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On Injective Envelopes of AF-Algebras

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Abstract In this paper, we prove that in the category of C^*-algebras and completely positive linear maps an injective AF-algebra must be finite dimensional. We also show that a separable essentially simple C^*-algebra whose injective envelope is a von Neumann algebra must be an AF-algebra. Further, we show that if the regular completion (or equivalently, the injective envelope) of an essentially simple AF-algebra is a W*-algebra, then the AF-algebra is isomorphic to a direct sum of elementary C^*-algebras.

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1. Introduction

The class of AF-algebras includes UHF-algebras studied in 60's by James Glimm [1], and that of the matroid C^*-algebras, which is stably isomorphic to UHF-algebras, studied around the same time by Dixmier [2] (see [3], [4] for more details).

Theory of injective envelopes has a long history in Functional Analysis. In 1964, Cohen showed that a unique injective envelope of a Banach space exists, and proved that an injective Banach space is linearly isometric to a function space $C(M)$, where M is a compact Hausdorff and extremely disconnected topological space, i.e., closure of every open subset in M is open). In 1978, Hamana proved that injective objects exist and are unique (up to isomorphism) in the category of Banach A-modules and continuous module homomorphisms, where A is an unital Banach algebra. He managed to do this by using seminorm admissible extensions of Banach A-modules and putting a partial order on the family of extensions.

In 1979, Hamana [5, Theorem 4.1], proved that any C^*-algebra has a unique injective envelope. Hamana used the Arveson extension theorem in [5]. In this setting, following Choi and Effros [6], he considered a completely positive linear map ϕ of C^*-algebra B into itself, and observed that $Im(\phi)$ with multiplication "$\circ$", $x \circ y = \phi(xy)$ for all $x, y \in Im(\phi)$, and involution and norm induced by those of B, is an unital C^*-algebra. The C^*-algebra $Im(\phi)$ is denoted by $C^*(\phi)$. Hamana proved that $C^*(\phi)$ is injective if B is injective in the category of C^*-algebras. Finally, if A is a C^*-algebra, there exists an injective C^*-algebra

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C containing A as a C^*-subalgebra, by Arveson extension theorem (which asserts that the algebra of bounded operators on a Hilbert space as a C^*-algebra is injective). By [5, Theorem 3.4], there exists a minimal A-projection ϕ on C. If $B = C^*(\phi)$ and κ is the canonical inclusion of A into B, then (B, κ) is an injective envelope of A.

The main purpose of the current paper is to make a detailed study of the injective envelopes of approximately finite-dimensional C^*-algebras (AF-algebras). These are the norm closure of an increasing sequence of finite-dimensional C^*-algebras. We do this via looking at essential ideals of AF-algebras, which has not been done by any other papers as we know.

In the second section, we set up terminologies and notations. In section 3, we discuss AF-algebras in more details and introduce the notion of a monotone complete C^*-algebra. Section 4 is devoted to the proof of the fact that no infinite-dimensional AF-algebra could be injective in the category of C^*-algebras. In section 5, we prove that a separable essentially simple C^*-algebra whose injective envelope is a von Neumann algebra must be an AF-algebra.

2. Terminologies and Notations

In this section, we recall standard definitions and results needed later in the text and discuss certain concrete examples. The main references for this section are [7], [3], and [8].

A C^*-algebra A is a W^*-algebra if and only if A, as a Banach space, is the dual space X^* of some (in fact, unique) Banach space X. It is a classical fact that a C^*-algebra A is said to be a W^*-algebra if A has a representation as a von Neumann algebra of operators acting on some Hilbert space. A C^*-algebra A is an AW^*-algebra if and only if the left annihilator of each right ideal in A has the form Ap, where $p \in A$ is a projection, or equivalently if every maximal abelian C^*-subalgebra $D \subseteq A$ is monotone complete [7]. Any W^*-algebra is an AW^*-algebra, but the converse is not true, i.e., there exists AW^*-algebras which fail have any faithful representation as a von Neumann algebra. A projection $p \in A$ is abelian if the AW^*-algebra pAp will be commutative algebra and an AW^*-algebra A is said to be of type I if every direct summand of A has an abelian projection.

Definition 2.1. [8] A C^*-algebra is an *approximately finite-dimensional algebra*, or shortly, an *AF-algebra*, if and only if it is an inductive limit of a countable direct sequence of C^*-algebras of finite-dimensional.

We recall that in [3], a C^*-algebra is an AF-algebra if A has a unit e, and there exists an increasing sequence $\{A_n\}_{n=1}^{\infty}$ of finite-dimensional subalgebras of A such that $\bigcup_{n=1}^{\infty} A_n$ is norm-dense in A. By [9, Example 6.2.4], this is clearly the same as the above definition.

For the separable C^*-algebra case, there also exists an equivalent local definition of AF-algebras (since finite-dimensional C^*-algebras are clearly separable, so are AF-algebras). If C^*-algebra is non-separable, AF-algebras are defined as the inductive limit of arbitrary inductive system of finite dimensional C^*-algebras, as done by Katsura in [10] (which again has a local characterization).

A collection $\{(A_n, \varphi_n)\}_{n=1}^{\infty}$ is a direct sequence of C^*-algebra, with each $\varphi_n : A_n \longrightarrow A_{n+1}$ $*$-homomorphism and A_n C^*-algebra. We often write

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \cdots$$

For instance, let $A_n = M_{2^n}$ be the $2^n \times 2^n$ complex matrices. Define $\varphi_n : M_{2^n} \longrightarrow M_{2^{n+1}}$ by sending a to

$$a \oplus a := \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

Then $M_2 \xrightarrow{\varphi_1} M_4 \xrightarrow{\varphi_2} M_8 \xrightarrow{\varphi_3} \cdots$ is a direct sequence of C^*-algebras.

A collection $\psi_n : A_n \longrightarrow B$ of $*$-homomorphism into a C^*-algebra B is called *compatible* (with the direct system) if

$$\begin{array}{ccc} A_n & \xrightarrow{\varphi_n} & A_{n+1} \\ & {\scriptstyle \psi_n} \searrow & \downarrow {\scriptstyle \psi_{n+1}} \\ & & B \end{array}$$

commutes for all n, where $\{(A_n, \varphi_n)\}_{n=1}^{\infty}$ a direct sequence of C^*-algebras. A C^*-algebra A with compatible $*$-homomorphism $\varphi^n : A_n \longrightarrow A$ is a direct limit for a direct sequence

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \cdots$$

if there is a unique $*$-homomorphism $\psi : A \longrightarrow B$ such that

$$\begin{array}{ccc} A & \xrightarrow{\psi} & B \\ {\scriptstyle \varphi^n} \uparrow & \nearrow {\scriptstyle \psi_n} & \\ A_n & & \end{array}$$

commutes for all n and given compatible $*$-homomorphism ψ_n into a C^*-algebra B.

Any direct sequence $\{(A_n, \varphi_n)\}_{n=1}^{\infty}$ has a direct limit $(A, \varphi^n) = \lim_{n\to\infty}(A_n, \varphi_n)$. A direct limit

$$A = \varinjlim (A_n, \varphi_n)$$

is an AF-algebra, where every A_n is finite-dimensional.

Example 2.2. [8] The C^*- algebra $C([0,1])$ is not an AF-algebra. Note that since $[0,1]$ is connected, the only projections on $C([0,1])$ are 0 and 1. Hence, $C([0,1])$ only has two finite-dimensional C^*-subalgebras, i.e., the subalgebras $\{0\}$ and $\mathbb{C}$. Therefore, $C([0,1])$ cannot be an AF-algebra.

Example 2.3. [8] The C^*- algebra $K(H)$ of compact operators on a separable infinite-dimensional Hilbert space H is an (infinite dimensional) AF-algebra. take orthonormal basis $\{e_k\}_{k=1}^{\infty}$ in H. Suppose that P_n is the projection onto the subspace spanned by $\{e_1, e_2, e_3, \cdots, e_n\}$, $A_n = P_n K(H) P_n$ and let φ_n be the inclusion map of A_n into A_{n+1}. Then since $\bigcup_{n\geq 1} A_n$ consists of the finite-rank operators on H, $K(H)$ is the direct limit of the A_n's.

Example 2.4. [3] $K(H)+\mathbb{C}I \subseteq B(H)$ is an AF-algebra. Let P_n be an increasing sequence of projections together with $\mathrm{rank}(P_n) = n$, which is converging strongly to the identity. Consider $A_n = \mathbb{C}P_n^{\perp} + P_n K(H) P_n \simeq \mathbb{C} \oplus M_n$. It is easy to check that $\bigcup_{n\geq 1} A_n$ is dense in $K(H) + \mathbb{C}I$. By definition of A_n, we see that $x \in A_n$ is the sum of a finite-rank operator and a multiple of the identity, therefore $A_n \subseteq K(H) + \mathbb{C}I$. Conversely, by using the fact that operators of finite-rank are norm dense in $K(H)$, and that the finite combinations of $e_1 e_2 \cdots$ are dense in H, it easy to show that $K(H) + \mathbb{C}I \subseteq \bigcup_{n\geq 1} A_n$. The embedding of A_n into A_{n+1} maps M_n once into M_{n+1}. Since $P_n^{\perp} = P_{n+1}^{\perp} + E_{n+1}$ where E_{n+1} is a rank 1 projection less than P_{n+1}, the scalars are imbedded once into each summand of A_{n+1}.

In all which follows, the equality $A = \overline{\cup_n A_n}$ always means that A is an AF-algebra with an increasing sequence of finite-dimensional subalgebras of A, $\{A_n\}_{n=1,2,\cdots}$, all containing the identity of A.

If we set, $A_0 = \mathbb{C}e$, so that $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$, then $A = \overline{\cup_{n=0}^{\infty} A_n}$ is an unital AF-algebra, and e is the unit of A. Consider that $A = \overline{\cup_n A_n}$, $B = \overline{\cup_n B_n}$. It is easy to check that $A \oplus B = \overline{\cup_n (A_n \oplus B_n)}$ and $A \otimes B = \overline{\cup_n (A_n \otimes B_n)}$. Let $A = \overline{\cup_n A_n}$, and ρ be a morphism of A onto a C^*-algebra B. Since $\|\rho(x)\| \leq \|x\|$ for all $x \in A$, then $B = \overline{\cup_n \rho(A_n)}$. Because, the C^*-algebras A_n are finite-dimensional, the C^*-subalgebras $B_n = \rho(A_n)$ of B are finite-dimensional, and because ρ maps unit of A into the unit of B, this gives that the B is an AF-algebra. This shows that the category of AF-algebras with their morphisms is closed under finite sums and (minimal) tensor products.

Definition 2.5. [8] A C^*-algebra A is called *local AF-algebra* if any finite subset $\{a_1, ..., a_n\} \subseteq A$ and $\varepsilon > 0$ there exists a C^*-subalgebra of finite-dimensional B of A and $b_1, \cdots, b_n$ in B such that for all $1 \leq j \leq n$,

$$\|a_j - b_j\| < \varepsilon.$$

The following result is proved in [8, Theorem 3.4].

Theorem 2.6. *For any given separable C^*-algebra A, the C^*-algebra A is AF-algebra if and only if it is a local AF-algebra.*

By [8, Proposition 1.4], if each isometry in C^*-algebra A is unitary, then A is *finite* C^*-algebra.

Proposition 2.7. [8, Proposition 3.6] *Every unital AF-algebra is a finite C^*-algebra.*

A C^*-algebra A is called *monotone complete* if and only if any bounded increasing net in A_{sa} has a least upper bound in A_{sa}, where A_{sa} denotes the real vector space of hermitian elements of A. The least upper bound of a bounded increasing net $\{h_\alpha\}_\alpha$ in A_{sa} is denoted by $\sup_\alpha h_\alpha$. A C^*-algebra A is *monotone σ-complete* if every bounded increasing sequence $\{h_n\}_{n\in\mathbb{N}}$ in A_{sa} has a least bound upper in A_{sa}.

Monotone complete C^*-algebras are unital, and every W^*-algebra is monotone complete. However, it is not known whether every AW^*-algebra is monotone complete. The following proposition gives the situation for the injective envelope of C^*-algebras.

Proposition 2.8. [7, Proposition 1.3] *Let $E \subseteq M$ be operator systems, with M monotone complete. If the linear map $\phi : M \longrightarrow E$ is positive and such that $\phi|_E = id_E$, then E is monotone complete.*

A subsequence of the above proposition is that the injective envelope $I(A)$ of any C^*-algebra A is monotone complete. Particular, $I(A)$ is an AW^*-algebra.

Let A be a C^*-subalgebra of B, we denote the smallest subset of B_{sa} that contains A_{sa} and is monotone closed in B by $m\text{-cl}_B A_{sa}$. The monotone closure of A in B is defined to be the set

$$m\text{-cl}_B A = (m\text{-cl}_B A_{sa}) + i(m\text{-cl}_B A_{sa}).$$

In particular, $m\text{-cl}_B A$ is a monotone complete C^*-subalgebra of B [11, Lemma 1.4].

A C^*-subalgebra C of B is called *monotone closed* if $m\text{-cl}_B C = C$. Because this property involves both C and B, it is possible for a C^*-subalgebra C of B to be monotone complete yet fail to be monotone closed. In fact, it is frequently the case that a von Neumann algebra $M \subset B(H)$ is not monotone closed in $B(H)$.

A *order dense* in C^*-algebra B is a C^*-subalgebra A of B if

$$h = \sup\{k \in A^+ : k \leq h\}, \quad \forall h \in B^+.$$

For example, C^*-algebra $K(H)$ in $B(H)$ is order dense.

Definition 2.9. A C^*-algebra B is *regular monotone completion* of a C^*-algebra A if
(1) A is a C^*-subalgebra of B,
(2) B is monotone complete,
(3) $m\text{-cl}_B A = B$, and
(4) A is order dense in B.

In [11], Hamana proved that a regular monotone completion exists for every C^*-algebra A and any two regular monotone completions of A are $*$-isomorphic. Here $\overline{A}$ is used to denote the regular monotone completion of A. Hamana's construction of $\overline{A}$ is via the injective envelope of A. Namely, $\overline{A}$ is the monotone closure of A in $I(A)$. For each C^*-algebra A there is a representation in which

$$A \subseteq \overline{A} \subseteq I(A),$$

where this containments are as a C^*-subalgebra. An important feature of this containment is that $\overline{A}$ is monotone closed in $I(A)$.

3. Injective Envelopes of AF-Algebras

In this section, we show that an injective AF-algebra has to be finite dimensional. After Hamana [5], we know that the category of C^*-algebras contain the injective envelope of its objects. Throughout this section, AF-algebras are assumed to be separable (not in the general local sense of Katsura).

Let A and B be C^*-algebras, and let $\phi : A \longrightarrow B$ be a linear map,

$$\phi^{(n)} : M_n(A) \longrightarrow M_n(B)$$

be the amplification map obtained by applying ϕ entrywise. The map ϕ is said *positive* and denotes $\phi \geq 0$ if a is in A_+, then $\phi(a)$ be in B_+, and if $\phi^{(n)} : M_n(A) \longrightarrow M_n(B)$ be a positive, the map ϕ n-positive, and finally, The ϕ is *completely positive*, if it is n-positive for given any $n \geq 1$. A positive map is automatically bounded (the proof is essentially the same as for positive linear functionals). Indeed, an unital positive map is a contraction. More generally, this kind of maps can be defined over operator systems.

An *order embedding* map ϕ is a map $\phi : A \longrightarrow B$ such that the ϕ is isometric and $\phi(x) \geq 0$ if and only if $x \geq 0$, equivalently, ϕ and $\phi^{-1} : \phi(A) \longrightarrow A$ are both positive contractions, and *complete order embedding* if $\phi^{(n)}$ is an order embedding for all n.

Definition 3.1. A C^*-algebra A is *injective* if given any subspace S of a C^*-algebra C, any completely positive linear map of S into A, there exists a completely positive linear map of C into A.

Definition 3.2. Let A be an AF-algebra. An AF-algebra $I(A)$ is injective envelope of A if and only if the $I(A)$ is injective and the only completely positive linear map $f : I(A) \longrightarrow I(A)$ for which $f|_A = id_A$ is the identity map $f = id_{I(A)}$.

The following proposition, we bring is a well-known result.

Proposition 3.3. [12, Proposition IV.2.1.7] *Any injective C^*-algebra is an AW^*-algebra.*

According to the proposition 2.8, we can also say that the injective envelope of any C^*-algebra is monotone complete. In particular, it is an AW^*-algebra. Now we want to prove that there is no injective AF-algebra except the finite dimensional ones.

Theorem 3.4. *In the category of C^*-algebras, the only injective AF-algebras are the ones which are of finite-dimensional.*

Proof. Let A be an injective AF-algebra. Then A is an AW^*-algebra by Proposition 3.3. On the other hand, since A is an AF-algebra, so $A = \overline{\cup_{n=0}^{\infty} A_n}$. Each A_n is finite-dimensional and the collection of A_n is countable, so A is a separable AW^*-algebra. Now, a maximal abelian C^*-subalgebra (masa) in AW^*-algebra A is $C(X)$ for an extermally disconnected compact X. Since, A is a separable so is $C(X)$. It follows that X must be metrizable and any extermally disconnected metrizable space should be discrete. It follows from compactness of X that X must be finite. So $C(X)$ will be finite-dimensional. Now, by [13], it follows that A is also finite-dimensional. ∎

A consequence of above theorem is the following corollary.

Corollary 3.5. *The injective envelope of an infinite-dimensional AF-algebras, could not be an AF-algebra.*

It is desirable to show that injective objects in the category of AF-algebras are also finite dimensional. At this point, we do not know if injective objects in the category of AF-algebras have to be AW^*-algebras. Also, we don't know yet if the category of all AF-algebras (including non-separable ones) contains injective envelopes of its objects.

4. Essential Ideals of AF-Algebras

The aim of this section is to discuss on essential ideals of AF-algebras and show that for a essentially simple AF-algebra, it is nice to have a hand on the minimal essential ideal. The motivation for this is coming from the result of Argerami and Farenick [14, Theorem 2.2].

An ideal I of a C^*-algebra A is called *essential* if $K \cap I \neq \{0\}$ for given any non-zero ideal K of A (or equivalently, $aI = 0$ implies $a = 0$, for all $a \in A$). An essential ideal is necessarily non-zero.

Example 4.1. (i) If Y is locally compact and Hausdorff, then $I = C_0(X)$ is an essential ideal of $C_0(Y)$ for some open dense subset $X \subseteq Y$ and is the only essential ideal of $C_0(Y)$.

(ii) let A be a type I AW^*-algebra and p is an abelian projection of A, then the ideal $I =< p >$ is an essential ideal in A.

Definition 4.2. A C^*-algebra A is called *essentially simple* if it has no proper closed essential ideal.

Clearly, each simple C^*-algebra is essentially simple, but the converse is not true as it is seen from the following example. If A is non-unital, then neither the multiplier algebra $M(A)$ nor the minimal unitization $A + \mathbb{C} \subseteq M(A)$ is essentially simple, as in both cases, A is a proper closed essential ideal. As we essentially deal with separable C^*-algebras in this paper, the former case is not such an interesting counterexample, as $M(A)$ is not usually sparable in infinite dimensional case. However, the first case is a good source of non essentially simple separable C^*-algebras, such as $K(H) + \mathbb{C}I \subseteq B(H)$, which also appeared in Example 2.4.

Example 4.3. (i) Finite dimensional C^*-algebras are essentially simple: Let B be a finite dimensional C^*-algebra and I be a closed ideal in B. Then B is a finite direct sum of matrix algebras, and since full matrix algebras are simple, I has to be a direct sum of a finite subfamily. Now if I is nontrivial, there is a full summand which is missing in I. Choose a non-zero element a in this full summand and regard it as an element of B. Clearly, $aI = 0$, while $a \neq 0$, that is, I could not be essential.

(ii) More generally, any direct sum of simple C^*-algebras (with more than one factor) is (non simple but) essentially simple, by an argument verbatim to (i).

Let I be a two-sided ideal in AF-algebra A then I also is an AF-algebra such that

$$I = \overline{\bigcup_{n\geq 1}(I \bigcap A_n)}.$$

where is $A = \overline{\bigcup_{n\geq 1} A_n}$ and each A_n is finite-dimensional C^*-subalgebra.

It is desirable if one could use the above characterization (along with alternative characterizations using the Bratteli diagrams) to characterize essentially simple AF-algebras. Also, it is desirable to have a characterization of those AF-algebras with a unique proper closed essential ideal (like the case of minimal unitization). Finally, for a non-essentially simple AF-algebra, it is nice to have a hand on the minimal essential ideal.

For the AF-algebra $A = K(H)$, the injective envelope of A is the W^*-algebra $B(H)$. We use the result of Argerami and Farenick [14, Theorem 2.2] to show that $K(H)$ (and its direct sums) are basically the only case that such phenomena could happen.

Definition 4.4. We say that a C^*-algebra is *elementary* if it is $*$-isomorphic to $K(H)$ for some Hilbert space H.

The next result follows directly from definition and [14, Theorem 2.2].

Proposition 4.5. *If A is a separable essentially simple C^*-algebra, then the followings are equivalent;*

(i) $\overline{A}$ is a W^*-algebra,

(ii) $I(A)$ is a W^*-algebra,

(iii) A is isomorphic to an at most countable direct sum of algebras of the form $K(H_i)$, where H_i is a Hilbert space.

Since, the $K(H)$ is AF-algebra, and AF-algebras are closed under taking countable direct sum. Therefore, we will have the following corollary.

Corollary 4.6. *A separable essentially simple C^*-algebra whose injective envelope is a von Neumann algebra must be an AF-algebra.*

Next, we discuss liminal and postliminal C^*-algebras in the context of injective envelopes. There exists examples of AF-algebras which are postliminal but not liminal, as well as examples which are not postliminal.

Definition 4.7. Let A be a C^*-algebra:

(i) A is *liminal* if $\varphi(A) = K(H)$, which (H, φ) is a non-zero irreducible representation of A (equivalently, $\varphi(A) \subseteq K(H)$).

(ii) A is called *postliminal* if $\varphi(A) \supseteq K(H)$, which (H, φ) is a non-zero irreducible representation (H, φ) of A (equivalently, $K(H) \cap \varphi(A) \neq 0$).

The liminal algebras are also called *CCR* stands for *completely continuous representations* being an old synonym *compact*, and the postliminal algebras are called *GCR* stands

for *generalized* CCR or Type I C^*-algebras. The Type I terminology should not be misunderstood with its von Neumann algebra counterpart, as Type I von Neumann algebras are not necessarily Type I as a C^*-algebra.
Every liminal C^*-algebra is abviously postliminal.

Example 4.8. (i) Every abelian C^*-algebra is liminal, for this let (H, φ) be a non-zero irreducible representation of A, then the comutant $\varphi(A)'$ equal to $\mathbb{C}1$. Furthers, since A is abelian, $\varphi(A) \subseteq \varphi(A)'$. Hence, $\varphi(A) = \mathbb{C}1$, so H is one dimensional. Since $\varphi(A)$ has no non-trivial invariant vector subspaces. Therefore, $\varphi(A) = K(H)$.

(ii) Let A be a finite-dimensional C^*-algebra, then it is liminal. Because if (H, φ) be a non-zero irreducible representation of A, then $H = \varphi(A)x$ for some non-zero vector $x \in H$, so H is finite-dimensional and therefore, $\varphi(A) \subseteq K(H) = B(H)$.

We know that every C^*-subalgebra of a liminal C^*-algebra and its quotient C^*-algebra is also liminal [9, Theorem 5.6.1]. Also, if I is a closed ideal in a C^*-algebra A, Then the postliminal of A equivalent to the postliminal of I and A/I [9, Theorem 5.6.2].

On the other hand, Teoplitz algebra $\mathcal{T}$ is postliminal, but it is not liminal. Since its commutator ideal $\mathcal{K} := K(H^2(\mathbb{T}))$ is liminal, and since the quotient $\mathcal{T}/\mathcal{K}$ is $*$-isomorphic to $C(\mathbb{T})$, i.e., it is abelian and so liminal. Hence, $\mathcal{K} := K(H^2(\mathbb{T}))$ and $\mathcal{T}/\mathcal{K}$ is postliminal. Finally, $\mathcal{T}$ is postliminal. However, $\mathcal{T}$ is not liminal, as the identity representation of $\mathcal{T}$ in $H^2(\mathbb{T})$ is irreducible but not finite-dimensional.

Next, let us observe that $K(H)$ is a liminal AF-algebra. Since the identity representation is the only non-zero irreducible representation that is unitarily equivalent to every non-zero irreducible representation of $K(H)$, and $K(H)' = \mathbb{C}1$, $(H, \varphi) = (H, i)$. Therefore, $K(H)$ is liminal.

On the other hand, the unital AF-algebra $A = K(H) + \mathbb{C}I \subseteq B(H)$ is not liminal, but it is postliminal. First, observe that finite-dimensional irreducible representations are only irreducible representations of an unital liminal C^*-algebra A. For this, let (H, φ) be a non-zero irreducible representation of A, then it is non-degenerate, and therefore $\varphi(1) = id_H$. Hence, $id_H \in \varphi(A) = K(H)$, so it is compact, and thus $dim(H) < \infty$. Now assume that H is a infinite-dimensional Hilbert space, then the C^*-algebra $A = K(H)+\mathbb{C}I$ is unital and contains an infinite-dimensional non-zero irreducible representation, namely, the identity representation on H. Hence, A is not liminal. But, A is postliminal. Since $K(H)$ and $\frac{A}{K(H)} = \mathbb{C}$ are liminal C^*-algebras, and so are postliminal. The last argument also reveals the fact that if I is a closed ideal of a C^*-algebra A such that I and $\frac{A}{I}$ are liminal, then it does not follow that A is also liminal (while the converse is known to be true [9, Theorem 5.6.1]).

It is easy to check that if a C^*-algebra is simple postliminal C^*-algebra, then it is elementary. Since A is a postliminal C^*-algebra, then $\varphi(A) \supseteq K(H)$. On the other hand, since A is simple, then $\varphi(A)$ is simple, and (H, φ) is non-zero irreducible representation. Hence, $\varphi(A) = K(H)$, i.e. A is *-isomorphic to $K(H)$. But if an elementary C^*-algebra is unital, then it is finite-dimensional and vice versa [9, Theorem 1.4.2]. Therefore, an infinite-dimensional unital simple C^*-algebra is not postliminal. Because, if C^*-algebra is postliminal, then it is elementary, Since it is simple. But we know that every unital elementary is finite-dimensional. For example, no UHF-algebras are postliminal. In particular, if H is a separable infinite-dimensional Hilbert space, then the Calkin algebra is an example of simple C^*-algebra, which contradicts the postliminal of $B(H)$. Namely, $B(H)$ is not postliminal.

Theorem 4.9. *Any separable essentially simple postliminal C^*-algebra is liminal.*

Proof. Let A satisfy the assumptions of the theorem. Since A is postliminal, it follows from [11, Theorem 6.6] that $\overline{A}$ is a type I AW^*-algebra. Since every type I AW^*-algebra is also injective, we get $\overline{A} = I(A)$. Now [7, Theorem 3.1] implies that A has a liminal essential ideal I, then $I = A$, by essential simplicity. That is, A is liminal. ∎

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