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On Common Fixed Point of Nonself Nonexpansive Mappings for Multistep Iteration in Banach Spaces

H. Kiziltunc[1] and I. Yildirim

Abstract : Suppose that E be a uniformly convex Banach space, let K be a nonempty closed convex subset of E with P as a nonexpansive retraction. Let $T : K \to E$ be a given nonself mapping. The modified multistep iterative scheme $\{x_n\}$ is defined by (1.10). We establish the weak convergence of a sequence of a modified multistep iteration of an nonself I-nonexpansive map in a Banach space which satisfies Opial's condition.

Keywords : Mann, Ishikawa, Noor and Multistep iterations; nonself nonexpansive maps
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1. Introduction and preliminaries

Let E be a normed linear space, K a nonempty, convex subset of E, and T a self map of K. Three most popular iteration procedures for obtaining fixed points of T, if they exist, are Mann iteration [12], defined by

$$u_1 \in K, \quad u_{n+1} = (1-\alpha_n)u_n + \alpha_n T u_n, \quad n \geq 1 \tag{1.1}$$

Ishikawa iteration [13], defined by

$$\begin{aligned} z_1 \in K, \quad z_{n+1} &= (1-\alpha_n)z_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)z_n + \beta_n T z_n, \quad n \geq 1, \end{aligned} \tag{1.2}$$

Noor iteration [14], defined by

$$\begin{aligned} v_1 \in K, \quad v_{n+1} &= (1-\alpha_n)v_n + \alpha_n T w_n, \\ w_n &= (1-\beta_n)v_n + \beta_n T t_n, \\ t_n &= (1-\gamma_n)v_n + \gamma_n T v_n, \quad n \geq 1, \end{aligned} \tag{1.3}$$

[1] Corresponding author.
E-mail address: hukmu@atauni.edu.tr

for certain choices of $\{\alpha_n\}$,$\{\beta_n\}$ and $\{\gamma_n\} \subset [0,1]$.

The multistep iteration [15], arbitrary fixed order $p \geq 2$, defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n^1, \tag{1.4}$$
$$y_n^i = (1-\beta_n^i)x_n + \beta_n^i T y_n^{i+1}, \quad i = 1,2,...,p-2$$
$$y_n^{p-1} = (1-\beta_n^{p-1})x_n + \beta_n^{p-1} T x_n.$$

The sequence $\{\alpha_n\}$ is such that for all $n \in \mathbb{N}$

$$\{\alpha_n\} \subset (0,1), \quad \lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty$$

and for all $n \in \mathbb{N}$

$$\left\{\beta_n^i\right\} \subset [0,1), \quad 1 \leq i \leq p-1, \quad \lim_{n\to\infty} \beta_n^1 = 0.$$

In the above taking $p = 3$ in (1.4) we obtain iteration (1.3). Taking $p = 2$ in (1.4) we obtain iteration (1.2).

The following definitions and statements will be needed for the proof of our theorem.

Let K be a subset of normed linear space $E = (E, \|\cdot\|)$ and T self-mappings of K. Then T is called nonexpansive on K if

$$\|Tx - Ty\| \leq \|x - y\| \tag{1.5}$$

for all $x, y \in K$. Let $F(T) := \{x \in K : Tx = x\}$ be denoted as the set of fixed points of a mapping T.

Let K be a subset of a normed linear space $E = (E, \|\cdot\|)$ and T and I self-mappings of K. Then T is called I-nonexpansive on K if

$$\|Tx - Ty\| \leq \|Ix - Iy\| \tag{1.6}$$

for all $x, y \in K$ [9].

T is called I-quasi-nonexpansive on K if

$$\|Tx - f\| \leq \|Ix - f\| \tag{1.7}$$

for all $x, y \in K$ and $f \in F(T) \cap F(I)$.

Let E be a real Banach space. A subset K of E is said to be a retract of E if there exists a continuous map $P : E \to K$ such that $Px = x$ for all $x \in K$. A map $P : E \to E$ is said to be a retraction if $P^2 = P$. It follows that if a map P is a retraction, then $Py = y$ for all y in the range of P. A set K is optimal if each point outside K can be moved to be closer to all points of K. Note that every nonexpansive retract is optimal. In strictly convex Banach spaces, optimal sets are closed and convex. However, every closed convex subset of a Hilbert space is optimal and also a nonexpansive retract.

Recall that a Banach space E is said to satisfy Opial's condition [6] if, for each sequence $\{x_n\}$ in E, the condition $x_n \to x$ implies that

$$\overline{\lim_{n\to\infty}} \|x_n - x\| < \overline{\lim_{n\to\infty}} \|x_n - y\| \tag{1.8}$$

for all $y \in E$ with $y \neq x$.

The first nonlinear ergodic theorem was proved by Baillon [1] for general nonexpansive mappings in Hilbert space H: if K is a closed and convex subset of H and T has a fixed point, then for every $x \in K$, $\{T^n x\}$ is weakly almost convergent, as $n \to \infty$, to a fixed point of T. It was also shown by Pazy [7] that if H is a real Hilbert space and $\left(\dfrac{1}{n}\right) \sum_{i=0}^{n-1} T^i x$ converges weakly, as $n \to \infty$, to $y \in K$, then $y \in F(T)$.

The concept of a quasi-nonexpansive mapping was initiated by Tricomi in 1941 for real functions. Diaz and Metcalf [2] and Dotson [3] studied quasi-nonexpansive mappings in Banach spaces. Recently, this concept was given by Kirk [5] in metric spaces which we adapt to a normed space as follows: T is called a quasi-nonexpansive mapping provided

$$\|Tx - f\| \leq \|x - f\| \tag{1.9}$$

for all $x \in K$ and $f \in F(T)$.

Remark 1.1. From the above definitions it is easy to see that if $F(T)$ is nonempty, a nonexpansive mapping must be quasi-nonexpansive, and linear quasi-nonexpansive mappings are nonexpansive. But it is easily seen that there exist nonlinear continuous quasi-nonexpansive mappings which are not nonexpansive.
There are many results on fixed points on nonexpansive and quasi-nonexpansive mappings in Banach spaces and metric spaces. For example, the strong and weak convergence of the sequence of certain iterates to a fixed point of quasi-nonexpansive maps was studied by Petryshyn and Williamson [8]. Their analysis was related to the convergence of Mann iterates studied by Dotson [3]. Subsequently, the convergence of Ishikawa iterates of quasi-nonexpansive mappings in Banach spaces was discussed by Ghosh and Debnath [4]. In [10], the weakly convergence theorem for I-asymptotically quasi-nonexpansive mapping defined in Hilbert space was proved. In [11], convergence theorems of iterative schemes for nonexpansive mappings have been presented and generalized.
In [16], Rhoades and Temir considered T and I self-mappings of K, where T is an I-nonexpansive mapping. They established the weak convergence of the sequence of Mann iterates to a common fixed point of T and I. More precisely, they proved the following theorem.
Theorem 1.2. (Rhoades and Temir [16]): Let K be a closed convex bounded subset of uniformly convex Banach space E, which satisfies Opial's condition, and let T, I self-mappings of K with T be an I-nonexpansive mapping, I a nonexpansive on K. Then, for $x_0 \in K$, the sequence $\{x_n\}$ of Mann iterates converges weakly to common fixed point of $F(T) \cap F(I)$.
In the above theorem, T remains self-mapping of a nonempty closed convex subset

K of a uniformly convex Banach space. If, however, the domain K of T is a proper subset of E and T maps K into E then, the iteration formula (1.1) may fail to be well defined. One method that has been used to overcome this in the case of single operator T is to introduce a retraction $P : E \to K$ in the recursion formula (1.1) as follows: $u_1 \in K$,

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n PTu_n, \quad n \geq 1,$$

In [17], Kızıltunc and Ozdemir considered T and I nonself-mappings of K, where T is an I-nonexpansive mapping. They established the weak convergence of the sequence of modified Ishikawa iterates to a common fixed point of T and I. More precisely, they proved the following theorem.

Theorem 1.3. (Kızıltunc and Ozdemir [17]): Let K be a closed convex bounded subset of uniformly convex Banach space E, which satisfies Opial's condition, and let T, I nonself mappings of K with T be an I-nonexpansive mapping, I a nonexpansive on K. Then, for $x_0 \in K$, the sequence $\{x_n\}$ of modified Ishikawa iterates converges weakly to common fixed point of $F(T) \cap F(I)$.

In this study, we consider T and I nonself mappings of K, where T is an I-nonexpansive mappings. We establish the weak convergence of the sequence of modified multistep iterates to a common fixed point of T and I.

Let E be a normed linear space, K be a nonempty convex subset of E with P as a nonexpansive retraction. Let $T : K \to E$ be a given nonself mapping. The modified multistep iterative scheme $\{x_n\}$ is defined by, arbitrary fixed order $p \geq 2$

$$\begin{aligned} x_{n+1} &= P\left((1 - \alpha_n)x_n + \alpha_n T y_n^1\right), \qquad (1.10) \\ y_n^i &= P\left((1 - \beta_n^i)x_n + \beta_n^i T y_n^{i+1}\right), \quad i = 1, 2, ..., p-2 \\ y_n^{p-1} &= P\left((1 - \beta_n^{p-1})x_n + \beta_n^{p-1} T x_n\right), \end{aligned}$$

where the sequence $\{\alpha_n\}$ is such that for all $n \in \mathbb{N}$,

$$\{\alpha_n\} \subset (0, 1), \quad \lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty$$

and for all $n \in \mathbb{N}$,

$$\left\{\beta_n^i\right\} \subset [0, 1), \quad 1 \leq i \leq p - 1, \quad \lim_{n\to\infty} \beta_n^1 = 0.$$

Clearly, if T is self maps, then (1.10) reduces to an iteration scheme (1.4).

2. The main result

Theorem 2.1. Let K be a closed convex bounded subset of uniformly convex Banach space E, which satisfies Opial's condition, and let T, I nonself mappings of K with T be an I-nonexpansive mapping, I a nonexpansive on K. Then, for $x_0 \in K$, the sequence $\{x_n\}$ of modified multistep iterates converges weakly to common fixed point of $F(T) \cap F(I)$.

Proof. If $F(T) \cap F(I)$ is nonempty and a singleton, then the proof is complete. We will assume that $F(T) \cap F(I)$ is not a singleton.

$$\|x_{n+1} - f\| = \left\|P\left((1-\alpha_n)x_n + \alpha_n T y_n^1\right) - f\right\|$$

$$\leq \left\|(1-\alpha_n)x_n + \alpha_n T y_n^1 - f\right\|$$

$$\leq \left\|(1-\alpha_n)(x_n - f) + \alpha_n\left[P\left((1-\beta_n^1)x_n + \beta_n^1 T y_n^2\right) - f\right]\right\|$$

$$\leq \left\|(1-\alpha_n)(x_n - f) + \alpha_n\left[(1-\beta_n^1)x_n + \beta_n^1 T y_n^2 - f\right]\right\|$$

$$\leq (1-\alpha_n)\|x_n - f\| + \alpha_n\left\|(1-\beta_n^1)x_n + \beta_n^1 T y_n^2 - f\right\|$$

$$\leq (1-\alpha_n)\|x_n - f\| + \alpha_n\left\|(1-\beta_n^1)(x_n - f) + \beta_n^1\left[P\left((1-\beta_n^2)x_n + \beta_n^2 T x_n\right) - f\right]\right\|$$

$$\leq (1-\alpha_n)\|x_n - f\| + \alpha_n\left\|(1-\beta_n^1)(x_n - f) + \beta_n^1\left[(1-\beta_n^2)x_n + \beta_n^2 T x_n - f\right]\right\|$$

$$\leq (1-\alpha_n)\|x_n - f\| + \alpha_n(1-\beta_n^1)\|x_n - f\| + \alpha_n\beta_n^1\left\|(1-\beta_n^2)x_n + \beta_n^2 T x_n - f\right\|$$

$$\leq (1-\alpha_n)\|x_n - f\| + \alpha_n(1-\beta_n^1)\|x_n - f\| + \alpha_n\beta_n^1\left\|(1-\beta_n^2)(x_n - f) + \beta_n^2(x_n - f)\right\|$$

$$\leq (1-\alpha_n)\|x_n - f\| + \alpha_n(1-\beta_n^1)\|x_n - f\| + \alpha_n\beta_n^1(1-\beta_n^2)\|x_n - f\| + \alpha_n\beta_n^1\beta_n^2\|x_n - f\|$$

$$= \|x_n - f\| \tag{2.1}$$

Thus, for $\alpha_n \neq 0$ and $\beta_n^i \neq 0$, $\{\|x_n - f\|\}$ is a nonincreasing sequence. Then, $\lim_{n\to\infty}\|x_n - f\|$ exists.

Now we show that $\{x_n\}$ converges weakly to a common fixed point of T and I. The sequence $\{x_n\}$ contains a subsequence which converges weakly to a point in K. Let $\{x_{n_k}\}$ and $\{x_{m_k}\}$ be two subsequences of $\{x_n\}$ which converge weakly to f and q, respectively. We will show that $f = q$. Suppose that E satisfies

Opial's condition and that $f \neq q$ is in weak limit set of the sequence $\{x_n\}$. Then $\{x_{n_k}\} \rightarrow f$ and $\{x_{m_k}\} \rightarrow q$, respectively. Since $\lim_{n\to\infty} \|x_n - f\|$ exists for any $f \in F(T) \cap F(I)$ by Opial's condition, we conclude that

$$\begin{aligned} \lim_{n\to\infty} \|x_n - f\| &= \lim_{k\to\infty} \|x_{n_k} - f\| \, 2.2 \\ &< \lim_{k\to\infty} \|x_{n_k} - q\| = \lim_{j\to\infty} \left\|x_{m_j} - q\right\| \\ &< \lim_{j\to\infty} \left\|x_{m_j} - f\right\| \\ &= \lim_{n\to\infty} \|x_n - f\| . \end{aligned} \quad (1)$$

This is a contradiction. Thus $\{x_n\}$ converges weakly to an element of $F(T) \cap F(I)$.

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H. KIZILTUNC and I. YILDIRIM
Department of Mathematics
Faculty of Arts and Sciences,
Ataturk University, 25240, Erzurum,
TURKEY.