Thai Journal of Mathematics
Volume 19 Number 4 (2021)
Pages 1685–1697
http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

Prime-Graceful Graphs

Sayan Panma and Penying Rochanakul*

Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand
e-mail : panmayan@yahoo.com (S. Panma); Penying.Rochanakul@cmu.ac.th (P. Rochanakul)

Abstract A graph G with n vertices and m edges, is said to be prime-graceful, if there is an injection $\psi : V(G) \to \{1, 2, ..., m+1\}$, where $\gcd(\psi(u), \psi(v)) = 1$ for all $e = \{u, v\} \in E(G)$ and the induced function $\psi^* : E(G) \to \{1, 2, ..., m\}$ defined as $\psi^*(e) = |\psi(u) - \psi(v)|$ is injective. In this paper, we introduce prime-graceful labeling and show that star $K_{1,n}$, bistar $B_{n,n}$, bistar $B_{n,p-2}$, where p is an odd prime, complete bipartite graph $K_{2,n}$, tristar $SL(3, n)$, triangular book graph $B_n^{(3)}$ and some spiders are prime-graceful, while path P_n, cycle C_n and complete graph K_n are not prime-graceful in general. We also extend the idea to k-prime-graceful labeling where the range of ψ is extended to $k \min\{n, m\}$ for $k > 1$. Next, we define the prime-graceful number to be the minimum k such that G is k-prime-graceful. Finally, we investigate the prime-graceful number of the underlying graphs.

MSC: 05C78; 05C30; 05C90
Keywords: graph labeling; prime-graceful labeling; k-prime-graceful; prime-graceful number

Submission date: 06.06.2020 / Acceptance date: 12.01.2021

1. Introduction

Graph labeling is one of the active fields in graph theory. There are numerous applications of graph labeling, including graph decomposition, coding for radar and missile guidance, X-ray crystallographic analysis, designing communications networks addressing, determining the optimal circuit layout, etc. More applications and details can be found in References [1–3].

A labeling of a graph is an assignment of labels to vertices and/or edges of the graph. The concept of graph labeling was first introduced by Rosa [4] in 1967. Since then, hundreds of graph labelings have been studied. Gallian has made a thorough survey on those labelings and gather them in a dynamic survey of graph labeling [5]. The term graceful labeling was first mentioned by Golomb [6] in 1972. While the term prime labeling was introduced by Tout, Dabboucy, and Howalla [7] in 1982.

In this paper, we introduce the prime-graceful labeling which is a combination of graceful labeling and prime labeling. The definitions of graceful labeling and prime labeling are varied. Here are the most commonly used versions.

*Corresponding author.

Definition 1.1. A *graceful labeling* of a graph $G = (V, E)$ with n vertices and m edges is a one-to-one mapping ψ of the vertex set $V(G)$ into the set $\{1, 2, \ldots, m + 1\}$ with the following property: If we define, for any edge $e = \{u, v\} \in E(G)$, the value $\psi^*(e) = |\psi(u)-\psi(v)|$ then $\psi^*(e)$ is a one-to-one mapping of the set E onto the set $\{1, 2, \ldots, m+1\}$. A graph is called *graceful* if it has a graceful labeling.

Definition 1.2. A *prime labeling* of a graph $G = (V, E)$ with n vertices and m edges is a one-to-one mapping ψ of the vertex set $V(G)$ into the set $\{1, 2, \ldots, n\}$ with the following property: for any edge $e = \{u, v\} \in E(G)$, the value $\gcd(\psi(u), \psi(v)) = 1$. A graph is called *prime* if it has a prime labeling.

We merge the constraints of both labelings and define prime-graceful graphs as follows.

Definition 1.3. A *prime-graceful labeling* of a graph $G = (V, E)$ with n vertices and m edges is a one-to-one mapping ψ of the vertex set $V(G)$ into the set $\{1, 2, \ldots, m+1\}$ with the following property: for any edge $e = \{u, v\} \in E(G)$,

(1) the value $\gcd(\psi(u), \psi(v)) = 1$,
(2) the induced function $\psi^* : E(G) \to \{1, 2, ..., m\}$, defined as $\psi^*(e) = |\psi(u) - \psi(v)|$, is injective.

A graph is called *prime-graceful* if it has a prime-graceful labeling.

Prior to our work, the weaker versions of prime-graceful labeling were studied as *prime graceful labeling* [8] and *3-prime graceful labeling* [9]. The range of ψ was extended from $n + 1$ to $\min\{2n, 2m\}$ and $\min\{3n, 3m\}$, respectively. This can be generalized as follows:

Definition 1.4. For $k \geq 2$, a *k-prime-graceful labeling* of a graph $G = (V, E)$ with n vertices and m edges is an injective function $\psi : V(G) \to \{1, 2, \ldots, \min\{kn, km\}\}$ with the following property: for any edge $e = \{u, v\} \in E(G)$,

(1) the value $\gcd(\psi(u), \psi(v)) = 1$,
(2) the induced function $\psi^* : E(G) \to \{1, 2, \ldots, k \min\{n, m\} + 1\}$, defined as $\psi^*(e) = |\psi(u) - \psi(v)|$, is injective.

A graph is called *k-prime-graceful* if it has a k-prime-graceful labeling.

Selvarajan and Subramoniam [8], have proved that path P_n, cycle C_n, star $K_{1,n}$, friendship graph F_n, bistar $B_{n,n}$, $C_4 \cup P_n$, $K_{m,2}$ and $K_{m,2} \cup P_n$ are 2-prime-graceful. Later, Pavithra and Mary [9] have proved that fan graph F_n, wheel graph W_n, helm graph H_n, gear graph G_n, flower graph Fl_n, sunflower graph SF_n, closed helm graph CH_n, and web graph Wb_n are 3-prime-graceful.

It would be interesting to find the smallest number k of each graph. Here we give the notion of the prime-graceful number.

Definition 1.5. The prime-graceful number of G, denoted $\Xi(G)$, is the minimum k such that G is k-prime-graceful. The prime-graceful number of a prime-graceful graph is defined to be 1.

Thus, we can conclude from prior works that the prime-graceful numbers of path P_n, cycle C_n, star $K_{1,n}$, friendship graph F_n, bistar $B_{n,n}$, $C_4 \cup P_n$, $K_{m,2}$ and $K_{m,2} \cup P_n$ are at most 2. While the prime-graceful numbers of fan graph F_n, wheel graph W_n, helm graph H_n, gear graph G_n, flower graph Fl_n, sunflower graph SF_n, closed helm graph CH_n, and web graph Wb_n are at most 3.

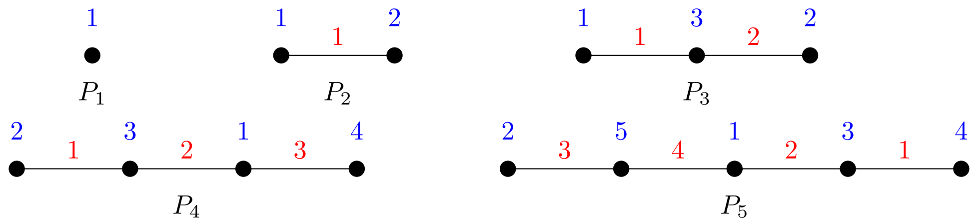
In the following section, star $K_{1,n}$, bistar $B_{n,n}$, bistar $B_{n,p-2}$, where p is an odd prime, complete bipartite graph $K_{2,n}$, tristar $SL(3,n)$, triangular book graph $B_n^{(3)}$ and some spiders are shown to be prime-graceful, while path P_n, cycle C_n and complete graph K_n are not prime-graceful in general.

2. Results

In this section, we show that some graphs are prime-graceful and give the prime-graceful numbers of some graphs.

Theorem 2.1. *For $n \in \mathbb{N}$, P_n is prime-graceful if and only if $n \leq 5$.*

Proof. It can be seen from the following diagram that P_1, P_2, P_3, P_4 and P_5 are prime-graceful.



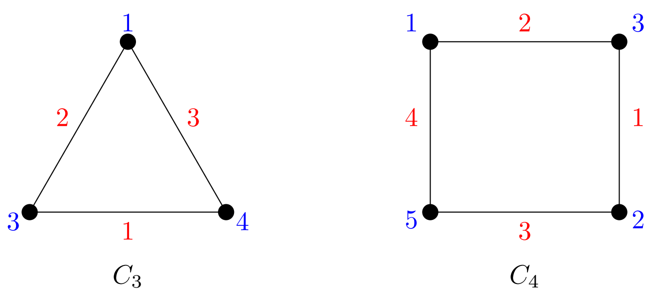
Consider graph P_n where $n > 5$. For any two adjacent vertices v and u, if both are labeled with even numbers, then $2 \leq \gcd(\psi(u), \psi(v))$. Hence, we need to avoid labeling two adjacent vertices with even numbers. To alternate between odd and even labels, the longest consecutive odd labeled vertices possible is 3. So, there are at most 2 edges with even label. This implies $|E| \leq 5$. Hence P_n is not prime-graceful when $n > 5$. ■

Theorem 2.2 (Theorem 1.10, [8]). *The path P_n is 2-prime-graceful.*

Corollary 2.3. *For any path P_n, $\Xi(P_n) = \begin{cases} 1, & n \leq 5 \\ 2, & n \geq 6 \end{cases}$.*

Theorem 2.4. *For $n \in \mathbb{N}$, C_n is prime-graceful if and only if $n \leq 4$.*

Proof. It can be seen from the following diagram that C_3 and C_4 are prime-graceful.



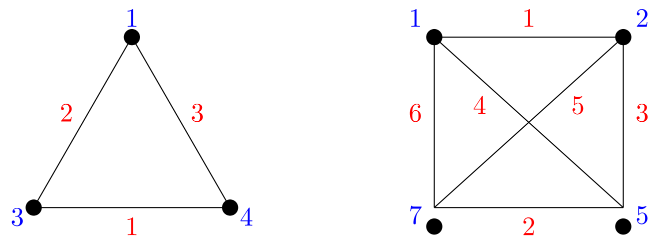
Assume $n \geq 5$. We have $|V(C_n)| = |E(C_n)| = n$. Similar to path graph, to avoid labeling two adjacent vertices with even numbers, we need to alternate between odd and even labels. Thus, at least $n - 1 > \lceil \frac{n}{2} \rceil$ edges labels are odd. Hence, $\psi^*(E) \neq \{1, 2, \ldots, |E|\}$. Therefore, C_n is prime-graceful if and only if $n \leq 4$. ■

Theorem 2.5 (Theorem 1.14, [8]). *The cycle C_n is 2-prime-graceful.*

Corollary 2.6. *For any cycle C_n, $\Xi(C_n) = \begin{cases} 1, & n \leq 4 \\ 2, & n \geq 5 \end{cases}$.*

Theorem 2.7. *For $n \in \mathbb{N}$, the complete graph K_n is prime-graceful if and only if $n \leq 4$.*

Proof. It can be seen from the following diagram that K_3 and K_4 are prime-graceful.



K_3 K_4

Assume $n \geq 5$. There are $\binom{n}{2}$ edges, and half of that must labeled even. Since all vetices are adjacent to each other, there can be at most 1 vertex with even label. Therefore, other $n-1$ vertices with odd labels can produce $\binom{n-1}{2}$ even edges label. Since $\binom{n-1}{2} > \frac{\binom{n}{2}}{2}$ for $n \geq 5$. K_n is prime-graceful if and only if $n \leq 4$. ∎

Corollary 2.8. *For any complete graph K_n, $\Xi(K_n) = 1$ if and only if $n \leq 4$.*

Corollary 2.9. *For any complete graph K_n with $n \geq 5$, $\Xi(K_n) \geq 2$.*

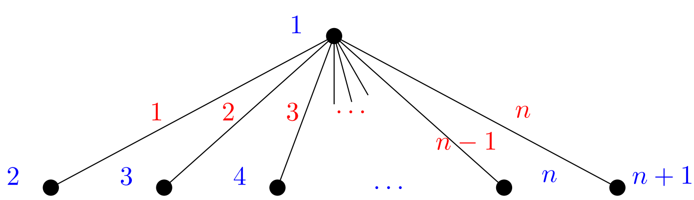
Theorem 2.10. *Stars $K_{1,n}$ are prime-graceful.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of star graph $K_{1,n}$ with v be the vertex with the largest degree of $K_{1,n}$, aka apex vertex. Define a function $\psi : V(K_{1,n}) \to \{1, 2, \ldots, n+1\}$ by

$$\psi(v) = 1,$$
$$\psi(v_i) = i + 1 \quad \text{if } 1 \leq i \leq n.$$

Then the edge label $\psi^*(\{v, v_i\}) = i$ for all $1 \leq i \leq n$. Thus, the edge label are all distinct. Also, $\gcd(\psi(v), \psi(v_i)) = 1$ for all $1 \leq i \leq n$. Therefore, $K_{1,n}$ is prime-graceful. ∎

Example 2.11. A prime-graceful labeling of $K_{1,n}$.



Corollary 2.12. *For any star $K_{1,n}$, $\Xi(K_{1,n}) = 1$.*

Definition 2.13. *Bistar* $B_{n,m}$ is the graph with $n+m+2$ vertices and $n+m+1$ edges obtained by joining the apex vertices of star $K_{1,n}$ and $K_{1,m}$.

Theorem 2.14. *For $n \in \mathbb{N}$, the bistar graph $B_{n,n}$ is prime-graceful.*

Proof. Let $u_1, u_2, \ldots, u_n$ be the vertices of star graph $K_{1,n}$ with u be the apex vertex, and let $v_1, v_2, \ldots, v_n$ be the vertices of star graph $K_{1,n}$ with v be the apex vertex.

By Bertand's Postulate, there exists a prime number p, where $n+1 < p < 2(n+1)-2$.

Case n is odd. Define a function $\psi : V(B_{n,n}) \to \{1, 2, \ldots, 2n+2\}$ by

$$
\begin{array}{ll}
\psi(u) = 1, & \psi(u) = p, \\
\psi(u_i) = i+1 & \text{if } 1 \le i \le \frac{p-n-2}{2}, \\
\psi(u_i) = i+n+1 & \text{if } \frac{p-n}{2} \le i \le p-n-2, \\
\psi(u_i) = i+n+2 & \text{if } p-n-1 \le i \le n, \\
\psi(v_i) = i + \frac{p-n}{2} & \text{if } 1 \le i \le n.
\end{array}
$$

Then, the resulting edge labels are

$$
\begin{aligned}
&\psi^*(uv) = p-1, \\
&\{\psi^*(uu_i) : \ 1 \le i \le n\} = \{1, 2, \ldots, \frac{p-n-2}{2}\} \cup \{\frac{p+n+2}{2}, \frac{p+n+4}{2}, \ldots, p-2\} \\
&\qquad\qquad \cup \{p, p+1, \ldots, 2n+1\}, \\
&\{\psi^*(vv_i) : \ 1 \le i \le n\} = \{\frac{p-n}{2}, \frac{p-n+2}{2}, \ldots, \frac{p+n}{2}\}.
\end{aligned}
$$

Thus, edge labels are all distinct. Moreover, we have

$$gcd(1,p) = gcd(1, \psi(u_i)) = gcd(p, \psi(v_i)) = 1$$

for all $1 \le i \le n$.

Case n is even. Define a function $\psi : V(B_{n,n}) \to \{1, 2, \ldots, 2n+2\}$ by

$$
\begin{array}{ll}
\psi(u) = 1, & \psi(v) = p, \\
\psi(u_i) = 2n+3-i & \text{if } 1 \le i \le 2n+2-p, \\
\psi(u_i) = \frac{3n+2}{2} - i & \text{if } 2n+3+p \le i \le n, \\
\psi(v_i) = i+1 & \text{if } 1 \le i \le n/2, \\
\psi(v_i) = i+p-n-1 & \text{if } (n/2)+1 \le i \le n.
\end{array}
$$

Then, the resulting edge labels are

$$
\begin{aligned}
&\psi^*(uv) = p-1, \\
&\{\psi^*(uu_i) : \ 1 \le i \le n\} = \{p, p+1, \ldots, 2n+1\} \cup \{\frac{n+2}{2}, \frac{n+4}{2}, \ldots, \frac{2p+n-4}{2}\}, \\
&\{\psi^*(vv_i) : \ 1 \le i \le n\} = \{\frac{2p-n-2}{2}, \frac{2p-n}{2}, \ldots, p-2\} \cup \{1, 2, \ldots, \frac{n}{2}\}.
\end{aligned}
$$

Thus, edge labels are all distinct. Moreover, we have

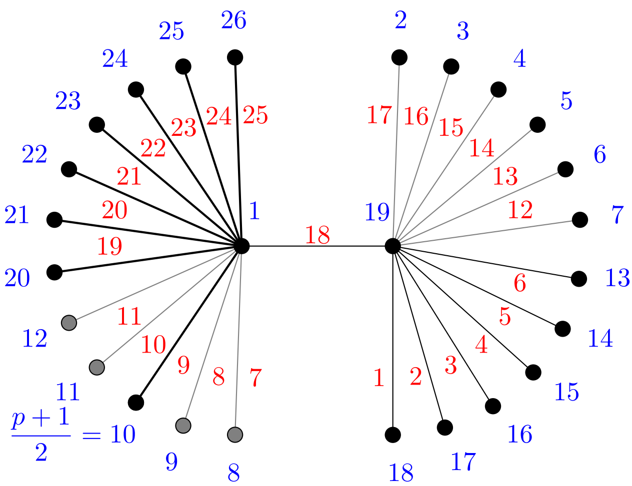
$$gcd(1,p) = gcd(1, \psi(u_i)) = gcd(p, \psi(v_i)) = 1$$

for all $1 \leq i \leq n$.

In both cases, the gcd of adjacent vertices are all 1 and edge labels are all distinct. Thus, the bistar graph $B_{n,n}$ is prime-graceful. ∎

Example 2.15. A prime-graceful labeling of $B_{12,12}$.

Choose $p = 19$.



Theorem 2.16. *For $n \in \mathbb{N}$ and a prime $p > 2$, the bistar graph $B_{n,p-2}$ is prime-graceful.*

Proof. Let $u_1, u_2, \ldots, u_n$ be the vertices of star graph $K_{1,n}$ with u be the apex vertex, and let $v_1, v_2, \ldots, v_{p-2}$ be the vertices of star graph $K_{1,p-2}$ with v be the apex vertex.

Define a function $\psi : V(B_{n,p-2}) \to \{1, 2, \ldots, n+p\}$ by

$$\psi(u) = 1, \qquad \psi(u) = p,$$
$$\psi(u_i) = i + p \qquad \text{if } 1 \leq i \leq n,$$
$$\psi(v_j) = j + 1 \qquad \text{if } 1 \leq j \leq p-2.$$

Then, the resulting edge labels are

$$\psi^*(uv) = p - 1,$$
$$\{\psi^*(uu_i) : \ 1 \leq i \leq n\} = \{p, p+1, \ldots, p+n-1\},$$
$$\{\psi^*(vv_j) : \ 1 \leq j \leq p-2\} = \{1, 2, \ldots, p-2\}.$$

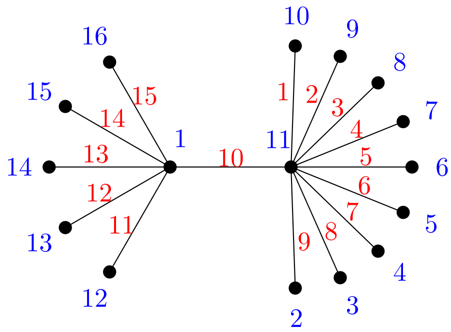
Thus, edge labels are all distinct. Moreover, we have

$$gcd(1, p) = gcd(1, \psi(u_i)) = gcd(p, \psi(v_j)) = 1$$

for all $1 \leq i \leq n$, $1 \leq j \leq p-2$.

Thus, if p is an odd prime, then the bistar graph $B_{n,p-2}$ is prime-graceful. ∎

Example 2.17. A prime-graceful labeling of $B_{5,9}$.



Corollary 2.18. *For any bistar $K_{n,m}$, $\Xi(K_{1,n}) = 1$ if $m = n$ or $m + 2$ is an odd prime.*

Definition 2.19. An $SF(n, m)$ is a graph consisting of a cycle C_n, where $n \geq 3$ and n set of m independent vertices where each set joined to a different vertex of C_n.

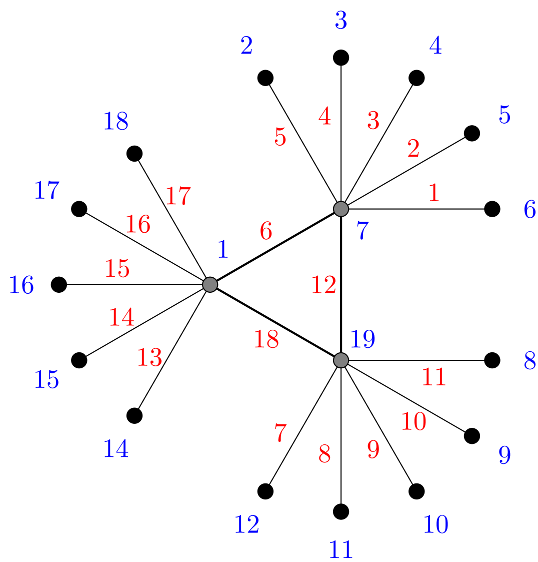
An $SF(3, m)$ is called the *tristar*.

Theorem 2.20. *For $n \in \mathbb{N}$, if $n + 2$ and $3n + 4$ are prime, the tristar graph $SF(3, n)$ is prime-graceful.*

Proof. Label the vertices on C_3 by $1, n + 2$ and $3n + 4$, respectively. Label the vertices that are adjacent with vertex label 1 by $2n + 4, 2n + 5, \ldots, 3n + 3$. Label the vertices that are adjacent with vertex label $n + 2$ by $2, 3, \ldots, n + 1$. Label the vertices that are adjacent with vertex label $3n + 4$ by $n + 3, n + 4, \ldots, 2n + 2$. The gcd of adjacent vertices are all 1 and edge labels are all distinct.

Thus, if $n + 2$ and $3n + 4$ are prime, the tristar graph $SF(3, n)$ is prime-graceful. ■

Example 2.21. A prime-graceful labeling of $SF(3, 5)$.



Theorem 2.22. *For $n \in \mathbb{N}$ where $n > 2$, the spider of n legs where all of its legs have lengths two is prime-graceful if and only if $2n+1$ or $2n+3$ is prime.*

Proof. Let v_i, u_i be the label of vertices on the ith leg, where u_i are leaf vertices, and v the label of the central vertex.

($\Leftarrow$) Assume $2n+1$ or $2n+3$ is prime.
Case $2n+1$ is prime. Define $\psi : \{v, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\} \to \{1, 2, 3, \ldots, 2n+1\}$ by

$$\begin{aligned} \psi(v) &= 2n+1, \\ \psi(v_i) &= 2i-1 && \text{if } 1 \le i \le n, \\ \psi(u_i) &= 2n+2-2i && \text{if } 1 \le i \le n. \end{aligned}$$

Then, the resulting edge labels are

$$\begin{aligned} \{\psi^*(vv_i) :\ 1 \le i \le n\} &= \{2, 4, \ldots, 2n\}, \\ \{\psi^*(v_iu_i) :\ 1 \le i \le n\} &= \{1, 3, \ldots, 2n-1\}. \end{aligned}$$

Therefore, all edge labels are distinct. Since $2n+1$ is prime, $\gcd(2n+1, 2i-1) = 1$ and $\gcd(2n+2-2i, 2i-1) = gcd(2n+1, 2i-1) = 1$. Thus, if $2n+1$ is prime, spiders of n legs with all legs have length two are prime-graceful.
Case $2n+3$ is prime. Define $\psi : \{v, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\} \to \{1, 2, 3, \ldots, 2n+1\}$ by

$$\begin{aligned} \psi(v) &= 1, \\ \psi(v_i) &= 2i+1 && \text{if } 1 \le i \le n, \\ \psi(u_i) &= 2n+2-2i && \text{if } 1 \le i \le n. \end{aligned}$$

Then, the resulting edge labels are

$$\begin{aligned} \{\psi^*(vv_i) :\ 1 \le i \le n\} &= \{2, 4, \ldots, 2n\}, \\ \{\psi^*(v_iu_i) :\ 1 \le i \le n\} &= \{1, 3, \ldots, 2n-1\}. \end{aligned}$$

Therefore, all edge labels are distinct. Since $2n+3$ is prime, $\gcd(2i+1, 1) = 1$ and $\gcd(2n+2-2i, 2i+1) = gcd(2n+3, 2i+1) = 1$. Thus, if $2n+3$ is prime, spiders of n legs with all legs have length two are prime-graceful.

($\Rightarrow$) Assume a spider of n legs where all of its legs have lengths two is prime-graceful. Then there exists a prime-graceful labeling

$$\phi : \{v, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\} \to \{1, 2, 3, \ldots, 2n+1\}.$$

We first show that $\phi(v)$ must be odd. If $\phi(v)$ is even, then to be relatively prime $\phi(v_i)$ must be odd for all $1 \le i \le n$. Thus, all n edge labels $\phi^*(vv_i)$ are odd and the remaining n edge labels $\phi^*(v_iu_i)$ must be even. This implies $\phi(u_i)$ is odd for all $1 \le i \le n$. Now we have $2n$ vertices that must be labeled with odd number. However, there are only $n+1$ odd numbers in the set $\{1, 2, 3, \ldots, 2n+1\}$. Since $n > 2$, this is clearly impossible. Hence, $\phi(v)$ is odd.

Next, we show that $\phi(v_i)$ is odd and $\phi(u_i)$ is even for all $1 \le i \le n$. Note that there are 3 possible label patterns of $v - v_i - u_i$, namely; $odd - odd - odd$, $odd - odd - even$, and $odd - even - odd$. Assume there are k legs with label pattern $odd - odd - odd$. To obtain the same number of odd and even edge labels, there must also be k legs with label pattern $odd - even - odd$. Thus, there are $n - 2k$ legs with label pattern $odd - odd - even$.

This implies there are exactly $0 + k + (n - 2k) = n - k$ vertices on spider's legs that are labeled with even number. However, the numbers of vertices of a spider graph is equal to the size of its label set. That means all labels from the set $\{1, 2, 3, \ldots, 2n+1\}$ must be used. Since there are n even number in the set, we have $n - k = n$. This makes $k = 0$ and implies all legs has label pattern $odd - odd - even$, i.e., $\phi(v_i)$ is odd and $\phi(u_i)$ is even for all $1 \leq i \leq n$.

Since the edge with label $2n$ must be incident to two vertices with odd labels 1 and $2n - 1$, then $\phi(v) \in \{1, 2n+1\}$.

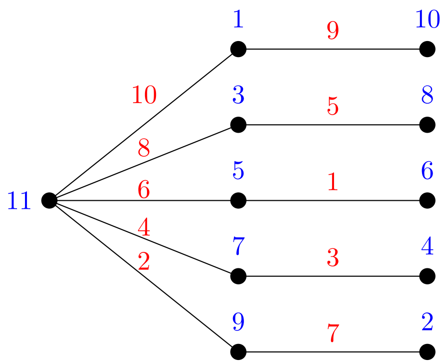
Case $\phi(v) = 2n + 1$**.** As $\phi(v_i)$ is odd for all $1 \leq i \leq n$ and all distinct, without lost of generality, we can define $\phi(v_i) = 2i - 1$. Since ϕ is a prime-graceful labeling, we have $gcd(\phi(v), \phi(v_i)) = gcd(2n + 1, 2i - 1) = 1$ for all $1 \leq i \leq n$. Thus, $2n + 1$ is prime.

Case $\phi(v) = 1$**.** As $\phi(v_i)$ is odd for all $1 \leq i \leq n$ and all distinct, without lost of generality, we can define $\phi(v_i) = 2i + 1$. Here we have $gcd(\phi(v), \phi(v_i)) = gcd(1, 2i + 1) = 1$ for all $1 \leq i \leq n$ and edge labels obtained from $vv_1, vv_2, \ldots, vv_n$ are $2, 4, 6, \ldots, 2n$.

Here, the edge with label $2n - 1$ must be incident to vertices with labels $2n$ and 1, aka v_1. So, we have $\phi(u_1) = 2n$. Now, the edge with label $2n - 3$ must be incident to vertices with labels 2 and $2n-1$, aka v_n. So, we have $\phi(u_n) = 2$. Similarly, we have $\phi(u_2) = 2n-2$, $\phi(u_{n-1}) = 4$, and so on. To put it simply, we have $\phi(u_i) = 2n + 2 - 2i$ for all $1 \leq i \leq n$. Since ϕ is a prime-graceful labeling, we have $gcd(\phi(v_i), \phi(u_i)) = gcd(2i+1, 2n+2-2i) = gcd(2i + 1, 2n + 3) = 1$ for all $1 \leq i \leq n$. Thus, $2n + 3$ is prime.

Therefore, the spider of n legs where all of its legs have lengths two is prime-graceful if and only if $2n + 1$ or $2n + 3$ is prime. ■

Example 2.23. A prime-graceful labeling of spider graph with 5 legs of length 2.



Theorem 2.24. *For $n \in \mathbb{N}$, the complete bipartite graph $K_{2,n}$ is prime-graceful.*

Proof. Let u and v be the two vertices of degree n, and $v_1, v_2, \ldots, v_n$ be other vertices of degree 2.

Define $\psi : \{u, v, v_1, v_2, \ldots, v_n\} \to \{1, 2, 3, \ldots, 2n+1\}$ by

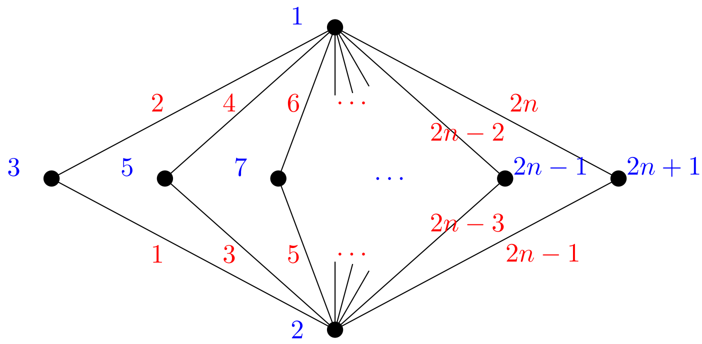
$$\psi(u) = 1, \qquad \psi(v) = 2,$$
$$\psi(v_i) = 2i + 1 \qquad \text{if } 1 \leq i \leq n.$$

Then, the resulting edge labels are

$$\{\psi^*(uv_i) : \ 1 \leq i \leq n\} = \{2, 4, \ldots, 2n\},$$
$$\{\psi^*(vv_i) : \ 1 \leq i \leq n\} = \{1, 3, \ldots, 2n-1\}.$$

Edge labels are all distinct. Moreover, $\gcd(2i+1,1) = 1$ and $\gcd(2i+1,2) = 1$ for all $1 \le i \le n$. Thus, the complete bipartite graph $K_{2,n}$ is prime-graceful. ∎

Example 2.25. A prime-graceful labeling of $K_{2,n}$.



Corollary 2.26. *For any complete bipartite graph $K_{2,n}$, $\Xi(K_{2,n}) = 1$.*

Definition 2.27. The *triangular book* graph $B_n^{(3)}$ is the graph with $n+2$ vertices $u, v, v_1, v_2, \ldots, v_n$ and $2n+1$ edges constructed by n triangles sharing a common edge uv. In other words, the triangular book graph $B_n^{(3)}$ is the complete tripartite graph $K_{1,1,n}$

Theorem 2.28. *For $n > 1$, the triangular book graph $B_n^{(3)}$ is prime-graceful if and only if $n+2$ is an odd prime or $n+1$ is a power of two.*

Proof. Let u and v be the two vertices of degree n, and $v_1, v_2, \ldots, v_n$ be other vertices of degree 2.

($\Leftarrow$) **Case $n+2$ is an odd prime.** Then n is odd. Define $\psi : \{u, v, v_1, v_2, \ldots, v_n\} \to \{1, 2, 3, \ldots, 2n+1\}$ by

$$\begin{aligned} \psi(u) &= 1, \\ \psi(v) &= n+2, \\ \psi(v_i) &= n+2+i \qquad \text{if } 1 \le i \le n. \end{aligned}$$

Then, the resulting edge labels are

$$\begin{aligned} \psi^*(uv) &= n+1, \\ \{\psi^*(uv_i) :\ 1 \le i \le n\} &= \{n+2, n+3, \ldots, 2n+2\}, \\ \{\psi^*(vv_i) :\ 1 \le i \le n\} &= \{1, 2, \ldots, n\}. \end{aligned}$$

Here, edge labels are all distinct. Since $n+2$ is prime, $\gcd(n+2+i,1) = 1$ and $\gcd(n+2, n+2+i) = gcd(n+2, i) = 1$ for all $1 \le i \le n$. Thus, if $n+2$ is odd prime, then the triangular book graph $B_n^{(3)}$ is prime-graceful.

Case $n+1$ is a power of two. Define $\psi : \{u, v, v_1, v_2, \ldots, v_n\} \to \{1, 2, 3, \ldots, 2n+2\}$ by

$$\begin{aligned} \psi(u) &= 1, \\ \psi(v) &= 2n+2, \\ \psi(v_i) &= 2i+1 \qquad \text{if } 1 \le i \le n. \end{aligned}$$

Then, the resulting edge labels are

$$\begin{aligned}\psi^*(uv) &= 2n+1,\\ \{\psi^*(uv_i):\ 1\le i\le n\} &= \{2,4,\dots,2n\},\\ \{\psi^*(vv_i):\ 1\le i\le n\} &= \{1,3,\dots,2n-1\}.\end{aligned}$$

Here, edge labels are all distinct. Since $n+1$ is a power of two, $n+1=2^k$ for some positive integer k. We the have $\gcd(1,2i+1)=1$ and $\gcd(2n+2,2i+1)=gcd(2^{k+1},2i+1)=1$ for all $1\le i\le n$. Thus, if $n+1$ is a power of two, then the triangular book graph $B_n^{(3)}$ is prime-graceful.

($\Rightarrow$) Assume a triangular book graph $B_n^{(3)}$ is prime-graceful. Then there exists a prime-graceful labeling $\phi:\{u,v,v_1,v_2,\dots,v_n\}\to\{1,2,3,\dots,2n+2\}$.

Because u and v are adjacent, $\phi(u)$ and $\phi(v)$ cannot be even at the same time. Hence, without lost of generality, we can assume $\phi(v)$ is odd.

Case $\phi(u)$ is even. The labels $\phi(v)$, $\phi(v_1)$, $\phi(v_2)$, $\dots$, $\phi(v_n)$ are all distinct odd labels from the set $\{1,3,5,\dots,2n+1\}$. Since the edge with label $2n+1$ only appear from incident vertices with labels 1 and $2n+2$, we have $\phi(u)=2n+2$. If there exists an odd prime p such that $p|(n+1)$, then $1<p\le n$ and there exists a vertex $x\in\{v,v_1,v_2,\dots,v_n\}$ that $\phi(x)=p$. Hence $gcd(\phi(u),\phi(x))\ge p>1$, which opposed to the property prime-graceful label. Therefore, there is no such p. That means $n+1$ is a power of two.

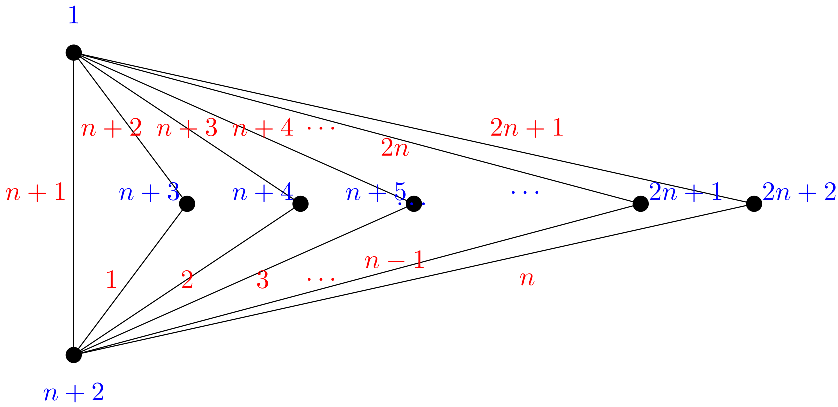
Case $\phi(u)$ is odd. Since both $\phi(u)$ and $\phi(v)$ are odd and the edge with label $2n+1$ only appear from incident vertices with labels 1 and $2n+2$, we have $\phi(u)=1$ or $\phi(v)=1$. Let's say $\phi(u)=1$. Assume $\phi(v)=a$, where a is odd and $3\le a\le 2n+1$. Then, to obtain edge with label a without violating the prime labeling property, there exists $j\in\{1,2,\dots,n\}$ where $\phi(v_j)=a+1$. Here, $\phi^*(vv_i)=1$. That eliminate 2 and $a-1$ from $\{\phi(v_i):\ 1\le i\le n\}$. Thus, to obtain edge with label $a-2$ without violating the graceful labeling property, there exists $\ell\in\{1,2,\dots,n\}$ where $\phi(v_\ell)=2a-2$. Then $2a-2\le 2n+2$. This implies $a\le n+2$. Then there exists a non-negative integer t where $a+t=n+2$.

Next, we show that $a=n+2$. Assume $t\ge 1$. Then $max\{\phi^*(vv_i):\ 1\le i\le n\}=max\{\phi(v)-3,2n+2-\phi(v)\}=max\{n-t-1,n+t\}=n+t$. That implies edge labels $n+t+1,n+t+2,\dots,2n+1$ can only be obtained from $\phi^*(uv_i)$. So, we can define $\phi(v_i)=n+2+i$ for $i=t,t+1,\dots,n$ and leave $\phi(v_i)\le a-1=n+1-t$ for all $1\le i\le t-1$. Then, the current resulting edge labels are

$$\begin{aligned}\{\psi^*(uv)\} &= \{a-1=n+1-t\},\\ \{\psi^*(uv_i):\ t\le i\le n\} &= \{n+t+1,n+t+2,\dots,2n+1\},\\ \{\psi^*(vv_i):\ t\le i\le n\} &= \{2t,2t+1,\dots,n+t\}.\end{aligned}$$

The sum of the remaining edge labels is $1+2+\cdots+2t-1=t(2t-1)$. Also, $\psi^*(uv)+\sum_{i=1}^{t-1}\psi^*(uv_i)+\sum_{i=1}^{t-1}\psi^*(vv_i)=t(a-1)$. So $t(2t-1)=t(a-1)$. That means $a=2t$, contradicts its parity. Therefore, $t=0$. Thus, $\phi(v)=n+2$ and $\{\phi(v_i):\ 1\le i\le n\}=\{n+3,n+4,\dots,2n+2\}$. Which implies $n+2$ must be prime. ■

Example 2.29. A prime-graceful labeling of $B_n^{(3)}$ where $n+2$ is prime.



Corollary 2.30. *For any triangular book graph $B_n^{(3)}$ with $n > 1$, $\Xi(B_n^{(3)}) = 1$ if and only if $n+2$ is an odd prime or $n+1$ is a power of two.*

3. Conclusions

We have introduced new vertex labelings, namely prime-graceful and k-prime-graceful, and proposed the notion of the prime-graceful number. We proved the existence of prime-graceful labelings for star $K_{1,n}$, bistar $B_{n,n}$, bistar $B_{n,m}$, where $m+2$ is an odd prime, complete bipartite graph $K_{2,n}$, tristar $SL(3,n)$, where $n+2$ and $3n+4$ are prime, triangular book graph $B_n^{(3)}$, where $n+2$ is an odd prime or $n+1$ is a power of two, and spider with n legs of length 2, where $2n+1$ or $2n+3$ is prime. Thus, the prime-graceful numbers of these graphs are 1.

The prime-graceful numbers of path P_n and cycle C_n are 2, when $n > 5$ and $n > 4$, respectively. While the prime-graceful number of complete graph K_n are not yet determined.

A possible direction of future research is to investigate the prime-graceful number of other graphs.

Acknowledgements

This work is supported by the Research Fund for DPST Graduate with First Placement [Grant no. 038/2015], The Institute for the Promotion of Teaching Science and Technology (IPST), Thailand.

References

[1] B.D. Acharya, S. Arumugam, A. Rosa, Labelings of Discrete Structures and Applications, Narosa, 2007.

[2] G.S. Bloom, S.W. Golomb, Applications of numbered undirected graphs, in Proceedings of the IEEE 65 (4) (1977) 562–570.

[3] G.S. Bloom, S.W. Golomb, Numbered complete graphs, unusual rulers, and assorted applications, Alavi Y., Lick D.R. (eds), Theory and Applications of Graphs, Lecture Notes in Mathematics 642, Springer, Berlin, Heidelberg, 1978.

[4] A. Rosa, On certain valuations of the vertices of a graph, Theory of Graphs (Internat. Symposium, Rome, July 1966), Gordon and Breach, N. Y. and Dunod Paris (1967), 349–355.

[5] J. Gallian, A dynamic survey of graph labeling, Electron. J. Combin. #DS6, http://www.combinatorics.org/ojs/index.php/eljc/article/view/DS6/pdf.

[6] S.W. Golomb, How to number a graph, in Graph Theory and Computing, R.C. Read, ed., Academic Press, New York (1972), 23–37.

[7] A. Tout, A.N. Dabboucy, K. Howalla, Prime labeling of graphs, Nat. Acad. Sci. Letters 11 (1982) 365–368.

[8] T.M. Selvarajan, R. Subramoniam, Prime graceful labeling, International Journal of Engineering & Technology 7 (4.36) (2018) 750–752.

[9] P. Pavithra, U. Mary, 3-Prime graceful labeling, Journal of Information and Computational Science 10 (1) (2020) 1381–1387.