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Some Fixed Point Theorems in b-Metric Spaces with b-Simulation Functions

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Abstract The more generalized idea of the triangle inequality was introduced so that the concept of metric space was extended to "b-metric space" in 1989 by Bakhtin. Many definitions and theories based on a metric space, e.g. convergent and cauchy sequences, a complete space, a simulation function, the contraction principle, the fixed point theorem, were considered in the b-metric spaces mentioned. In this article the notions of b-simulation functions and generalized $\mathcal{Z}_b$-contraction mappings were proposed. Also the existence of a fixed point for such a mapping in a complete b-metric space was presented.

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1. Introduction

The existence of a fixed points for contraction mappings in complete metric spaces was first investigated by Banach himself who established the well known Banach contraction principle [1] in 1922. It was applied for the existence theory of differential, integral, partial differential and functional equations [2]. It is a tool for providing the existence of solutions in game theory, mathematical economic and some biological models [2, 3]

Since then many authors have extended and improved this and other fixed point results.

In 1989, Bakhtin [4] (see also Czerwik [5]) introduced the concept of a b-metric space (a more general type of metric space) and proved some fixed point theorems for some contraction mappings in b-metric spaces which generalize Banach's contraction principle in metric spaces.

In 2015, Khojasteh et al. [6] introduced the notion of a simulation function in connection with generalization of Banach's contraction principle.

In 2016, Olgun et al. [7] introduced the notion of a generalized $\mathcal{Z}$-contraction and proved the existence of fixed points, using the concept of a simulation function.

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Recently, Roldán-López-de-Hierroet et al. [8] modified the notion of a simulation function and guaranteed the existence and uniqueness of a coincidence point of two nonlinear mappings, using the concept of a simulation function.

Very recently, Demma et al. [9] introduced the notion of b-simulation functions in the setting of b-metric spaces and established the existence and uniqueness of a fixed point in b-metric spaces.

In this paper, we introduce the notion of generalized $\mathcal{Z}_b$-contraction with b-simulation function and prove some fixed point theorems in complete b-metric spaces. Furthermore, we give an example to illustrate the main result. As consequences of this study, several related results of fixed point theory in metric space and b-metric space were deduced.

2. Preliminaries

We begin by giving some notations and preliminaries that we shall need to state our results.

In the sequel, the letters $\mathbb{R}$ and $\mathbb{N}$ will denote the set of all real numbers and the set of all natural numbers, respectively.

Definition 2.1. [10] (Metric space) Let X be a nonempty set. A function $d : X \times X \to [0, \infty)$ is said to be a metric on X if, for all $x, y, z \in X$ the following are condition

(m1) $d(x, y) = 0$ if and only if $x = y$;
(m2) $d(x, y) = d(y, x)$;
(m3) $d(x, y) \leq d(x, z) + d(z, y)$;

The pair (X, d) is called a *metric space.*

Definition 2.2. [4] (b-Metric Space) Let X be a nonempty set and let $b \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is said to be a *b-metric* if for all $x, y, z \in X$ the following conditions are satisfies:

(b1) $d(x, y) = 0$ if and only if $x = y$;
(b2) $d(x, y) = d(y, x)$;
(b3) $d(x, y) \leq b[d(x, z) + d(z, y)]$.

The pair (X, d) is called a *b-metric space* (in short bMS).

Example 2.3. [11] Let the function $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ defined by $d(x, y) = |x - y|^2$. Then d is a b-metric on $\mathbb{R}$ with $b = 2$, but it is not a metric on $\mathbb{R}$, as

$$d(1, 3) = 4 > 2 = d(1, 2) + d(2, 3).$$

Let us show that d is a b-metric on $\mathbb{R}$ with $b = 2$. Consider

$$\begin{aligned} d(x, y) = |x - y|^2 &\leq (|x - z| + |z - y|)^2 \\ &= |x - z|^2 + (2|x - z||z - y|) + |z - y|^2 \\ &\leq |x - z|^2 + (|x - z|^2 + |z - y|^2) + |z - y|^2 \text{ (Remark 2.4)} \\ &= 2\Big(|x - z|^2 + |z - y|^2\Big) \\ &= 2\Big(d(x, z) + d(z, y)\Big). \end{aligned}$$

Remark 2.4. Let $A, B \in \mathbb{R}$.
Since $0 \leq (|A| - |B|)^2 = |A|^2 - 2|A||B| + |B|^2$, $2|A||B| \leq |A|^2 + |B|^2$.

Definition 2.5. [5] (Convergent, Cauchy sequence and Complete) Let $\{x_n\}$ be a sequence in a $b-$metric space (X, d).

(i) $\{x_n\}$ is called $b-$convergent if and only if there is $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$.

(ii) $\{x_n\}$ is a $b-$Cauchy sequence if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

(iii) The $b-$metric space is *Complete* if every Cauchy sequence convergent.

Proposition 2.6. [5] *In a $b-$metric space (X, d), the following assertions hold:*

(i) A $b-$convergent sequence has a unique limit.

(ii) Each $b-$convergent sequence is $b-$Cauchy.

(iii) In general, a $b-$metric is not continuous.

Definition 2.7. [6] (Simulation function) Let $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be a mapping. Then ζ is called a *simulation function* if it satisfies the following conditions:

($\zeta 1$) $\zeta(0, 0) = 0$;

($\zeta 2$) $\zeta(t, s) < s - t$ for all $t, s > 0$;

($\zeta 3$) if $\{t_n\}, \{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$

We denote the set of all simulation functions by $\mathcal{Z}$.

Example 2.8. [6] Let $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be defined by

$$\zeta(t, s) = \lambda s - t$$

for all $t, s \in [0, \infty)$ and $\lambda \in [0, 1)$. Then ζ is a simulation function.

Proof. ($\zeta 1$) $\zeta(0, 0) = \lambda(0) - (0) = 0$.

($\zeta 2$) Let $t, s > 0$

$$\zeta(t, s) = \lambda s - t < s - t.$$

($\zeta 3$) Let $\{t_n\}, \{s_n\}$ be sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = C$ for some $C \in \mathbb{R}^+$.

Then

$$\begin{aligned}\limsup_{n\to\infty} \zeta(t_n, s_n) &= \limsup_{n\to\infty} (\lambda s_n - t_n) \\ &= \lambda \limsup_{n\to\infty} (s_n) - \limsup_{n\to\infty} (t_n) = \lambda C - C < 0.\end{aligned}$$

∎

Example 2.9. [6] (Generalization of Example 2.8) Let $\zeta_1 : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be defined by

$$\zeta_1(t, s) = \psi(s) - \phi(t)$$

for all $t, s \in [0, \infty)$, where $\psi, \phi : [0, \infty) \to [0, \infty)$ are two continuous functions such that $\psi(t) = \phi(t) = 0$ if and only if $t = 0$ and $\psi(t) < t \leq \phi(t)$ for all $t > 0$.

Then ζ_1 is a simulation function.

Proof. ($\zeta 1$) $\zeta_1(0, 0) = \psi(0) - \phi(0) = 0$.

($\zeta 2$) Let $t, s > 0$

$$\zeta_1(t, s) = \psi(s) - \phi(t) < s - t.$$

($\zeta 3$) Let $\{t_n\}, \{s_n\}$ be sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = C$ for some $C \in \mathbb{R}^+$.
Then

$$\begin{aligned}\limsup_{n\to\infty} \zeta_1(t_n, s_n) &= \limsup_{n\to\infty}(\psi(s_n) - \phi(t_n)) \\ &= \limsup_{n\to\infty} \psi(s_n) - \limsup_{n\to\infty} \phi(t_n) \\ &= \psi(\limsup_{n\to\infty} s_n) - \phi(\limsup_{n\to\infty} t_n) \\ &= \psi(C) - \phi(C) < 0.\end{aligned}$$

■

Definition 2.10. [6] ($\mathcal{Z}$-contraction) Let (X, d) be a metric space, $T : X \to X$ a mapping and $\zeta \in \mathcal{Z}$. Then T is called a *$\mathcal{Z}$-contraction* with respect to ζ if the following condition is satisfied

$$\zeta(d(Tx, Ty), d(x, y)) \geq 0, \text{ for all } x, y \in X.$$

If T is a $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$, then $d(Tx, Ty) < d(x, y)$ for all distinct $x, y \in X$.

Theorem 2.11. [6] *Let (X, d) be a complete metric space and $T : X \to X$ be a $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$. Then T has a unique fixed point u in X and for every $x_0 \in X$ the Picard sequence $\{x_n\}$; where $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$ converges to the fixed point of T.*

Definition 2.12. [7] (Generalized $\mathcal{Z}$-contraction) Let (X, d) be a metric space, $T : X \to X$ be a mapping, and $\zeta \in \mathcal{Z}$. Then T is called *generalized $\mathcal{Z}$- contraction* with respect to ζ if the following condition is satisfied

$$\zeta(d(Tx, Ty), M(x, y)) \geq 0 \text{ for all } x, y \in X,$$

where

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}\Big(d(x, Ty) + d(y, Tx)\Big)\right\}.$$

Remark 2.13. [7] Every generalized $\mathcal{Z}$-contraction on a metric space has at most one fixed point. Indeed, let z and w be two fixed points of T , which is a generalized $\mathcal{Z}$-contraction self map of a metric space (X, d). Then

$$0 \leq \zeta(d(Tz, Tw), M(z, w)) = \zeta(d(z, w), d(z, w)),$$

which is a contradiction.

Theorem 2.14. [7] *Let (X, d) be a complete metric space and $T : X \to X$ be a generalized $\mathcal{Z}$-contraction with respect to $\zeta \in \mathbb{Z}$. Then T has a fixed point in X. Moreover, for every $x_0 \in X$, the Picard sequence $\{T^n x_0\}$ converges to this fixed point.*

Definition 2.15. [9] (b-simulation function) Let (X, d) be a b-metric space with a constant $b \geq 1$. A b-simulation function is a function $\xi : [0, \infty) \times [0, \infty) \to \mathbb{R}$, satisfying the following conditions:

($\xi 1$) $\xi(t, s) < s - t$ for all $t, s > 0$;
($\xi 2$) if $\{t_n\}, \{s_n\}$ are sequences in $(0, \infty)$ such that

$$0 < \lim_{n\to\infty} t_n \le \liminf_{n\to\infty} s_n \le \limsup_{n\to\infty} s_n \le b \lim_{n\to\infty} t_n < \infty,$$

then

$$\limsup_{n\to\infty} \xi(bt_n, s_n) < 0.$$

We denote the set of all b-simulation functions by $\mathcal{Z}_b$.

Example 2.16. [9] Let $\xi : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be defined by

$$\xi(t,s) = \lambda s - t$$

for all $t, s \in [0,\infty)$ and $\lambda \in [0,1)$. Then ξ is a b-simulation function.

Proof. (ξ1) Let $t, s > 0$

$$\xi(t,s) = \lambda s - t < s - t.$$

(ξ2) Let $\{t_n\}, \{s_n\}$ be sequences in $(0,\infty)$ such that

$$0 < C = \lim_{n\to\infty} t_n \le \liminf_{n\to\infty} s_n \le \limsup_{n\to\infty} s_n \le b \lim_{n\to\infty} t_n = bC < \infty,$$

for some $C \in \mathbb{R}^+$

Then

$$\begin{aligned}\limsup_{n\to\infty} \xi(bt_n, s_n) &= \limsup_{n\to\infty} (\lambda s_n - bt_n)\\ &= \lambda \limsup_{n\to\infty}(s_n) - b \limsup_{n\to\infty}(t_n) \le \lambda bC - bC < 0.\end{aligned}$$

■

Theorem 2.17. [9] *Let (X,d) be a complete b-metric space with a constant $b \ge 1$ and let $T : X \to X$ be a mapping. Suppose that there exists a b-simulation function ξ such that*

$$\xi(bd(Tx,Ty), d(x,y)) \ge 0$$

for all $x, y \in X$. Then T has a unique fixed point.

3. Main Results

In this section, we define the generalized $\mathcal{Z}_b$-contraction and prove the existence of a fixed point for such mapping in complete b-metric spaces.

Definition 3.1. Let (X,d) be a b-metric spaces with a constant $b \ge 1, T : X \to X$ be a mapping, and $\xi \in \mathcal{Z}_b$. Then T is called generalized $\mathcal{Z}_b$-contraction with respect to ξ if the following condition is satisfied

$$\xi(bd(Tx,Ty), M_b(x,y)) \ge 0 \text{ for all } x, y \in X, \tag{3.1}$$

where

$$M_b(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2b}\Big(d(x,Ty) + d(y,Tx)\Big)\right\}.$$

Lemma 3.2. *Let (X,d) be a b-metric space with constant $b \ge 1$ and let $T : X \to X$ be a generalized $\mathcal{Z}_b$-contraction with be respect to $\xi \in \mathcal{Z}_b$. Let $\{x_n\}$ be a Picard sequence with initial point $x_0 \in X$. Then*

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$$

Proof. Let $x_0 \in X$ be arbitrary and $\{x_n\}$ be a Picard sequence in X, that is, $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$.
If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$ then x_{n_0} is a fixed point of T and the assertion follows. On the other hand, suppose that $d(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$. Then, since

$$\begin{aligned} M_b(x_n, x_{n-1}) &= \max\left\{ \begin{array}{c} d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n), \\ \frac{1}{2b}\Big(d(x_n, x_n) + d(x_{n-1}, x_{n+1})\Big) \end{array} \right\} \\ &= \max\Big\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\Big\}. \end{aligned}$$

From (3.1) and property ($\xi 1$), we have

$$\begin{aligned} 0 &\le \xi\Big(bd(x_{n+1}, x_n), M_b(x_n, x_{n-1})\Big) \\ &= \xi\Big(bd(x_{n+1}, x_n), \max\Big\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\Big\}\Big) \\ &< \max\Big\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\Big\} - bd(x_{n+1}, x_n). \end{aligned} \tag{3.2}$$

If $d(x_n, x_{n+1}) \ge d(x_{n-1}, x_n)$ for some $n \in \mathbb{N}$, then from (3.2), we get

$$0 < d(x_n, x_{n+1}) - bd(x_{n+1}, x_n),$$

so

$$bd(x_{n+1}, x_n) < d(x_{n+1}, x_n),$$

hence

$$b < 1,$$

which is a contradiction. Thus $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$ and

$$0 \le \xi\Big(bd(x_n, x_{n+1}), d(x_{n-1}, x_n)\Big). \tag{3.3}$$

So, the sequence $\{d(x_n, x_{n+1})\}$ is a decreasing sequence of nonnegative real numbers. Hence there exist $r \ge 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$. Assume $r > 0$. Applying the property ($\xi 2$), with $t_n = d(x_n, x_{n+1})$ and $s_n = d(x_{n-1}, x_n)$, it follows that

$$\limsup_{n\to\infty} \xi\Big(bd(x_n, x_{n+1}), d(x_{n-1}, x_n)\Big) < 0,$$

which contradicts (3.3). Therefore,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

■

Lemma 3.3. *Let (X, d) be a b-metric space with constant $b \ge 1$ and let $T : X \to X$ be a generalized $\mathcal{Z}_b$-contraction with respect to $\xi \in \mathcal{Z}_b$. Let $\{x_n\}$ be a Picard sequence with initial point $x_0 \in X$. Suppose that $x_{n-1} \ne x_n$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ is a bounded sequence.*

Proof. Assume that $\{x_n\}$ is not a bounded sequence. Then there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $n_1 = 1$ and, for each $k \in \mathbb{N}, n_{k+1}$ is the minimum integer such that

$$d(x_{n_{k+1}}, x_{n_k}) > 1 \tag{3.4}$$

and

$$d(x_m, x_{n_k}) \leq 1 \text{ for all integers } m \text{ such that } n_k \leq m \leq n_{k+1} - 1. \tag{3.5}$$

By ($b3$) of Definition 2.2 and (3.4), we get

$$\begin{aligned} 1 < d(x_{n_{k+1}}, x_{n_k}) &\leq bd(x_{n_{k+1}}, x_{n_{k+1}-1}) + bd(x_{n_{k+1}-1}, x_{n_k}) \\ &\leq bd(x_{n_{k+1}}, x_{n_{k+1}-1}) + b. \end{aligned} \tag{3.6}$$

Letting $k \to \infty$ in (3.6) and using Lemma 3.2, we obtain

$$1 \leq \liminf_{k\to\infty} d(x_{n_{k+1}}, x_{n_k}) \leq \limsup_{k\to\infty} d(x_{n_{k+1}}, x_{n_k}) \leq b.$$

From (3.1) and property ($\xi 1$), we have

$$\begin{aligned} 0 &\leq \xi\Big(bd(x_{n_{k+1}}, x_{n_k}), M_b(x_{n_{k+1}-1}, x_{n_k-1})\Big) \\ &< M_b(x_{n_{k+1}-1}, x_{n_k-1}) - bd(x_{n_{k+1}}, x_{n_k}) \\ bd(x_{n_{k+1}}, x_{n_k}) &< M_b(x_{n_{k+1}-1}, x_{n_k-1}). \end{aligned} \tag{3.7}$$

Since

$$\begin{aligned} M_b(x_{n_{k+1}-1}, x_{n_k-1}) &= \max\left\{ \begin{array}{c} d(x_{n_{k+1}-1}, x_{n_k-1}), d(x_{n_{k+1}-1}, x_{n_{k+1}}), d(x_{n_k-1}, x_{n_k}, \\ \frac{1}{2b}\Big(d(x_{n_{k+1}-1}, x_{n_k}) + d(x_{n_k-1}, x_{n_{k+1}})\Big) \end{array} \right\} \\ &\leq \max\left\{ \begin{array}{c} b\Big(d(x_{n_{k+1}-1}, x_{n_k}) + d(x_{n_k}, x_{n_k-1})\Big), d(x_{n_{k+1}-1}, x_{n_{k+1}}), d(x_{n_k-1}, x_{n_k}), \\ \frac{1}{2b}\Big(d(x_{n_{k+1}-1}, x_{n_k}) + d(x_{n_k-1}, x_{n_{k+1}})\Big) \end{array} \right\} \\ &\leq \max\left\{ \begin{array}{c} b\Big(1 + d(x_{n_k}, x_{n_k-1})\Big), d(x_{n_{k+1}-1}, x_{n_{k+1}}), d(x_{n_k-1}, x_{n_k}), \\ \frac{1}{2b}\Big(d(x_{n_{k+1}-1}, x_{n_k}) + d(x_{n_k-1}, x_{n_{k+1}})\Big) \end{array} \right\} \\ &\leq \max\left\{ \begin{array}{c} b\Big(1 + d(x_{n_k}, x_{n_k-1})\Big), d(x_{n_{k+1}-1}, x_{n_{k+1}}), d(x_{n_k-1}, x_{n_k}), \\ \frac{1}{2b}\Big(1 + d(x_{n_k-1}, x_{n_{k+1}})\Big) \end{array} \right\} \\ &\leq \max\left\{ \begin{array}{c} b\Big(1 + d(x_{n_k}, x_{n_k-1})\Big), d(x_{n_{k+1}-1}, x_{n_{k+1}}), d(x_{n_k-1}, x_{n_k}), \\ \frac{1}{2b}\Big(1 + b\Big(d(x_{n_k-1}, x_{n_k}) + d(x_{n_k}, x_{n_{k+1}})\Big)\Big) \end{array} \right\} \\ &\leq \max\left\{ \begin{array}{c} b\Big(1 + d(x_{n_k}, x_{n_k-1})\Big), d(x_{n_{k+1}-1}, x_{n_{k+1}}), d(x_{n_k-1}, x_{n_k}), \\ \frac{1}{2b}\Big(b + b\Big(d(x_{n_k-1}, x_{n_k}) + d(x_{n_k}, x_{n_{k+1}})\Big)\Big) \end{array} \right\} \\ &= \max\left\{ \begin{array}{c} b\Big(1 + d(x_{n_k}, x_{n_k-1})\Big), d(x_{n_{k+1}-1}, x_{n_{k+1}}), d(x_{n_k-1}, x_{n_k}), \\ \frac{1}{2}\Big(1 + d(x_{n_k-1}, x_{n_k}) + d(x_{n_k}, x_{n_{k+1}})\Big) \end{array} \right\} \end{aligned}$$

$$\leq \max\left\{\begin{array}{l} b\Big(1+d(x_{n_k},x_{n_k-1})\Big), d(x_{n_{k+1}-1},x_{n_{k+1}}), d(x_{n_k-1},x_{n_k}), \\ \frac{1}{2}\Big(1+d(x_{n_k-1},x_{n_k})+b\Big(d(x_{n_k},x_{n_{k+1}-1})+d(x_{n_{k+1}-1},x_{n_{k+1}})\Big)\Big) \end{array}\right\}$$

$$\leq \max\left\{\begin{array}{l} b\Big(1+d(x_{n_k},x_{n_k-1})\Big), d(x_{n_{k+1}-1},x_{n_{k+1}}), d(x_{n_k-1},x_{n_k}), \\ \frac{1}{2}\Big(b+d(x_{n_k-1},x_{n_k})+b\Big(d(x_{n_k},x_{n_{k+1}-1})+d(x_{n_{k+1}-1},x_{n_{k+1}})\Big)\Big) \end{array}\right\}. \quad (3.8)$$

From (3.4) and (3.8), we get

$$\begin{aligned} b &< bd(x_{n_{k+1}},x_{n_k}) \\ &< M_b(x_{n_{k+1}-1},x_{n_k-1}) \\ &\leq \max\left\{\begin{array}{l} b\Big(1+d(x_{n_k},x_{n_k-1})\Big), d(x_{n_{k+1}-1},x_{n_{k+1}}), d(x_{n_k-1},x_{n_k}), \\ \frac{1}{2}\Big(b+d(x_{n_k-1},x_{n_k})+b\Big(d(x_{n_k},x_{n_{k+1}-1})+d(x_{n_{k+1}-1},x_{n_{k+1}})\Big)\Big) \end{array}\right\}, \end{aligned}$$

taking $k\to\infty$, then

$$\begin{aligned} b &\leq \lim_{k\to\infty} M_b(x_{n_{k+1}-1},x_{n_k-1}) \\ &\leq \lim_{k\to\infty}\max\left\{\begin{array}{l} b\Big(1+d(x_{n_k},x_{n_k-1})\Big), d(x_{n_{k+1}-1},x_{n_{k+1}}), d(x_{n_k-1},x_{n_k}), \\ \frac{1}{2}\Big(b+d(x_{n_k-1},x_{n_k})+b\Big(d(x_{n_k},x_{n_{k+1}-1})+d(x_{n_{k+1}-1},x_{n_{k+1}})\Big)\Big) \end{array}\right\} \\ &= b, \end{aligned}$$

that is,

$$\lim_{k\to\infty} M_b(x_{n_{k+1}-1},x_{n_k-1}) = b.$$

Thus by (3.7) and property ($\xi 2$), with $t_k = d(x_{n_{k+1}},x_{n_k})$ and $s_k = M_b(x_{n_{k+1}-1},x_{n_k-1})$, we have

$$0 \leq \limsup_{k\to\infty} \xi\Big(bd(x_{n_{k+1}},x_{n_k}), M_b(x_{n_{k+1}-1},x_{n_k-1})\Big) < 0,$$

which is a contradiction. Hence the sequence $\{x_n\}$ is bounded. ■

Lemma 3.4. *Let (X,d) be a b-metric space with constant $b \geq 1$ and let $T : X \to X$ be a generalized $\mathcal{Z}_b$-contraction with respect to $\xi \in \mathcal{Z}_b$. Let $\{x_n\}$ be a Picard sequence initial point $x_0 \in X$. Suppose that $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ is a Cauchy sequence.*

Proof. Let

$$C_n = \sup\{d(x_i,x_j) : i,j \geq n\}, n \in \mathbb{N}.$$

Since the sequence $\{x_n\}$ is bounded (Lemma 3.3), $C_n < \infty$ for every $n \in \mathbb{N}$ and since $\{C_n\}$ is a positive decreasing sequence, there exist $C \geq 0$ such that

$$\lim_{n\to\infty} C_n = C.$$

Suppose $C > 0$. By the definition of C_n, for every $k \in \mathbb{N}$ there exists $n_k, m_k \in \mathbb{N}$ such that $m_k > n_k \geq k$ and

$$C_k - \frac{1}{k} < d(x_{m_k},x_{n_k}) \leq C_k. \quad (3.9)$$

Letting $k \to \infty$ in (3.9), we have

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = C, \tag{3.10}$$

and

$$\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k-1}) = C. \tag{3.11}$$

By (3.1) and property ($\xi 1$), we have

$$\begin{aligned} 0 &\le \xi\Big(bd(x_{m_k}, x_{n_k}), M_b(x_{m_k-1}, x_{n_k-1})\Big) \\ &< M_b(x_{m_k-1}, x_{n_k-1}) - bd(x_{m_k}, x_{n_k}), \end{aligned}$$

so

$$\begin{aligned} & bd(x_{m_k}, x_{n_k}) < M_b(x_{m_k-1}, x_{n_k-1}) \\ &= \max\left\{ \begin{array}{l} d(x_{m_k-1}, x_{n_k-1}), d(x_{m_k-1}, x_{m_k}), d(x_{n_k-1}, x_{n_k}), \\ \frac{1}{2b}\Big(d(x_{m_k-1}, x_{n_k}) + d(x_{n_k-1}, x_{m_k})\Big) \end{array} \right\} \\ &\le \max\left\{ \begin{array}{l} d(x_{m_k-1}, x_{n_k-1}), d(x_{m_k-1}, x_{m_k}), d(x_{n_k-1}, x_{n_k}), \\ \frac{1}{2b}\Big(b(d(x_{m_k-1}, x_{m_k}) + d(x_{m_k}, x_{n_k})) + b(d(x_{n_k-1}, x_{n_k}) + d(x_{n_k}, x_{m_k}))\Big) \end{array} \right\} \\ &= \max\left\{ \begin{array}{l} d(x_{m_k-1}, x_{n_k-1}), d(x_{m_k-1}, x_{m_k}), d(x_{n_k-1}, x_{n_k}), \\ \frac{1}{2}\Big(d(x_{m_k-1}, x_{m_k}) + d(x_{m_k}, x_{n_k}) + d(x_{n_k-1}, x_{n_k}) + d(x_{n_k}, x_{m_k})\Big) \end{array} \right\}. \end{aligned} \tag{3.12}$$

Letting $k \to \infty$ in (3.12), using Lemma 3.2, (3.10) and (3.11), we have

$$\begin{aligned} bC &= \lim_{k\to\infty} bd(x_{m_k}, x_{n_k}) \le \lim_{k\to\infty} M_b(x_{m_k-1}, x_{n_k-1}) \\ &\le \lim_{k\to\infty} \max\left\{ \begin{array}{l} d(x_{m_k-1}, x_{n_k-1}), d(x_{m_k-1}, x_{m_k}), d(x_{n_k-1}, x_{n_k}), \\ \frac{1}{2}\Big(d(x_{m_k-1}, x_{m_k}) + d(x_{m_k}, x_{n_k}) + d(x_{n_k-1}, x_{n_k}) + d(x_{n_k}, x_{m_k})\Big) \end{array} \right\} \\ &= C, \end{aligned}$$

then

$$bC \le \liminf_{k\to\infty} M_b(x_{m_k-1}, x_{n_k-1}) \le \limsup_{k\to\infty} M_b(x_{m_k-1}, x_{n_k-1}) \le C. \tag{3.13}$$

From (3.13) we see that, Since $C > 0$ that $b = 1$. Then by the property ($\xi 2$) with $t_k = d(x_{m_k}, x_{n_k})$ and $s_k = M_b(x_{m_k-1}, x_{n_k-1})$, we get

$$0 \le \limsup_{k\to\infty} \xi\Big(bd(x_{m_k}, x_{n_k}), M_b(x_{m_k-1}, x_{n_k-1})\Big) < 0,$$

which is a contradiction. Thus $C = 0$, that is,

$$\lim_{n\to\infty} C_n = 0 \text{ for all } b \ge 1.$$

This proves that $\{x_n\}$ is a Cauchy sequence. ∎

Theorem 3.5. *Let (X, d) be a complete b-metric space with constant $b \ge 1$ and let $T : X \to X$ be a generalized $\mathcal{Z}_b$-contraction with respect to $\xi \in \mathcal{Z}_b$. Then T has a fixed point.*

Proof. Let $x_0 \in X$ and $\{x_n\}$ be a Picard sequence with initial point x_0. if $x_m = x_{m+1}$ for some $m \in \mathbb{N}$, then $x_m = x_{m+1} = Tx_m$, that is x_m is a fixed point of T. In this case, the existence of a fixed point is proved. So, we can suppose that $x_n \neq x_{n+1}$ for every $n \in \mathbb{N}$. Now by Lemma 3.4, the sequence $\{x_n\}$ is Cauchy and since (X, d) is complete, then there exists some $z \in X$ such that

$$\lim_{n\to\infty} x_n = z. \tag{3.14}$$

We shall prove that z is a fixed point of T. Assume $z \neq Tz$, then $d(z, Tz) = k > 0$ for some $k \in \mathbb{R}$.
Since

$$\begin{aligned} d(z,Tz) \leq M_b(x_n,z) &= \max \left\{ \begin{array}{l} d(x_n,z), d(x_n,Tx_n), d(z,Tz), \\ \frac{1}{2b}\Big(d(x_n,Tz)+d(z,Tx_n)\Big) \end{array} \right\} \\ &\leq \max \left\{ \begin{array}{l} d(x_n,z), d(x_n,Tx_n), d(z,Tz), \\ \frac{1}{2b}\Big(b(d(x_n,z)+d(z,Tz))+b(d(z,x_n)+d(x_n,Tx_n))\Big) \end{array} \right\} \\ &= \max \left\{ \begin{array}{l} d(x_n,z), d(x_n,Tx_n), d(z,Tz), \\ \frac{1}{2}\Big(d(x_n,z)+d(z,Tz)+d(z,x_n)+d(x_n,Tx_n)\Big) \end{array} \right\} \end{aligned} \tag{3.15}$$

taking $n \to \infty$, we get

$$\lim_{n\to\infty} M_b(x_n, z) = d(z, Tz) = k > 0.$$

Using (3.1), (3.15) and property ($\xi 1$), we obtain

$$\begin{aligned} 0 &\leq \xi(bd(Tx_n,Tz), M_b(x_n,z)) \\ &< M_b(x_n,z) - bd(Tx_n,Tz) \end{aligned} \tag{3.16}$$

$$bd(Tx_n,Tz) < M_b(x_n,z)$$

$$d(Tx_n,Tz) < \frac{M_b(x_n,z)}{b}. \tag{3.17}$$

By ($b3$) of Definition (2.2), we get

$$d(z,Tz) \leq b[d(z,Tx_n)+d(Tx_n,Tz)]$$

$$\frac{d(z,Tz)}{b} \leq d(Tx_n,Tz). \tag{3.18}$$

Letting $n \to \infty$ in (3.17) and (3.18), we have

$$\frac{k}{b} = \lim_{n\to\infty} \frac{d(z,Tz)}{b} \leq \lim_{n\to\infty} d(Tx_n,Tz) \leq \lim_{n\to\infty} \frac{M_b(x_n,z)}{b} = \frac{k}{b}.$$

Then

$$\lim_{n\to\infty} d(Tx_n,Tz) = \frac{k}{b} > 0.$$

Therefore by (3.16) and property ($\xi 2$), with $t_n = d(Tx_n, Tz)$ and $s_n = M_b(x_n, z)$. Then

$$0 \leq \limsup_{n\to\infty} \xi\Big(bd(Tx_n,Tz), M_b(x_n,z)\Big) < 0,$$

which is a contradiction, we get $d(z, Tz) = 0$, that is z is a fixed point of T. This complete the proof. ∎

Corollary 3.6. *Let (X,d) be a complete b-metric space with a constant $b \geq 1$ and let $T : X \to X$ be a mapping. Suppose that there exists $\lambda \in (0,1)$ such that*

$$bd(Tx,Ty) \leq \lambda M_b(x,y) \text{ for all } x,y \in X.$$

Then T has a fixed point.

Proof. The result follows from Theorem 3.5, by taking as b-simulation function

$$\xi(t,s) = \lambda s - t$$

for all $t,s \leq 0$. ■

Note If $M_b(x,y) = d(x,y)$, this corollary gives a result of Banach type [12].

Corollary 3.7. [7] *Let (X,d) be a complete metric space, $T : X \to X$ be a mapping. Suppose that there exists a simulation function ξ such that*

$$\zeta(d(Tx,Ty), M(x,y)) \geq 0 \text{ for all } x,y \in X,$$

where

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}\Big(d(x,Ty) + d(y,Tx)\Big)\right\}.$$

Then T has a fixed point.

Proof. It follows from Theorem 3.5 with $b = 1$. ■

Example 3.8. Let $X = [0,1]$ and $d : X \times X \to \mathbb{R}$ defined by $d(x,y) = (x-y)^2$. Then (X,d) is a complete b-metric space with $b = 2$. Define $T : X \to X$ by

$$Tx = \frac{ax}{1+x} \text{ for all } x \in X \text{ and } a \in (0, \frac{1}{\sqrt{2}}].$$

Let $\xi : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be defined by $\xi(t,s) = \frac{s}{s+1} - t$. Then ξ is a b-simulation function. Indeed, we obtain

$$\begin{aligned}
\xi(2d(Tx,Ty), M_b(x,y)) &= \frac{M_b(x,y)}{M_b(x,y)+1} - 2d(Tx,Ty) \\
&\geq \frac{d(x,y)}{d(x,y)+1} - 2d(Tx,Ty) \\
&= \frac{(x-y)^2}{(x-y)^2+1} - 2\Big[\frac{ax}{1+x} - \frac{ay}{1+y}\Big]^2 \\
&= \frac{(x-y)^2}{(x-y)^2+1} - \frac{2a^2(x-y)^2}{[(1+x)(1+y)]^2} \\
&\geq \frac{(x-y)^2}{(x-y)^2+1} - \frac{2a^2(x-y)^2}{(x-y)^2+1} \\
&= \frac{(x-y)^2 - 2a^2(x-y)^2}{(x-y)^2+1} \\
&= \frac{(1-2a^2)(x-y)^2}{(x-y)^2+1} \geq 0, \text{ for all } x,y \in X.
\end{aligned}$$

Thus all the conditions of Theorem 3.5 are satisfied. Hence T has a fixed point (at $x = 0$).

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