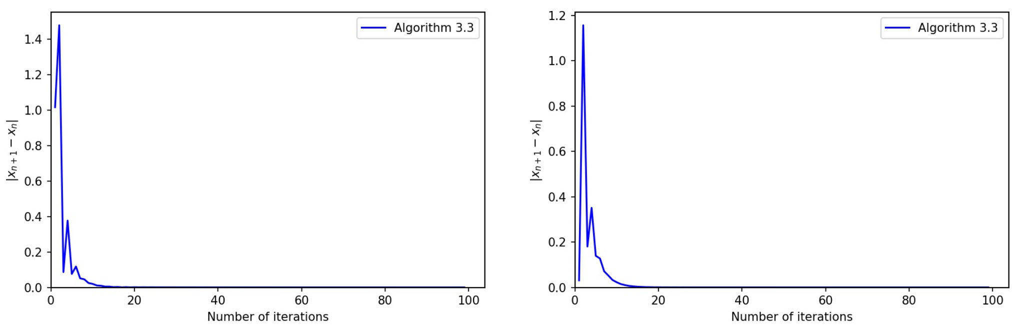
It is easy to verify that T and S are quasi-ϕ-nonexpansive and, $(I - T)$ and $(I - S)$ are demiclosed at zero. Furthermore, since $0 \in \Omega$, $\Omega \neq \emptyset$. In algorithm 3.3, we take $\gamma = 0.3$, $\alpha_n = \frac{1}{(n+1)^2}$ as our parameters. Clearly, these parameters satisfy the hypothesis of Theorem 3.4. Using a tolerance 10^{-8} and setting maximum number of iterations $n = 100$, we obtain the following iterates:

Table 1. Numerical results of Example 5.1

Table of values choosing $x_1 = -3$ and $y_1 = (2,-2)^T$		
	Algorithm 3.3	
n	$\lvert x_{n+1} - x_n \rvert$	$\lVert y_{n+1} - y_n \rVert$
1	1.0156	3.1784
20	9.36E-4	0.0475
40	2.14E-4	0.0164
60	7.41E-5	5.69E-3
80	2.56E-5	1.97E-3
99	9.37E-6	7.19E-4

Table 2. Numerical results of Example 5.1

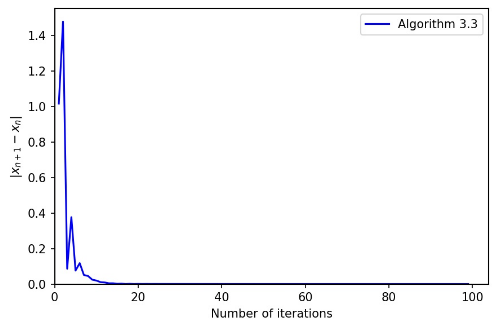
Table of values choosing $x_1 = 1.25$ and $y_1 = (0.5, 2.15)^T$		
	Algorithm 3.3	
n	$\lvert x_{n+1} - x_n \rvert$	$\lVert y_{n+1} - y_n \rVert$
1	0.0309	4.3415
20	4.61E-4	6.89E-3
40	2.68E-5	2.04E-3
60	9.22E-6	7.07E-4
80	3.19E-6	2.45E-4
99	1.16E-6	8.94E-5



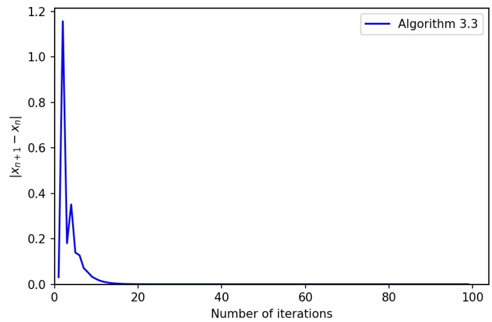
(a) Graph of the first 99 iterates of Algorithm 3.3 choosing $x_1 = -3$

(b) Graph of the first 99 iterates of Algorithm 3.3 choosing $x_1 = 1.25$

Figure 1. Graph of the iterates of $\{x_n\}$ generated by Algorithm 3.3



(A) Graph of the first 99 iterates of Algorithm 3.3 choosing $y_1 = (2,$-$2)^T$



(B) Graph of the first 99 iterates of Algorithm 3.3 choosing $(0.5, 2.15)^T$

FIGURE 2. Graph of the iterates of $\{y_n\}$ generated by Algorithm 3.3

Example 5.2.

In Theorems and 3.2, set $E_1 = E_2 = E_3 = L_2([0,1])$. Let $A : E_1 \to E_3$ and $B : E_2 \to E_3$ be defined by

$$(Ax)(t) = 2x(t), \quad \text{and} \quad (Bx)(t) = x(t), \quad \text{then} \quad A^* = A \quad \text{and} \quad B^* = B.$$

Let $T : E_1 \to E_1$ and $S : E_2 \to E_2$ be defined by

$$(Tx)(t) = \frac{x(t)}{8} \quad \text{and} \quad (Sx)(t) = \frac{x(t)}{2}.$$

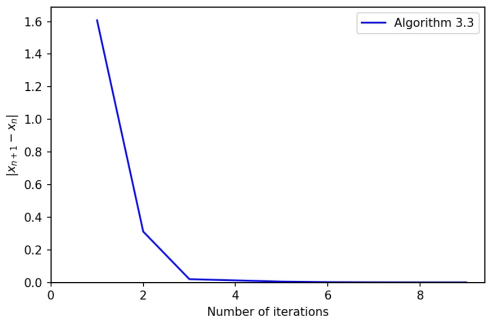
It is easy to verify that T and S are quasi-ϕ-nonexpansive and, $(I - T)$ and $(I - S)$ are demiclosed at zero. Furthermore, since $0 \in \Omega$, $\Omega \neq \emptyset$. In algorithm 3.3 $\gamma = 0.01$, $\alpha_n = \frac{1}{(n+1)^2}$ as our parameters. Clearly, these parameters satisfy the hypothesis of Theorems . Using a tolerance 10^{-8} and setting maximum number of iterations $n = 10$, we obtain the following iterates:

TABLE 3. Numerical results of Example 5.2

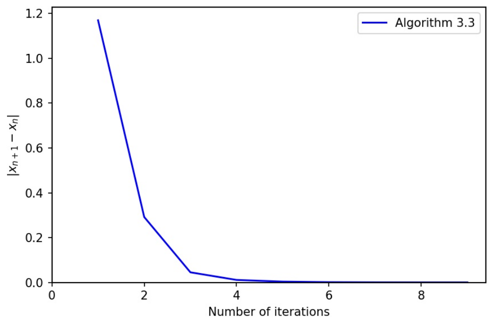
Table of values choosing $x_1(t) = e^t$ and $y_1(t) = \sin t$		
	Algorithm 3.3	
n	$\lvert x_{n+1} - x_n \rvert$	$\lVert y_{n+1} - y_n \rVert$
1	1.608	0.9811
2	0.3116	0.3132
3	0.0196	0.2127
4	0.0123	0.0805
5	4.78E-3	0.0256
6	1.46E-3	7.78E-3
7	4.22E-4	2.33E-3
8	1.21E-4	7.03E-3
9	3.55E-5	2.12E-4

TABLE 4. Numerical results of Example 5.2

Table of values choosing $x_1(t) = t + \cos t$ and $y_1(t) = 2t$		
	Algorithm 3.3	
n	$\|x_{n+1} - x_n\|$	$\|\|y_{n+1} - y_n\|\|$
1	1.1701	1.3055
2	0.2916	0.4081
3	0.0449	0.1607
4	0.019	0.0562
5	3.42E-3	0.0176
6	1.01E-3	5.33E-3
7	2.89E-4	1.59E-3
8	8.33E-4	4.81E-4
9	2.42E-5	1.45E-4

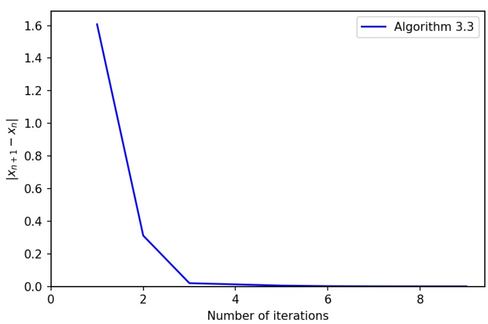


(A) Graph of the first 9 iterates of Algorithm 3.3 choosing $x_1 = e^t$

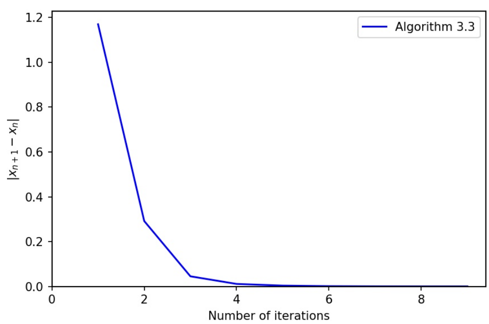


(B) Graph of the first 9 iterates of Algorithm 3.3 choosing $x_1 = t + \cos t$

FIGURE 3. Graph of the iterates of $\{x_n\}$ generated by Algorithm 3.3



(A) Graph of the first 9 iterates of Algorithm 3.3 choosing $y_1 = \sin t$



(B) Graph of the first 9 iterates of Algorithm 3.3 choosing $y_1 = 2t$

FIGURE 4. Graph of the iterates of $\{y_n\}$ generated by Algorithm 3.3

Observations. From the numerical illustrations presented in Examples 5.1 and 5.2 (see Tables 1, 2, 3 and 4 and Figures 1, 2, 3 and 4) we observe that the sequence $\{x_n\}$ approaches the solution faster than $\{y_n\}$ and the performance of our proposed algorithm is relatively the same as we vary the starting points.

6. Conclusion

This paper presents a *new* iterative algorithm for solving SEFPP for quasi-ϕ-nonexpansive mappings in real Banach spaces that will include *all* l_p, for $1 < p < \infty$. Weak convergence of the sequence generated by the algorithm is proved and strong convergence is proved under the assumption that the operators are semi-compact. Furthermore, applications of the theorem to split equality problem and split variational inclusion problem are also presented. Finally, numerical implementation of the algorithm is presented.

Declarations

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Conflict of Interest

The authors declare that they have no conflict of interest.

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