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Full Formulas Induced by Full Terms

Thodsaporn Kumduang[1] **and Sorasak Leeratanavalee**[2,*]

[1] *Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand*
e-mail : kumduang01@gmail.com

[2] *Research Center in Mathematics and Applied Mathematics, Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand*
e-mail : sorasak.l@cmu.ac.th

Abstract The algebraic system is a well-established structure of classical universal algebra. An algebraic system is a triple consisting a nonempty set together with the sequence of operation symbols and the sequence of relation symbols. To express the primary properties of algebraic systems one needs the notion of formulas. The paper is devoted to studying of the structures related to full formulas which are extensional concepts constructed from full terms. Defining a superposition operation on the set of full formulas one obtains a many-sorted algebra which satisfies the superassociative law. In particular, we introduce a natural concept of a full hypersubstitution for algebraic systems which extends the concept of full hypersubstitutions of algebras, i.e., the mappings which send operation symbols to full terms of the same arities and relation symbols to full formulas of the corresponding arities. Together with one associative operation on the collection of full hypersubstitutions for algebraic systems, we obtain a semigroup of full hypersubstitutions for algebraic systems.

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1. Introduction and Preliminary Definitions

The idea of terms is one of the mathematical basic concepts. Terms may be regarded as words formed by letters. Let I be a nonempty indexed set and $(f_i)_{i\in I}$ be a sequence of operation symbols. To every operation symbol f_i, we assign a natural number $n_i \in \mathbb{N}^+ := \{1,2,3,\ldots\}$, the arity of f_i. The sequence $\tau := (n_i)_{i\in I}$ is called a *type*. We denote by $X_n := \{x_1,\ldots,x_n\}$ is a finite set called an *alphabet* and its elements are called *variables* and for each $n \geq 1$, let $X_n := \{x_1,\ldots,x_n\}$. An *$n$-ary term of type τ* is defined inductively by:

(1) Every variable $x_j \in X_n$ is an n-ary term of type τ.
(2) If $t_1,\ldots,t_{n_i}$ are n-ary terms of type τ and f_i is an n_i-ary operation symbol, then $f_i(t_1,\ldots,t_{n_i})$ is an n-ary term of type τ.

*Corresponding author.

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Let $W_\tau(X_n)$ be the set of all n-ary terms of type τ and $W_\tau(X) := \bigcup_{n\in\mathbb{N}^+} W_\tau(X_n)$ be the set of all terms of type τ. The study of terms in various directions can be found, for instance, in [1, 2].

Now, we recall the concept of superposition operation of terms. For each natural numbers $m, n \geq 1$, the superposition operation of terms is a many-sorted mapping

$$S^n_m : W_\tau(X_n) \times W_\tau(X_m)^n \to W_\tau(X_m)$$

defined by

(1) $S^n_m(x_j, t_1, \ldots, t_n) := t_j$, if $x_j, 1 \leq j \leq n$ is a variable from X_n.
(2) $S^n_m(f_i(s_1, \ldots, s_{n_i}), t_1, \ldots, t_n) := f_i(S^n_m(s_1, t_1, \ldots, t_n), \ldots, S^n_m(s_{n_i}, t_1, \ldots, t_n))$.

Then the many-sorted algebra can be defined by

$$\text{clone}\tau = ((W_\tau(X_n))_{n\in\mathbb{N}^+}, (S^n_m)_{n,m\in\mathbb{N}^+}, (x_i)_{i\leq n\in\mathbb{N}^+}),$$

which is called *the clone of all terms of type* τ. In this case, the variables $x_1, \ldots, x_n$ act as the nullary operations. The primary result of the clone of all terms of type τ is a satisfying identities $(C1), (C2), (C3)$ (see [3]).

One of important structures on universal algebra is an algebraic system. Here, we would like to generalize all above concepts to algebraic systems. For more information about algebraic systems, we refer the reader to [4, 5]. To approach this, we begin by giving some basic definitions. Let J be a nonempty indexed set and let $(\gamma_j)_{j\in J}$ be a sequence of relation symbols. Let $\tau' := (n_j)_{j\in J}$ where n_j is the arity of γ_j for every $j \in J$.

Definition 1.1. ([5]) *An algebraic system of type* (τ, τ') is a triple consisting a nonempty set A together with a sequence $(f_i^{\mathcal{A}})_{i\in I}$ of operations on A where $f_i^{\mathcal{A}}$ is n_i-ary for $i \in I$ and a sequence $(\gamma_j^{\mathcal{A}})_{j\in J}$ of relations on A where $\gamma_j^{\mathcal{A}}$ is n_j-ary for $j \in J$, i.e,
$\mathcal{A} := (A, (f_i^{\mathcal{A}})_{i\in I}, (\gamma_j^{\mathcal{A}})_{j\in J})$.

Here, some standard examples will be provided. Ordered semigroups can be regarded as algebraic systems of type $((2), (2))$ such as an algebraic system $(\mathbb{N}^+, +, \leq)$ of type $((2), (2))$ consisting the set of all natural numbers, one binary operation on $\mathbb{N}^+$ and one binary relation on $\mathbb{N}^+$, say $+$ and $\leq$, respectively.

Not only all of the terms in the second-order language will be used to express properties of algebraic systems but also another one is called *quantifier free formulas*. The concept of quantifier free formulas is first introduced by A.I. Mal'cev who is the Russian mathematician in 1973 [5]. For more detail see also [4, 5]. Next, we recall the formal definition of n-ary quantifier free formulas which is defined by K. Denecke and D. Phusanga in 2008.

Definition 1.2. ([6]) Let $n \in \mathbb{N}^+$. *An n-ary quantifier free formula of type* (τ, τ') *(for simply, formula)* is defined in the following way:

(1) If t_1, t_2 are n-ary terms of type τ, then the equation $t_1 \approx t_2$ is an n-ary quantifier free formula of type (τ, τ').
(2) If $j \in J$ and $t_1, \ldots, t_{n_j}$ are n-ary terms of type τ and γ_j is an n_j-ary relation symbol, then $\gamma_j(t_1, \ldots, t_{n_j})$ is an n-ary quantifier free formula of type (τ, τ').
(3) If F is an n-ary quantifier free formula of type (τ, τ'), then $\neg F$ is an n-ary quantifier free formula of type (τ, τ').
(4) If F_1 and F_2 are n-ary quantifier free formulas of type (τ, τ'), then $F_1 \vee F_2$ is an n-ary quantifier free formula of type (τ, τ').

Let $\mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))$ be the set of all n-ary quantifier free formulas of type (τ,τ') and let

$$\mathcal{F}_{(\tau,\tau')}(W_\tau(X)) := \bigcup_{n\in\mathbb{N}^+} \mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))$$

be the set of all quantifier free formulas of type (τ,τ').

The definition of superposition of formulas R^n_m was already defined. For this defining see [6]. These operations are used to define a many-sorted algebra

$$Formclone(\tau,\tau') := ((W_\tau(X_n) \cup \mathcal{F}_{(\tau,\tau')}(X_n))_{n\geq 1}, (R^n_m)_{m,n\geq 1}, (x_i)_{1\leq i\leq n, i, n\in\mathbb{N}}),$$

which is called the *formula-term clone* of type (τ,τ').

In 2017, T. Kumduang and S. Leeratanavalee [7] introduced the definition of linear formulas which extended the idea of linear terms. Some fundamental properties of linear formulas were studied. This topic also investigated by K. Denecke in 2019. (see, [1, 8]).

Now, we recall the concept of a hypersubstitution, which was introduced by K. Denecke, D. Lau, R. Poschel and D. Schweigert in [3]. We begin with a number of definitions and some notations. Throughout of these preliminaries, we assume a fixed type $\tau = (n_i)_{i\in I}$, with operation symbols f_i for $i \in I$. See, e.g., [3, 9] for more background on hypersubstitutions and hyperidentities.

The concept of a hypersubstitution for algebras can be extended in the canonical way to a hypersubstitution for algebraic systems of type (τ,τ'). In 2018, J. Koppitz and D. Phusanga [10] introduced the concept of a hypersubstitution for algebraic systems.

Definition 1.3. ([10]) A hypersubstitution for algebraic systems of type (τ,τ') is a mapping

$$\sigma : \{f_i \mid i \in I\} \cup \{\gamma_j \mid j \in J\} \to W_\tau(X) \cup \mathcal{F}_{(\tau,\tau')}(W_\tau(X))$$

which maps operation symbols to terms and relation symbols to quantifier free formulas preserving arities. Let $Hyp(\tau,\tau')$ be the set of all hypersubstitutions for algebraic systems of type (τ,τ').

Any hypersubstitution for algebraic systems of type (τ,τ') σ induces an extensional mapping $\widehat{\sigma}$ defined on the set $W_\tau(X) \cup \mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))$, as follows.

Definition 1.4. Let $\sigma \in Hyp(\tau,\tau')$ and let $n \in \mathbb{N}$. Then σ induces a mapping

$$\widehat{\sigma} : W_\tau(X) \cup \mathcal{F}_{(\tau,\tau')}(W_\tau(X)) \to W_\tau(X) \cup \mathcal{F}_{(\tau,\tau')}(W_\tau(X))$$

by setting

(1) $\widehat{\sigma}[x_i] := x_i$ for every $i = 1,\ldots,n$.
(2) $\widehat{\sigma}[f_i(t_1,\ldots,t_{n_i})] := S^{n_i}_n(\sigma(f_i), \widehat{\sigma}[t_1],\ldots,\widehat{\sigma}[t_{n_i}])$ for $i \in I$.
(3) $\widehat{\sigma}[s \approx t] := \widehat{\sigma}[s] \approx \widehat{\sigma}[t]$.
(4) $\widehat{\sigma}[\gamma_j(t_1,\ldots,t_{n_j})] := R^{n_j}_n(\sigma(\gamma_j), \widehat{\sigma}[t_1],\ldots,\widehat{\sigma}[t_{n_j}])$ for $j \in J$.
(5) $\widehat{\sigma}[\neg F] := \neg\widehat{\sigma}[F]$ for every $F \in \mathcal{F}_{(\tau,\tau')}(W_\tau(X))$.
(6) $\widehat{\sigma}[F_1 \vee F_2] := \widehat{\sigma}[F_1] \vee \widehat{\sigma}[F_2]$ for every $F_1, F_2 \in \mathcal{F}_{(\tau,\tau')}(W_\tau(X))$.

We recall that $\circ$ denotes the usual composition of mappings and it is easy to verify that $\widehat{\sigma}_1 \circ \sigma_2 \in Hyp(\tau,\tau')$, whenever $\sigma_1, \sigma_2 \in Hyp(\tau,\tau')$. In the study of hypersubstitutions for algebras, the multiplication $\circ_h$ of hypersubstitutions was introduced. For algebraic systems, J. Koppitz and D. Phusanga defined a binary operation $\circ_r$ on $Hyp(\tau,\tau')$ by $\sigma_1 \circ_r \sigma_2 := \widehat{\sigma}_1 \circ \sigma_2$ where $\sigma_1, \sigma_2 \in Hyp(\tau,\tau')$.

Let σ_{id} be the hypersubstitution for algebraic systems which maps each n_i-ary operation symbol f_i to the term $f_i(x_1,,...,x_{n_i})$ for all $i \in I$, and maps each n_j-ary relation symbol γ_j to the quantifier free formula $\gamma_j(x_1, \ldots, x_{n_j})$ for all $j \in J$. In particular, the authors proved that $\mathcal{H}yp(\tau, \tau') := (Hyp(\tau, \tau'), \circ_r, \sigma_{id})$ is a monoid. To approach linear hypersubstitutions for algebraic systems see [11].

Throughout this paper, we assume that all operation symbols have the same fixed arity n for some $n \in \mathbb{N}^+$. Let $\tau_{n,i}$ be such a fixed n-ary type with operation symbols $(f_i)_{i\in I}$ indexed by some nonempty set I. That is $\tau_{n,i}$ is a sequnce of i-tuple of fixed n-ary operation symbols. For instance, $\tau_{2,3} = (2, 2, 2)$ this means we have three binary operation symbols. For the type of relation symbols which fixed arity m, we denoted by $\tau'_{m,j}$ be a sequence of such relation symbols with the same fixed arity m indexed by some set J. This means that, this paper is devoted to study the algebraic systems of type $(\tau_{n,i}, \tau'_{m,j})$.

This paper is motivated by several recent studies [8, 10, 11] of such research area. We restrictly focus on full terms of type $\tau_{n,i}$ (the definition will be given on the next section) for a natural number $n \geq 1$ and $i \in I$, i.e., there are infinitely many n-ary operation symbols. We define the definition of full formulas of type $(\tau_{n,i}, \tau'_{m,j})$ for natural numbers $n, m \geq 1$. Our first aim is to construct the many-sorted algebra as in the same situation of $Formclone(\tau, \tau')$, and then define the superposition operation R^n on the set of full formulas. Furthermore, the canonical concept of full hypersubstitutions for algebraic systems is introduced. To construct the algebra of all full hypersubstitutions for algebraic systems, the theorem of superassociative law, endomorphism of clone, and some significantly properites of the extension of full hypersubstitutions for algebraic systems are investigated.

2. Full Formulas and the Clone of Full Formulas

Now, it comes to our main results. The concept of a full term was already defined for algebras by K. Denecke and P. Jampachon in 2004 [12]. In particular, they formed a Menger algebra of such terms [13]. Let H_n be the set of all mapping $\alpha : \{1, \ldots, n\} \to \{1, \ldots, n\}$.

Definition 2.1. An *n-ary full term of type* $\tau_{n,i}$ is inductively defined by:

(1) Let $\alpha \in H_n$ be an arbitrary function and let f_i be an operation symbol of type $\tau_{n,i}$. Then $f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$ is an n-ary full term of type $\tau_{n,i}$.
(2) If $t_1, \ldots, t_n$ are n-ary full terms of type $\tau_{n,i}$ and f_i is an operation symbol of type $\tau_{n,i}$, then $f_i(t_1, \ldots, t_n)$ is an n-ary full term of type $\tau_{n,i}$.

Let $W^F_{\tau_{n,i}}(X_n)$ be the set of all n-ary full terms of type $\tau_{n,i}$ and let

$$W^F_{\tau_{n,i}}(X) := \bigcup_{n\in\mathbb{N}^+} W^F_{\tau_{n,i}}(X_n)$$

be the set of all full terms of type $\tau_{n,i}$.

Example 2.2. Let $\tau_{2,1}$ be the type with only one binary operation symbol f and $H_2 = \{\alpha : \{1, 2\} \to \{1, 2\}\}$ be the set of all mappings on $\{1, 2\}$. Then $f(x_1, x_1), f(x_1, x_2), f(x_2, x_1), f(x_2, x_2), f(f(x_1, x_1), f(x_1, x_1)), f(f(x_2, x_1), f(x_2, x_2))$ are examples of binary full terms of type $\tau_{2,1}$.

We now define an $(n + 1)$-superposition operation S^n on $W^F_{\tau_{n,i}}(X_n)$, i.e., a mapping

$$S^n : W^F_{\tau_{n,i}}(X_n) \times W^F_{\tau_{n,i}}(X_n)^n \to W^F_{\tau_{n,i}}(X_n)$$

(1) $S^n(f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)}), s_1, \ldots, s_n) := f_i(s_{\alpha(1)}, \ldots, s_{\alpha(n)})$ where $\alpha \in H_2$,

(2) $S^n(f_i(t_1,\ldots,t_n),s_1,\ldots,s_n) := f_i(S^n(t_1,s_1,\ldots,s_n),\ldots,S^n(t_n,s_1,\ldots,s_n))$.

Now we can form the algebra $clone_F\tau_{n,i} := (W^F_{\tau_{n,i}}(X_n), S^n)$ of type $(n+1)$. Furthermore, the algebra $clone_F\tau_{n,i}$ is a Menger algebra of rank n.

Theorem 2.3. ([12]) *The algebra $clone_F\tau_{n,i}$ satisfies the superassociative law;*

$$S^n(S^n(t,t_1,\ldots,t_n),s_1,\ldots,s_n) = S^n(t,S^n(t_1,s_1,\ldots,s_n),\ldots,S^n(t_n,s_1,\ldots,s_n))$$

where $t,t_1,\ldots,t_n,s_1,\ldots,s_n \in W^F_{\tau_{n,i}}(X_n)$.

Our main purpose is to define quantifier free full formulas which are induced by full terms. Using the definition of full terms in Definition 2.1, we define the new concept of a quantifier free full formula of type $(\tau_{n,i},\tau'_{m,j})$ for natural numbers $n,m \geq 1$.

Let us consider $\tau'_{m,j} = (m,\ldots,m)$, i.e., we have j-tuple of m-ary relation symbols. As an example we consider type $\tau'_{3,4}$, that means $\tau'_{3,4} = (3,3,3,3)$ so that we have four ternary relation symbols.

Definition 2.4. Let $n \in \mathbb{N}^+$. *An n-ary quantifier free full formula of type $(\tau_{n,i},\tau'_{m,j})$ (for simply, full formula)* is defined in the following way:

(1) If t_1,t_2 are n-ary full terms of type $\tau_{n,i}$, then the equation $t_1 \approx t_2$ is an n-ary quantifier free full formula of type $(\tau_{n,i},\tau'_{m,j})$.

(2) If $t_1,\ldots,t_m$ are n-ary full terms of type $\tau_{n,i}$ and γ_j is a relation symbol of type $\tau'_{m,j}$, then $\gamma_j(t_1,\ldots,t_m)$ is an n-ary quantifier free full formula of type $(\tau_{n,i},\tau'_{m,j})$.

(3) If F is an n-ary quantifier free full formula of type $(\tau_{n,i},\tau'_{m,j})$, then $\neg F$ is an n-ary quantifier free full formula of type $(\tau_{n,i},\tau'_{m,j})$.

(4) If F_1 and F_2 are n-ary quantifier free full formulas of type $(\tau_{n,i},\tau'_{m,j})$, then $F_1 \vee F_2$ is an n-ary quantifier free full formula of type $(\tau_{n,i},\tau'_{m,j})$.

Let $\mathcal{F}^F_{(\tau_{n,i},\tau'_{m,j})}(W^F_{\tau_{n,i}}(X_n))$ be the set of all n-ary quantifier free full formulas of type $(\tau_{n,i},\tau'_{m,j})$ and let

$$\mathcal{F}^F_{(\tau_{n,i},\tau'_{m,j})}(W_{\tau_{n,i}}(X)) := \bigcup_{n\in\mathbb{N}^+} \mathcal{F}^F_{(\tau_{n,i},\tau'_{m,j})}(W_{\tau_{n,i}}(X_n))$$

be the set of all quantifier free full formulas of type $(\tau_{n,i},\tau'_{m,j})$.

Example 2.5. Let $(\tau_{2,2},\tau'_{2,1})$ be a type of algebraic systems, i.e., we have two binary operation symbols f and g and one binary relation symbol γ. Then the examples of binary full formulas of type $(\tau_{2,2},\tau'_{2,1})$ are $f(x_1,x_2) \approx g(x_2,x_1), f(g(x_2,x_2),f(x_1,x_1)) \approx f(x_1,x_1), \gamma(f(x_2,x_2),g(x_2,x_1)), \gamma(f(x_1,x_1),f(g(x_2,x_2),f(x_1,x_1)))$. Moreover, we obtain other full formulas of type $(\tau_{2,2},\tau'_{2,1})$ from these by using the connectives $\neg$ and $\vee$.

Now, we generalize the definition of superposition operation of full terms to quantifier free full formulas by substituting variables occurring in a quantifier free full formula by full terms, then we get quantifier free full formulas. We explain this by the following operation R^n where $n \geq 1$.

Definition 2.6. The operation

$$R^n : \mathcal{F}^F_{(\tau_{n,i},\tau'_{m,j})}(W^F_{\tau_{n,i}}(X_n)) \times W^F_{\tau_{n,i}}(X_n)^n \to \mathcal{F}^F_{(\tau_{n,i},\tau'_{m,j})}(W^F_{\tau_{n,i}}(X_n))$$

where $m,n \in \mathbb{N}^+$, are defined by the following inductive steps:

(1) If $t_1,t_2 \in W^F_{\tau_{n,i}}(X_n)$, then
$R^n(t_1 \approx t_2,s_1,\ldots,s_n) := S^n(t_1,s_1,\ldots,s_n) \approx S^n(t_2,s_1,\ldots,s_n)$.

(2) If $t_1, \ldots, t_m \in W^F_{\tau_{n,i}}(X_n)$, then
$R^n(\gamma_j(t_1, \ldots, t_m), s_1, \ldots, s_n) := \gamma_j(S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_m, s_1, \ldots, s_n))$.

(3) If $F \in \mathcal{F}^F_{(\tau_{n,i},\tau'_{m,j})}(W^F_{\tau_{n,i}}(X_n))$, then $R^n(\neg F, s_1, \ldots, s_n) := \neg R^n(F, s_1, \ldots, s_n)$.

(4) If $F_1, F_2 \in \mathcal{F}^F_{(\tau_{n,i},\tau'_{m,j})}(W^F_{\tau_{n,i}}(X_n))$, then
$R^n(F_1 \vee F_2, s_1, \ldots, s_n) := R^n(F_1, s_1, \ldots, s_n) \vee R^n(F_2, s_1, \ldots, s_n)$.

Below we provide an example of Definition 2.6 that demonstrates a method for substituting full formulas by full terms.

Example 2.7. Let $(\tau_{2,1}, \tau'_{3,1})$ be a type of algebraic systems with a binary operation symbol and a ternary relation symbol say g and λ, respectively. Consider the superposition R^2 and two binary full terms $s_1 = g(x_{\alpha(1)}, x_{\alpha(2)}), s_2 = g(x_{\beta(1)}, x_{\beta(2)})$ in $W^F_{\tau_{2,1}}(X_2)$ where α and β are mappings from $\{1, 2\}$ to $\{1, 2\}$ defined by $\alpha = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$. Then we have the following:

(1) If $t_1 \approx t_2$ has the form $g(x_2, x_2) \approx g(x_1, x_2)$, then $R^2(g(x_2, x_2) \approx g(x_1, x_2), s_1, s_2)$ is a full formula $S^2(g(x_2, x_2), s_1, s_2) \approx S^2(g(x_1, x_2), s_1, s_2)$. By the superposition S^2, we have $g(g(x_1, x_1), g(x_1, x_1)) \approx g(g(x_2, x_1), g(x_1, x_1))$.

(2) If $\lambda(g(x_1, x_2), g(x_2, x_1), g(x_2, x_2))$ is a full formula in $\mathcal{F}^F_{(\tau_{2,1},\tau'_{3,1})}(W^F_{\tau_{2,1}}(X_2))$, then we obtain $R^2(\lambda(g(x_1, x_2), g(x_2, x_1), g(x_2, x_2)), s_1, s_2)$. By Definition 2.6 and S^2, it is equal to $\lambda(g(g(x_2, x_1), g(x_1, x_1)), g(g(x_1, x_1), g(x_2, x_1)), g(g(x_1, x_1), g(x_1, x_1)))$.

Furthermore, applying logical connectors $\neg$ and $\vee$ we obtain other full formulas.

If we form the set of all n-ary quantifier free full formulas of type $(\tau_{n,i}, \tau'_{m,j})$, the computation with full formulas is fully described by the following algebra,

$$Formclone^F(\tau_{n,i}, \tau'_{m,j}) := (W^F_{\tau_{n,i}}(X_n), \mathcal{F}^F_{(\tau_{n,i},\tau'_{m,j})}(W^F_{\tau_{n,i}}(X_n)), S^n, R^n).$$

This algebraic structure is called the *clone of full formulas* of type $(\tau_{n,i}, \tau'_{m,j})$.

Theorem 2.8. *The algebra $Formclone^F(\tau_{n,i}, \tau'_{m,j})$ satisfies the following equation.*

$$R^n(R^n(F, t_1, \ldots, t_n), s_1, \ldots, s_n) = R^n(F, S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_n, s_1, \ldots, s_n))$$

where $t_1, \ldots, t_n, s_1, \ldots, s_n \in W^F_{\tau_{n,i}}(X_n)$ and $F \in \mathcal{F}^F_{(\tau_{n,i},\tau'_{m,j})}(W^F_{\tau_{n,i}}(X_n))$.

Proof. We give a proof by the definition of full formulas F. If F is a full formula $s \approx t$, then

$$\begin{aligned}
&R^n(R^n(s \approx t, t_1, \ldots, t_n), s_1, \ldots, s_n) \\
&= S^n(S^n(s, t_1, \ldots, t_n), s_1, \ldots, s_n) \approx S^n(S^n(t, t_1, \ldots, t_n), s_1, \ldots, s_n) \\
&= S^n(s, S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_n, s_1, \ldots, s_n)) \\
&\quad \approx S^n(t, S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_n, s_1, \ldots, s_n)) \\
&= R^n(s \approx t, S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_n, s_1, \ldots, s_n)).
\end{aligned}$$

If F is a full formula $\gamma_j(u_1, \ldots, u_n)$, where $u_1, \ldots, u_n \in W^F_{\tau_{n,i}}(X_n)$, then

$$\begin{aligned}
&R^n(R^n(\gamma_j(u_1, \ldots, u_n), t_1, \ldots, t_n), s_1, \ldots, s_n) \\
&= R^n(\gamma_j(S^n(u_1, t_1, \ldots, t_n), \ldots, S^n(u_n, t_1, \ldots, t_n)), s_1, \ldots, s_n) \\
&= \gamma_j(S^n(u_1, S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_n, s_1, \ldots, s_n)) \\
&\quad , \ldots, S^n(u_n, S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_n, s_1, \ldots, s_n))) \\
&= R^n(\gamma_j(u_1, \ldots, u_n), S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_n, s_1, \ldots, s_n)).
\end{aligned}$$

Assume that a full formula F satisfies the statement of the theorem. Then
$R^n(R^n(\neg F, t_1, \ldots, t_n), s_1, \ldots, s_n)$

$$\begin{aligned} &= R^n(\neg R^n(F, t_1, \ldots, t_n), s_1, \ldots, s_n) \\ &= \neg R^n(R^n(F, t_1, \ldots, t_n), s_1, \ldots, s_n) \\ &= \neg R^n(F, S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_n, s_1, \ldots, s_n)) \\ &= R^n(\neg F, S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_n, s_1, \ldots, s_n)). \end{aligned}$$

Assume that full formulas F_1 and F_2 satisfy the statement of the theorem. Then $R^n(R^n(F_1 \vee F_2, t_1, \ldots, t_n), s_1, \ldots, s_n)$

$$\begin{aligned} &= R^n(R^n(F_1, t_1, \ldots, t_n), s_1, \ldots, s_n) \vee R^n(R^n(F_2, t_1, \ldots, t_n), s_1, \ldots, s_n) \\ &= R^n(F_1, S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_n, s_1, \ldots, s_n)) \\ &\quad \vee R^n(F_2, S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_n, s_1, \ldots, s_n)) \\ &= R^n(F_1 \vee F_2, S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_n, s_1, \ldots, s_n)). \end{aligned}$$

This shows that

$$R^n(R^n(F, t_1, \ldots, t_n), s_1, \ldots, s_n) = R^n(F, S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_n, s_1, \ldots, s_n))$$

for all $F \in \mathcal{F}^F_{(\tau_{n,i}, \tau'_{m,j})}(W^F_{\tau_{n,i}}(X_n))$. ■

3. Full Hypersubstitutions of Type $(\tau_{n,i}, \tau'_{m,j})$

The major intention of this section is to introduce the notion of a full hypersubstitution for algebraic systems of type $(\tau_{n,i}, \tau'_{m,j})$. Such concept is a powerful tool to study hyperidentities. See, e.g., [3, 14] for more inside story on hyperidentities. In [12] the full hypersubstitution of type $\tau_{n,i}$ was given by K. Denecke and P. Jampachon. To fulfil the understanding, the following details are necessary.

For a full term we need the concept of a full term t_β arising from the original full term t where $\beta \in H_n$. Such full term can be defined inductively as follows:

(1) If $t = f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$ for $i \in I, \alpha \in H_n$, then $t_\beta = f_i(x_{\beta(\alpha(1))}, \ldots, x_{\beta(\alpha(n))})$.

(2) If $t = f_i(t_1, \ldots, t_n)$, then $t_\beta = f_i((t_1)_\beta, \ldots, (t_n)_\beta)$.

Obviously, t_β is a full term for any term t and $\beta \in H_n$.

The concept of ordinary hypersubstitutions was mentioned in the previous section. Now, we recall the notion of full hypersubstitution which is a restriction of hypersubstitutions.

Definition 3.1. A *full hypersubstitution* of type $\tau_{n,i}$ is a mapping

$$\sigma : \{f_i \mid i \in I\} \to W^F_{\tau_{n,i}}(X_n).$$

Every full hypersubstitution σ can be extended to a mapping

$$\widehat{\sigma} : W^F_{\tau_{n,i}}(X_n) \to W^F_{\tau_{n,i}}(X_n)$$

defined on the set of full terms by the following identities:

(1) $\widehat{\sigma}[f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})] := (\sigma(f_i))_\alpha$ for every $\alpha \in H_n$.

(2) $\widehat{\sigma}[f_i(t_1, \ldots, t_n)] := S^n(\sigma(f_i), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n])$.

Let $Hyp^F(\tau_{n,i})$ be the set of all full hypersubstitutions of type $\tau_{n,i}$. On this set, one can defined a binary operation $\circ_h$ by $\sigma_1 \circ_h \sigma_2 := \widehat{\sigma}_1 \circ \sigma_2$ where $\circ$ denotes the usual composition. Together with the identity element σ_{id} defining by $\sigma_{id}(f_i) = f_i(x_1, \ldots, x_n)$ we can form the monoid $(Hyp^F(\tau_{n,i}), \circ_h, \sigma_{id})$. For more detail on a full hypersubstitution see [12].

By both ideas of full hypersubstitutions of type $\tau_{n,i}$ and hypersubstitutions for algebraic systems of type (τ, τ') as we mentioned in Definition 1.3, we will combine these ideas and introduce the new algebraic structure of a semigroup of full hypersubstitutions for algebraic systems in a natural way by using many tools from the previous results. We will start with giving the concept of full formulas generated by $\beta \in H_n$. For any full formula F of type $(\tau_{n,i}, \tau'_{m,j})$, a full formula F_β arising from F and the term t_β by mapping all variables corresponding to a mapping $\beta \in H_n$ inductively by the following steps.

(1) If F is $s \approx t$, then $F_\beta = (s \approx t)_\beta := s_\beta \approx t_\beta$.
(2) If F is $\gamma_j(t_1, \ldots, t_m)$, then $F_\beta = (\gamma_j(t_1, \ldots, t_m))_\beta := \gamma_j((t_1)_\beta, \ldots, (t_m)_\beta)$.
(3) $(\neg F)_\beta = \neg(F_\beta)$ where F_β is already defined.
(4) $(F_1 \vee F_2)_\beta = (F_1)_\beta \vee (F_2)_\beta$ where $(F_1)_\beta, (F_2)_\beta$ are already defined.

Definition 3.2. A *full hypersubstitution* for algebraic systems of type $(\tau_{n,i}, \tau'_{m,j})$, for short, a full hypersubstitution of type $(\tau_{n,i}, \tau'_{m,j})$ is a mapping

$$\sigma : \{f_i \mid i \in I\} \cup \{\gamma_j \mid j \in J\} \to W^F_{\tau_{n,i}}(X_n) \cup \mathcal{F}^F_{(\tau_{n,i},\tau'_{m,j})}(W^F_{\tau_{n,i}}(X_m))$$

which sends n_i-ary operation symbols to n_i-ary full terms and n_j-ary relation symbols to n_j-ary quantifier free full formulas, respectively. We denote the set of all full hypersubstitutions of type $(\tau_{n,i}, \tau'_{m,j})$ by $Hyp^F(\tau_{n,i}, \tau'_{m,j})$.

To define a binary operation on $Hyp^F(\tau_{n,i}, \tau'_{m,j})$, we extend a full hypersubstitution of type $(\tau_{n,i}, \tau'_{m,j})$ to a mapping $\widehat{\sigma}$ defining on the set $W^F_{\tau_{n,i}}(X_n) \cup \mathcal{F}^F_{(\tau_{n,i},\tau'_{m,j})}(W^F_{\tau_{n,i}}(X_m))$ of all full terms of type $\tau_{n,i}$ and all full formulas of type $(\tau_{n,i}, \tau'_{m,j})$ as follows.

Definition 3.3. Let σ be a full hypersubstitution of type $(\tau_{n,i}, \tau'_{m,j})$. Then σ induces a mapping

$$\widehat{\sigma} : W^F_{\tau_{n,i}}(X_n) \cup \mathcal{F}^F_{(\tau_{n,i},\tau'_{m,j})}(W^F_{\tau_{n,i}}(X_m)) \to W^F_{\tau_{n,i}}(X_n) \cup \mathcal{F}^F_{(\tau_{n,i},\tau'_{m,j})}(W^F_{\tau_{n,i}}(X_m)),$$

by the following settings.

(1) $\widehat{\sigma}[f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})] := (\sigma(f_i))_\alpha$ for every $\alpha \in H_n$.
(2) $\widehat{\sigma}[f_i(t_1, \ldots, t_n)] := S^n(\sigma(f_i), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n])$.
(3) $\widehat{\sigma}[s \approx t] := \widehat{\sigma}[s] \approx \widehat{\sigma}[t]$.
(4) $\widehat{\sigma}[\gamma_j(t_1, \ldots, t_m)] := R^m(\sigma(\gamma_j), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_m])$ for $t_1, \ldots, t_m \in W^F_{\tau_{n,i}}(X_m)$.
(5) $\widehat{\sigma}[\neg F] := \neg\widehat{\sigma}[F]$ for $F \in \mathcal{F}^F_{(\tau_{n,i},\tau'_{m,j})}(W^F_{\tau_{n,i}}(X_m))$.
(6) $\widehat{\sigma}[F_1 \vee F_2] := \widehat{\sigma}[F_1] \vee \widehat{\sigma}[F_2]$ for $F_1, F_2 \in \mathcal{F}^F_{(\tau_{n,i},\tau'_{m,j})}(W^F_{\tau_{n,i}}(X_m))$.

Example 3.4. Let $(\tau_{3,1}, \tau'_{2,1})$ be a type, i.e., we have one ternary operation symbol, say f and one binary relation symbol, say γ. Let $\sigma : \{f\} \cup \{\gamma\} \to W^F_{\tau_{3,2}}(X_3) \cup \mathcal{F}^F_{(\tau_{3,1},\tau'_{2,1})}(W^F_{\tau_{3,1}}(X_2))$ where $\sigma(f) = f(x_2, x_1, x_3)$ and $\sigma(\gamma) = f(x_1, x_2, x_1) \approx f(x_2, x_2, x_1)$. Then we have

$$\begin{aligned}
\widehat{\sigma}[f(x_3, x_2, x_2)] &= S^3(\sigma(f), \widehat{\sigma}[x_3], \widehat{\sigma}[x_2], \widehat{\sigma}[x_2]) \\
&= S^3(f(x_2, x_1, x_3), x_3, x_2, x_2) \\
&= f(x_2, x_3, x_2), \\
\text{and } \widehat{\sigma}[\gamma(x_2, x_1)] &= R^2(\sigma(\gamma), \widehat{\sigma}[x_2], \widehat{\sigma}[x_1]) \\
&= R^2(f(x_1, x_2, x_1) \approx f(x_2, x_2, x_1), x_2, x_1) \\
&= f(x_2, x_1, x_2) \approx f(x_1, x_1, x_2).
\end{aligned}$$

By using the definition of $\widehat{\sigma}$ and the usual composition of mappings, we can set a binary operation $\circ_r$ on $Hyp^F(\tau_{n,i}, \tau'_{m,j})$ as follows:

Definition 3.5. Let $\sigma_1, \sigma_2 \in Hyp^F(\tau_{n,i}, \tau'_{m,j})$. A binary operation $\circ_r$ on $Hyp^F(\tau_{n,i}, \tau'_{m,j})$ is defined by the following assertion:

$$\sigma_1 \circ_r \sigma_2 := \widehat{\sigma}_1 \circ \sigma_2.$$

By this defining, $\widehat{\sigma}_1 \circ \sigma_2$ sends $\{f_i \mid i \in I\}$ to $W^F_{\tau_{n,i}}(X_n)$ and sends $\{\gamma_j \mid j \in J\}$ to $\mathcal{F}^F_{(\tau_{n,i},\tau'_{m,j})}(W^F_{\tau_{n,i}}(X_m))$, respectively. So we obtain the result of any two elements in $Hyp^F(\tau_{n,i}, \tau'_{m,j})$ operating by $\circ_r$.

In order to prove the associativity of the binary operation $\circ_r$, we need some preparations.

Lemma 3.6. *Let $\sigma \in Hyp^F(\tau_{n,i}, \tau'_{m,j})$ and $\beta \in H_n$. Then*

(1) $S^n(t, \widehat{\sigma}[t_{\beta(1)}], \ldots, \widehat{\sigma}[t_{\beta(n)}]) = S^n(t_\beta, \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n])$.
(2) $R^m(F, \widehat{\sigma}[t_{\beta(1)}], \ldots, \widehat{\sigma}[t_{\beta(m)}]) = R^m(F_\beta, \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_m])$.

Proof. (1) Let $t \in W^F_{\tau_{n,i}}(X_n)$. We give a proof by induction on the complexity of the full term t. If $t = f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$ where $\alpha \in H_n$, then

$$\begin{aligned} &S^n(f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)}), \widehat{\sigma}[t_{\beta(1)}], \ldots, \widehat{\sigma}[t_{\beta(n)}]) \\ &= f_i(\widehat{\sigma}[t_{\beta(\alpha(1))}], \ldots, \widehat{\sigma}[t_{\beta(\alpha(n))}]) \\ &= S^n(f_i(x_{\beta(\alpha(1))}, \ldots, x_{\beta(\alpha(n))}), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n])) \\ &= S^n((f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)}))_\beta, \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n])). \end{aligned}$$

If $t = f_i(s_1, \ldots, s_n)$, where $s_1, \ldots, s_n \in W^F_{\tau_{n,i}}(X_n)$, and assume that

$$S^n(s_i, \widehat{\sigma}[t_{\beta(1)}], \ldots, \widehat{\sigma}[t_{\beta(n)}]) = S^n((s_i)_\beta, \widehat{\sigma}[t_1, \ldots, t_n])$$

for all $i = 1, \ldots, n$, then

$$\begin{aligned} &S^n(f_i(s_1, \ldots, s_n), \widehat{\sigma}[t_{\beta(1)}], \ldots, \widehat{\sigma}[t_{\beta(n)}]) \\ &= f_i(S^n(s_1, \widehat{\sigma}[t_{\beta(1)}], \ldots, \widehat{\sigma}[t_{\beta(n)}]), \ldots, S^n(s_n, \widehat{\sigma}[t_{\beta(1)}], \ldots, \widehat{\sigma}[t_{\beta(n)}])) \\ &= f_i(S^n((s_1)_\beta, \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n]), \ldots, S^n((s_n)_\beta, \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n])) \\ &= S^n(f_i((s_1)_\beta, \ldots, (s_n)_\beta), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n]) \\ &= S^n((f_i(s_1, \ldots, s_n))_\beta, \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n]). \end{aligned}$$

Therefore, we have $S^n(t, \widehat{\sigma}[t_{\beta(1)}], ..., \widehat{\sigma}[t_{\beta(n)}]) = S^n(t_\beta, \widehat{\sigma}[t_1], ..., \widehat{\sigma}[t_n])$ for all $t \in W^F_{\tau_{n,i}}(X_n)$.

(2) Let $F \in \mathcal{F}^F_{(\tau_{n,i},\tau'_{m,j})}(W^F_{\tau_{n,i}}(X_m))$. We give a proof by the following steps. If F is a full formula $s \approx t$, then

$$\begin{aligned} &R^m(s \approx t, \widehat{\sigma}[t_{\beta(1)}], \ldots, \widehat{\sigma}[t_{\beta(m)}]) \\ &= S^m(s, \widehat{\sigma}[t_{\beta(1)}], \ldots, \widehat{\sigma}[t_{\beta(m)}]) \approx S^m(t, \widehat{\sigma}[t_{\beta(1)}], \ldots, \widehat{\sigma}[t_{\beta(m)}]) \\ &= S^m(s_\beta, \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_m]) \approx S^m(t_\beta, \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_m]) \\ &= R^m((s \approx t)_\beta, \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_m]). \end{aligned}$$

If F is a full formula $\gamma_j(s_1, \ldots, s_m)$, where $s_1, \ldots, s_m \in W^F_{\tau_{n,i}}(X_m)$, then

$$\begin{aligned} &R^m(\gamma_j(s_1, \ldots, s_m), \widehat{\sigma}[t_{\beta(1)}], \ldots, \widehat{\sigma}[t_{\beta(m)}]) \\ &= \gamma_j(S^m(s_1, \widehat{\sigma}[t_{\beta(1)}], \ldots, \widehat{\sigma}[t_{\beta(m)}]), \ldots, S^m(s_m, \widehat{\sigma}[t_{\beta(1)}], \ldots, \widehat{\sigma}[t_{\beta(m)}])) \\ &= \gamma_j(S^m((s_1)_\beta, \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_m]), \ldots, S^m((s_m)_\beta, \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_m])) \\ &= R^m(\gamma_j((s_1)_\beta, \ldots, (s_m)_\beta), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_m]) \\ &= R^m((\gamma_j(s_1, \ldots, s_m))_\beta, \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_m]). \end{aligned}$$

Assume that a full formula F satisfies the statement (2). Then

$R^m(\neg F, \widehat{\sigma}[t_{\beta(1)}], \ldots, \widehat{\sigma}[t_{\beta(m)}])$

$$
\begin{aligned}
&= \neg R^m(F, \widehat{\sigma}[t_{\beta(1)}], \ldots, \widehat{\sigma}[t_{\beta(m)}]) \\
&= \neg R^m(F_\beta, \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_m]) \\
&= R^m(\neg(F_\beta), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_m]) \\
&= R^m((\neg F)_\beta, \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_m]).
\end{aligned}
$$

Assume that full formulas F_1 and F_2 satisfy the statement (2). Then

$$
\begin{aligned}
&R^m(F_1 \vee F_2, \widehat{\sigma}[t_{\beta(1)}], \ldots, \widehat{\sigma}[t_{\beta(m)}]) \\
&\quad= R^m(F_1, \widehat{\sigma}[t_{\beta(1)}], \ldots, \widehat{\sigma}[t_{\beta(m)}]) \vee R^n(F_2, \widehat{\sigma}[t_{\beta(1)}], \ldots, \widehat{\sigma}[t_{\beta(m)}]) \\
&\quad= R^m((F_1)_\beta, \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_m]) \vee R^m((F_2)_\beta, \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_m]) \\
&\quad= R^m((F_1 \vee F_2)_\beta, \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_m]).
\end{aligned}
$$

Therefore, we have $R^m(F, \widehat{\sigma}[t_{\beta(1)}], \ldots, \widehat{\sigma}[t_{\beta(m)}]) = R^m(F_\beta, \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_m])$ for all $F \in \mathcal{F}^F_{(\tau_{n,i},\tau'_{m,j})}(W^F_{\tau_{n,i}}(X_m))$. ∎

The following theorem shows that σ is an endomorphism on $Formclone^F(\tau_{n,i}, \tau'_{m,j})$. We apply the results of Theorem 2.8 and 3.6 to prove this fact.

Theorem 3.7. *Let $\sigma \in Hyp^F(\tau_{n,i}, \tau'_{m,j})$. Then the following assertions hold:*

(1) $\widehat{\sigma}[S^n(t, t_1, \ldots, t_n)] = S^n(\widehat{\sigma}[t], \widehat{\sigma}(t_1), \ldots, \widehat{\sigma}(t_n))$.
(2) $\widehat{\sigma}[R^n(F, t_1, \ldots, t_m)] = R^n(\widehat{\sigma}[F], \widehat{\sigma}(t_1), \ldots, \widehat{\sigma}(t_m))$.

Proof. (1) We can give a proof by induction on the complexity of a full term t by applying the results of Lemma 3.6 and Theorem 2.8.
(2) Let $F \in \mathcal{F}^F_{(\tau_{n,i},\tau'_{m,j})}(W^F_{\tau_{n,i}}(X_m))$. We give a proof by the following steps. If F is a full formula $s \approx t$, then

$$
\begin{aligned}
&\widehat{\sigma}[R^m(s \approx t, t_1, \ldots, t_m)] \\
&\quad= \widehat{\sigma}[S^m(s, t_1, \ldots, t_m)] \approx \widehat{\sigma}[S^m(t, t_1, \ldots, t_m)] \\
&\quad= S^m(\widehat{\sigma}[s], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_m]) \approx S^m(\widehat{\sigma}[t], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_m]) \\
&\quad= R^m(\widehat{\sigma}[s \approx t], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_m]).
\end{aligned}
$$

If F is a full formula $\gamma_j(s_1, \ldots, s_m)$, where $s_1, \ldots, s_m \in W^F_{\tau_{n,i}}(X_m)$, then

$$
\begin{aligned}
&\widehat{\sigma}[R^m(\gamma_j(s_1, \ldots, s_m), t_1, \ldots, t_m)] \\
&\quad= R^m(\sigma(\gamma_j), \widehat{\sigma}[S^m(s_1, t_1, \ldots, t_m)], \ldots, \widehat{\sigma}[S^m(s_m, t_1, \ldots, t_m)]) \\
&\quad= R^m(\sigma(\gamma_j), S^m(\widehat{\sigma}[s_1], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_m]), \ldots, S^m(\widehat{\sigma}[s_m], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_m])) \\
&\quad= R^m(R^m(\sigma(\gamma_j), \widehat{\sigma}[s_1], \ldots, \widehat{\sigma}[s_m]), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_m]) \\
&\quad= R^m(\widehat{\sigma}[\gamma_j(s_1, \ldots, s_m)], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_m]).
\end{aligned}
$$

Assume that a full formula F satisfies the statement (2). Then we have to show that it is also satisfied for $\neg F$. Finally, assume that full formulas F_1 and F_2 satisfy the statement (2). Then we have to show that it is also satisfied for $F_1 \vee F_2$. ∎

The proof of the following two lemmas is straightforward and its will be used to prove the next two theorems.

Lemma 3.8. *Let $t \in W^F_{\tau_{n,i}}(X_n), F \in \mathcal{F}^F_{(\tau_{n,i},\tau'_{m,j})}(W^F_{\tau_{n,i}}(X_m)), \beta, \gamma \in H_n$. Then*

$$t_{\gamma \circ \beta} = (t_\beta)_\gamma \text{ and } F_{\gamma \circ \beta} = (F_\beta)_\gamma.$$

Lemma 3.9. *Let $t \in W^F_{\tau_{n,i}}(X_n), F \in \mathcal{F}^F_{(\tau_{n,i},\tau'_{m,j})}(W^F_{\tau_{n,i}}(X_m)), \beta \in H_n$. Then*

$$\widehat{\sigma}[t_\beta] = \widehat{\sigma}[t]_\beta \text{ and } \widehat{\sigma}[F_\beta] = \widehat{\sigma}[F]_\beta.$$

The following theorem shows that the extension of a multiplication is equal to the composition of the extensions by induction on the complexity of a full term t and the definition of a full formula F.

Theorem 3.10. *Let $\sigma_1, \sigma_2 \in Hyp^F(\tau_{n,i}, \tau'_{m,j})$. Then*

$$(\sigma_1 \circ_r \sigma_2)\widehat{} = \widehat{\sigma}_1 \circ \widehat{\sigma}_2.$$

Proof. Let $t \in W^F_{\tau_{n,i}}(X_n)$. We give a proof by induction on the complexity of a full term t. If $t = f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$ where $\alpha \in H_n$, then

$$\begin{aligned}(\sigma_1 \circ_r \sigma_2)\widehat{}[f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})] &= ((\sigma_1 \circ_r \sigma_2)(f_i))_\alpha \text{ (by Definition 3.3)}\\ &= \widehat{\sigma}_1[\widehat{\sigma}_2(f_i)]_\alpha\\ &= \widehat{\sigma}_1[\widehat{\sigma}_2(f_i)_\alpha] \text{ (by Lemma 3.9)}\\ &= \widehat{\sigma}_1[\widehat{\sigma}_2[f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})]] \text{ (by Definition 3.3)}\\ &= (\widehat{\sigma}_1 \circ \widehat{\sigma}_2)[f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})].\end{aligned}$$

If $t = f_i(s_1, \ldots, s_n)$, where $s_1, \ldots, s_n \in W^F_{\tau_{n,i}}(X_n)$, and assume that

$$(\sigma_1 \circ_r \sigma_2)\widehat{}[s_i] = (\widehat{\sigma}_1 \circ \widehat{\sigma}_2)[s_i]$$

for all $i = 1, \ldots, n$, then

$$\begin{aligned}(\sigma_1 \circ_r \sigma_2)\widehat{}[f_i(s_1, \ldots, s_n)] &= S^n((\widehat{\sigma}_1 \circ \widehat{\sigma}_2)(f_i), (\widehat{\sigma}_1 \circ \widehat{\sigma}_2)[s_1], \ldots, (\widehat{\sigma}_1 \circ \widehat{\sigma}_2)[s_n])\\ &= S^n(\widehat{\sigma}_1[\widehat{\sigma}_2(f_i)], \widehat{\sigma}_1[\widehat{\sigma}_2[s_1]], \ldots, \widehat{\sigma}_1[\widehat{\sigma}_2[s_n]])\\ &= \widehat{\sigma}_1[S^n([\sigma_2(f_i)], \widehat{\sigma}_2[s_1], \ldots, \widehat{\sigma}_2[s_n])] \text{ (by Lemma 3.7)}\\ &= (\widehat{\sigma}_1 \circ \widehat{\sigma}_2)[f_i(s_1, \ldots, s_n)].\end{aligned}$$

Let $F \in \mathcal{F}^F_{(\tau_{n,i}, \tau'_{m,j})}(W^F_{\tau_{n,i}}(X_m))$. We give a proof by the following steps. If F is a full formula $s \approx t$, then

$$\begin{aligned}(\sigma_1 \circ_r \sigma_2)\widehat{}[s \approx t] &= (\sigma_1 \circ_r \sigma_2)\widehat{}[s] \approx (\sigma_1 \circ_r \sigma_2)\widehat{}[t]\\ &= (\widehat{\sigma}_1 \circ \widehat{\sigma}_2)[s] \approx (\widehat{\sigma}_1 \circ \widehat{\sigma}_2)[t]\\ &= (\widehat{\sigma}_1 \circ \widehat{\sigma}_2)[s \approx t].\end{aligned}$$

If F is a full formula $\gamma_j(s_1, \ldots, s_m)$, where $s_1, \ldots, s_m \in W^F_{\tau_{n,i}}(X_m)$, then

$$\begin{aligned}(\sigma_1 \circ_r \sigma_2)\widehat{}[\gamma_j(s_1, \ldots, s_m)] &= R^m((\widehat{\sigma}_1 \circ \widehat{\sigma}_2)(\gamma_j), (\widehat{\sigma}_1 \circ \widehat{\sigma}_2)[s_1], \ldots, (\widehat{\sigma}_1 \circ \widehat{\sigma}_2)[s_m])\\ &= R^m(\widehat{\sigma}_1[\widehat{\sigma}_2(\gamma_j)], \widehat{\sigma}_1[\widehat{\sigma}_2[s_1]], \ldots, \widehat{\sigma}_1[\widehat{\sigma}_2[s_m]])\\ &= \widehat{\sigma}_1[R^m([\sigma_2(\gamma_j)], \widehat{\sigma}_2[s_1], \ldots, \widehat{\sigma}_2[s_m])] \text{ (by Lemma 3.7)}\\ &= (\widehat{\sigma}_1 \circ \widehat{\sigma}_2)[\gamma_j(s_1, \ldots, s_m)].\end{aligned}$$

Assume that a full formula F satisfies the theorem. Then we have to show that it is also satisfied for $\neg F$. Finally, assume that full formulas F_1 and F_2 satisfy. Then we have to show that it is also satisfied for $F_1 \vee F_2$. ∎

The following theorem is the most important result because the paper is presenting the new algebraic structure.

Theorem 3.11. *The algebra $(Hyp^F(\tau_{n,i}, \tau'_{m,j}), \circ_r)$ is a semigroup.*

Proof. Using Lemma 3.10 and the natural fact that the usual composition $\circ$ is associative, it can be shown that $\circ_r$ is an associative binary operation on $Hyp^F(\tau_{n,i}, \tau'_{m,j})$. In fact, for any $\sigma_1, \sigma_2, \sigma_3 \in Hyp^F(\tau_{n,i}, \tau'_{m,j})$, we have

$$\begin{aligned}(\sigma_1 \circ_r \sigma_2) \circ_r \sigma_3 &= (\sigma_1 \circ_r \sigma_2)\widehat{} \circ \sigma_3\\ &= (\widehat{\sigma}_1 \circ \widehat{\sigma}_2) \circ \sigma_3\\ &= \widehat{\sigma}_1 \circ (\widehat{\sigma}_2 \circ \sigma_3)\\ &= \sigma_1 \circ_r (\sigma_2 \circ_r \sigma_3).\end{aligned}$$

It implies that the structure $(Hyp^F(\tau_{n,i}, \tau'_{m,j}), \circ_r)$ forms a semigroup. ∎

4. Conclusions and Recommendations for the Future Work

For the first main result of the paper, full formulas of type $(\tau_{n,i}, \tau'_{m,j})$ induced by full terms of type $\tau_{n,i}$ were introduced. The algebraic structure which is called the clone of full formulas of type $(\tau_{n,i}, \tau'_{m,j})$ was constructed. Such algebra is a couple of the set of full formulas and the $(n+1)$-ary superposition operation defined on this set. Furthemore, the concept of full hypersubstitutions for algebraic systems of type $(\tau_{n,i}, \tau'_{m,j})$ and their extensions were established. This mapping is closely connected to endomorphism of the clone of full formulas. Finally, we proved that the set of all full hypersubstitutions for algebraic systems of type $(\tau_{n,i}, \tau'_{m,j})$ with one associative binary operation forms a semigroup.

Finally, a number of suggestions for future research work in this area are given.

(1) Study some fundamental properties of the semigroup of all full hypersubstitutions for algebraic systems of type $(\tau_{n,i}, \tau'_{m,j})$. Find the order of its elements. Charaterize the idempotent elements and determine several kinds of regular elements. Investigate the Green's relations.

(2) Apply another type of terms to define the formulas genereted by those terms. Construct the clone of such formulas and study some algebraic properties of its.

(3) Define the concepts of nondeterministic full hypersubstitutions for algebraic systems. Giving the connection between them and the results presented in this paper (see the paper [15] for this research area).

(4) Classify all varieties $\mathcal{V}$ of algebraic systems of some types by replace the operation by the term operation induced by a full term and replace the relation by the relation induced by a full formula (see for example [16]).

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