Thai Journal of Mathematics
Volume 19 Number 4 (2021)
Pages 1613–1624

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

On Weak Subdifferential and Augmented Normal Cone

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Abstract In this paper, we study augmented normal cone and investigate relation between weak subdifferential and augmented normal cone. we define augmented normal cone via weak subdifferential and vice versa. The necessary condition for having the global maximum is given in the paper. We find the preliminary properties of augmented normal cones including investigating them for Fréchet differentiable functions. In the sequel , some properties of Weak subdifferential and Fréchet subdifferentil are considered. It is also compared optimality condition via weak subifferential and optimality condition via Fréchet subifferential.

MSC: 49K35; 47H10; 20M12

Keywords: subgradient; weak subgradient; normal cone; augmented normal cone; Fréchet differentiable function; locally Lipchitz function

Submission date: 27.04.2018 / Acceptance date: 28.04.2019

1. Introduction

Recall that, a convex set has a supporting hyperplane at each boundary point [1]. This leads to one of the central notions in convex analysis, that of a subgradient of a possible nonsmooth even extended real valued function [2, 3]. Subgradient plays an important role in the deriving of optimality conditions and duality theorems [4–8]. Since a nonconvex set has no supporting hyperplane at each boundary point, the notion of subgradient have been generalized by most researchers on optimality conditions for nonconvex problems,for more details on this study see [2, 9, 10]. The variety of different subdifferentials can be divided into two large groups:

- “simple” subdifferentials
- “strict” subdifferentials.

A simple subdifferential is defined at a given point and it does not take into account ”differential” properties of a function in its neighborhood. They are not widely used directly because of rather poor calculus. Contrary to the simple subdifferentials, the definitions of strict subdifferentials incorporate differential properties of a function near a given point.

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Published by The Mathematical Association of Thailand.

The notion of weak subdifferential which is a generalization of the classic subdifferential, is introduced by Azimov and Gasimov [11, 12]. It uses explicitly defined supporting conic surfaces instead of supporting hyperplanes. The main reason of difficulties arising when passing from the convex analysis to the nonconvex one is that, the nonconvex cases may arise in many different forms and each case may require special approach. The main ingredient is the method of supporting the given nonconvex set. Subgradient plays an important role in deriving of optimality conditions and duality theorems. The first canonical generalized gradient introduced by Clarke [2, 3]. He applied this generalized gradient systematically to study nonsmooth problems in a variety of problems. Since a nonconvex set has no supporting hyperplane at each boundary point , the notion of subgradient have been generalized by most researchers on optimality conditions for nonconvex problems [2, 3, 13]. By using the notion of subgradients, a collection of zero duality gap conditions for a wise class of nonconvex optimization problems was derived [11, 12]. Augmented normal cone via weak subdifferential defined by Kasimbeyli and Mammadov in [14, 15]. In this study some important properties of the augmented normal cones via the weak subdifferentials are given. Some theorems, by using the definition and properties of the weak subdifferential which are described in [1, 7, 10, 14–21], concerning the augmented normal cone and weak subdifferential in nonsmooth and nonconvex analysis are presented.

2. Preliminaries

Let X be a real normed space and let X^* be the topological dual of X. By $\|\cdot\|$ we denote the norm of X and by $\langle x^*, x\rangle$ the value of the linear functional $x^* \in X^*$ at the point $x \in X$. Let S be a nonempty subset of X and $\bar{x} \in S$.

Definition 2.1 ([14, 15])**.** Let $f : X \to \mathbb{R}$ be a function and $\bar{x} \in X$ be a given point. The set

$$\partial f(\bar{x}) = \Big\{x^* \in X^* : (\forall x \in X) \;\; f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x}\rangle\Big\},$$

is called the subdifferential of f at $\bar{x} \in X$.

The next definition generalized the notion of subdifferential.

Definition 2.2 ([14, 15])**.** Let $f : X \to \mathbb{R}$ be a function and $\bar{x} \in X$ be a given point. A pair $(x^*, c) \in X^* \times \mathbb{R}^+$ where $\mathbb{R}^+$, the set of nonnegative real numbers, is called weak subgradient of f at $\bar{x} \in X$ if the following inequality holds:

$$(\forall x \in X) \qquad f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x}\rangle - c\|x - \bar{x}\|.$$

The set

$$\partial^w f(\bar{x}) = \Big\{(x^*, c) \in X^* \times \mathbb{R}^+ : (\forall x \in X) \qquad f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x}\rangle - c\|x - \bar{x}\| \;\; \Big\}$$

of all weak subgradients of f at $\bar{x} \in X$ is called the weak subdifferential of f at $\bar{x} \in X$. If $\partial^w f(\bar{x}) \neq \emptyset$, then f is called weakly subdifferentiable at $\bar{x}$.

Remark 2.3 ([9])**.** It is clear when f is subdifferentiable at $\bar{x}$, then f is also weakly subdifferentiable at $\bar{x}$; that is, if $x^* \in \partial f(\bar{x})$, then by the definition of weak subgradient we get $(x^*, c) \in \partial^w f(\bar{x})$ for every $c \geq 0$. Note that the converse may fail (consider $f(x) = - \mid x \mid, X = \mathbb{R}$).

The next definition is needed in the sequel.

Definition 2.4 ([22]). Let $f : X \to \mathbb{R}$ be a function. If there is a continuous linear map $f'(\bar{x}) : X \to \mathbb{R}$ with the property

$$\lim_{\|h\|\to 0} \frac{|f(\bar{x}+h) - f(\bar{x}) - (f'(\bar{x}))(h)|}{\|h\|} = 0,$$

then $f'(\bar{x}) : X \to \mathbb{R}$ is called the Fréchet derivative of f at $\bar{x} \in X$ and f is called the Fréchet differentiable at $\bar{x}$.

Remark 2.5 ([9]). It follows from Definition 2.2 that the pair $(x^*, c) \in X^* \times \mathbb{R}^+$ is a weak subdifferential of f at $\bar{x} \in X$ if and only if there exists the continuous (super linear) concave function $g : X \to \mathbb{R}$ defined by $g(x) = f(\bar{x}) + \langle x^*, x - \bar{x}\rangle - c\|x - \bar{x}\|, x \in X$, satisfies

$$(\forall x \in X) \quad g(x) \le f(x) \quad \text{and} \quad g(\bar{x}) = f(\bar{x}).$$

This condition means that g supports f from below. Hence , it follows that , if f is weakly subdifferentiable at $\bar{x}$ and $(x^*, c) \in \partial^w f(\bar{x})$, then the graph of function g becomes a supporting surface to epigraph of f on X at the point $(\bar{x}, f(\bar{x}))$.

Theorem 2.6 ([14]). *Let the weak subdifferential of $f : X \to \mathbb{R}$ at $\bar{x}$ be nonempty. Then the set $\partial^w f(\bar{x})$ is closed and convex.*

3. Main Results

In this section we first recall the definition of augmented normal cone that presented in [6] and then we state the main results.

Definition 3.1. The set

$$N_S(\bar{x}) = \{x^* \in X^* : \langle x^*, x - \bar{x}\rangle \le 0 \quad (\forall x \in S)\}$$

is called a normal cone to S at $\bar{x}$.

Definition 3.2. The set

$$N_S{}^a(\bar{x}) = \{(x^*, c) \in X^* \times \mathbb{R}^+; \langle x^*, x - \bar{x}\rangle - c \parallel x - \bar{x} \parallel \le 0 \quad (\forall x \in S)\}$$

is called an augmented normal cone to S at $\bar{x}$. Note that if there exists $x^* \in X^*$ such that $(x^*, 0) \in N_S{}^a(\bar{x})$, then $x^* \in N_S(\bar{x})$.

Remark 3.3. From the definitions of normal and augmented normal cones , we have

$$x^* \in N_S(\bar{x}) \Longrightarrow (x^*, c) \in N_S{}^a(\bar{x}) \qquad (\forall c \ge 0).$$

Remark 3.4. If $(x^*, c) \in N_S{}^a(\bar{x})$ with $\parallel x^* \parallel \le c$, then it is obvious for all $x \in S$ that

$$\langle x^*, x - \bar{x}\rangle - c \parallel x - \bar{x} \parallel \le 0.$$

This means that $(x^*, c) \in N_S{}^a(\bar{x})$. An augmented normal cone consisting of only such elements is called trivial and denoted by $N_S{}^{triv}(\bar{x})$. Obviously

$$N_S{}^{triv}(\bar{x}) \subset N_S{}^a(\bar{x}).$$

Note: If $\bar{x} \in X$ then

$$N_X{}^a(\bar{x}) = \{(x^*, c) \in X^* \times \mathbb{R}^+; \langle x^*, x - \bar{x}\rangle - c \parallel x - \bar{x} \parallel \le 0 \quad (\forall x \in S)\} =$$
$$\{(x^*, c) \in X^* \times \mathbb{R}^+; \parallel x^* \parallel \le c\} = N_X{}^{triv}(\bar{x}).$$

Proposition 3.5. *If $c_1 \leq c_2$, then*

$$(x^*, c_1) \in N_S{}^a(\bar{x}) \Longrightarrow (x^*, c_2) \in N_S{}^a(\bar{x}).$$

Proof. Let $(x^*, c_1) \in N_S{}^a(\bar{x})$, then by the definition of augmented normal cone, we have

$$\langle x^*, x - \bar{x} \rangle - c_1 \parallel x - \bar{x} \parallel \leq 0 \quad (\forall\ x \in S)$$

so that by assumption $c_1 \leq c_2$, we obtain

$$\langle x^*, x - \bar{x} \rangle - c_2 \parallel x - \bar{x} \parallel \leq 0 \quad (\forall\ x \in S).$$

Therefore $(x^*, c_1) \in N_S{}^a(\bar{x})$ which is the desired result. ■

Note: If $\bar{x} \in S$, then it is clear that $(0, 0) \in N_S{}^a(\bar{x})$ and so the augmented normal cone is a nonempty.

Proposition 3.6. *The set $N_S{}^a(\bar{x})$ is a closed convex cone.*

Proof. The proof directly follows from the definition of $N_S{}^a(\bar{x})$. ■

Proposition 3.7. *$(x^*, c) \in N_S{}^a(\bar{x})$ if and only if the function $g : X \longrightarrow \mathbb{R}$ defined by*

$$g(x) = \langle x^*, x - \bar{x} \rangle - c \parallel x - \bar{x} \parallel$$

satisfied in:

$$g(x) \leq 0 \quad (\forall x \in S), g(\bar{x}) = 0.$$

Proof. The proof is straightforward from the definition of $N_S{}^a(\bar{x})$. ■

The next proposition states the necessary condition for having the global maximum.

Proposition 3.8. *Let $f : X \longrightarrow \mathbb{R}$ be a function that attains a global maximum at $\bar{x}$, then we have*

$$\partial^w f(\bar{x}) \subset N_X{}^{triv}(\bar{x}) \subset N_X{}^a(\bar{x}).$$

Proof. If $\partial^w f(\bar{x}) \neq \emptyset$, then there exists a pair (x^*, c) such that

$$f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x} \rangle - c \parallel x - \bar{x} \parallel \quad (\forall\ x \in X).$$

With assumption f attains a global maximum at $\bar{x}$, therefore

$$\langle x^*, x - \bar{x} \rangle - c \parallel x - \bar{x} \parallel \leq 0 \quad (\forall\ x \in X).$$

So that

$$\parallel x^* \parallel \leq c$$

and we have $(x^*, c) \in N_X{}^{triv}(\bar{x})$ and proof is completed by $N_S{}^{triv}(\bar{x}) \subset N_S{}^a(\bar{x})$. ■

Corollary 3.9. *Let $f : X \longrightarrow \mathbb{R}$ be a function that attains a global minimum at $\bar{x}$, then we have*

$$\partial^w(-f(\bar{x})) \subset N_X{}^{triv}(\bar{x}).$$

The following example shows that the inclusion in the Proposition 3.4 can be strict.

Example 3.10. Let $X = \mathbb{R}, f(x) = - \mid x \mid$, then we have

$$\partial^w f(0) = \{(\alpha, c); \mid \alpha \mid \leq c - 1\}$$

and

$$N_{\mathbb{R}}{}^{triv}(0) = \{(\alpha, c); \mid \alpha \mid \leq c\}.$$

Therefore $\partial^w f(\bar{x}) \neq N_X{}^{triv}(\bar{x})$, and we note that f has a global maximum at $\bar{x} = 0$.

The following example shows that the converse of the Proposition 3.4 may fail.

Example 3.11. Let

$$f(x) = \begin{cases} 0 & x \in Q \\ 1 & x \in Q^c. \end{cases},$$

then

$$\partial^w f(0) = N_X{}^{triv}(0) = \{(\alpha, c); \mid \alpha \mid \leq c\}$$

while f attains a global minimum at $\bar{x} = 0$.

Proposition 3.12. *Let $f : X \longrightarrow \mathbb{R}$ be a function that attains a global minimum at $\bar{x}$, then we have*

$$N_X{}^a(\bar{x}) \subset \partial^w f(\bar{x}).$$

Proof. Let $(x^*, c) \in N_X{}^a(\bar{x})$, then we have

$$\langle x^*, x - \bar{x}\rangle - c \parallel x - \bar{x} \parallel \leq 0 \quad (\forall\, x \in X).$$

Since f attains a global minimum at $\bar{x}$, then we obtain

$$f(x) - f(\bar{x}) \geq 0 \quad (\forall\, x \in X),$$

from the above inequalities, we get

$$f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x}\rangle - c \parallel x - \bar{x} \parallel \leq 0 \quad (\forall\, x \in X)$$

so that $(x^*, c) \in \partial^w f(\bar{x})$ and proof is completed. ∎

The next proposition states a link between weak subdifferential of f , $-f$ and augmented normal cone at $\bar{x}$ for the functions that attain a global minimum at $\bar{x}$. This is a necessary condition in optimality conditions.

Proposition 3.13. *Let $f : X \longrightarrow \mathbb{R}$ be a function that attains a global minimum at $\bar{x}$, then we have*

$$\partial^w(-f(\bar{x})) \subset N_X{}^a(\bar{x}) \subset \partial^w f(\bar{x}).$$

Proof. The proof directly follows from the Corollary 3.1 and the Proposition 3.5. ∎

Corollary 3.14. *Let f is a constant function . Then we have*

$$N_X{}^c(\bar{x}) = \partial^w f(\bar{x}) = \partial^w(-f(\bar{x})).$$

Proof. The proof follows from the Propositions 3.6. ∎

As a particular case , consider the weak subdifferentiability of an indicator function. Let δ_S be an indicator function of a set $S \subset X$, such that

$$\delta_S(x) = \begin{cases} 0 & x \in S \\ \infty & o.w. \end{cases},$$

Kasimbeily in [16] generalized one of the well-known theorems in convex analysis that stating a relationship between the subdifferentiability of the indicator function and the supporting hyperplane to a convex set. Now we similarly establish a relationship between the weak subdifferential of the indicator function of any set and its augmented normal cone.

Proposition 3.15. *Let δ_S be an indicator function of a set $S \subset X$. Then we have:*

$$N_S{}^a(\bar{x}) = \partial^w \delta_S(\bar{x}).$$

Proof. Assume that $(x^*, c) \in N_S{}^a(\bar{x})$, therefore we have

$$\langle x^*, x - \bar{x}\rangle - c \parallel x - \bar{x} \parallel \leq 0 \quad (\forall\ x \in S).$$

We know that

$$\delta_S(x) - \delta_S(\bar{x}) = 0 \quad (\forall\ x \in S),$$
$$\delta_S(x) - \delta_S(\bar{x}) = \infty \quad (\forall\ x \notin S),$$

so that we obtain

$$\delta_S(x) - \delta_S(\bar{x}) \geq \langle x^*, x - \bar{x}\rangle - c \parallel x - \bar{x} \parallel \quad (\forall\ x \in X)$$

i.e, $(x^*, c) \in \partial^w \delta_S(\bar{x})$. Conversely, if $(x^*, c) \in \partial^w \delta_S(\bar{x})$, then we have

$$\delta_S(x) - \delta_S(\bar{x}) \geq \langle x^*, x - \bar{x}\rangle - c \parallel x - \bar{x} \parallel \quad (\forall\ x \in X).$$

If $x \in S$, then we obtain

$$\delta_S(x) - \delta_S(\bar{x}) = 0 \quad ,$$

and consequently

$$\langle x^*, x - \bar{x}\rangle - c \parallel x - \bar{x} \parallel \leq 0 \quad (\forall\ x \in S).$$

This means that $(x^*, c) \in N_S{}^a(\bar{x})$, and the proof is completed. ∎

In the sequel we state some important properties of the augmented normal cone.

Proposition 3.16. *Let $S_1 \subset S_2$, then we have*

$$N_{S_2}{}^a(\bar{x}) \subset N_{S_1}{}^a(\bar{x}).$$

Proof. Assume that $(x^*, c) \in N_{S_2}{}^a(\bar{x})$, then

$$\langle x^*, x - \bar{x}\rangle - c \parallel x - \bar{x} \parallel \leq 0 \quad (\forall\ x \in S_2).$$

It follows from $S_1 \subset S_2$ that

$$\langle x^*, x - \bar{x}\rangle - c \parallel x - \bar{x} \parallel \leq 0 \quad (\forall\ x \in S_1)$$

i.e, $(x^*, c) \in N_{S_1}{}^a(\bar{x})$. The proof is completed. ∎

Remark 3.17. In Proposition 3.16 , If $S_1 = S_2$ then we have $N_{S_2}{}^c(\bar{x}) = N_{S_1}{}^c(\bar{x})$, while the following example shows that the converse may drop.

Example 3.18. Let $S_1 = [0,1], S_2 = [0,2]$. It is easy to check that

$$N_{S_1}{}^a(0) = N_{S_2}{}^a(0) = \{(\alpha, c) : \alpha \leq c\},$$

while $S_1 \neq S_2$.

Proposition 3.19. $N_S{}^a(\bar{x}) = N_{clS}{}^a(\bar{x})$.

Proof. Since $S \subset clS$ then Proposition 3.16 implies that $N_{clS}{}^a(\bar{x}) \subset N_S{}^a(\bar{x})$. To see the the reverse inclusion we take $x \in cl, S$ then there exists $\{x_n\} \subset S$ such that $x_n \longrightarrow x$. Now assume that $(x^*, c) \in N_S{}^a(\bar{x})$. Hence

$$\langle x^*, x_n - \bar{x}\rangle - c \parallel x_n - \bar{x} \parallel \leq 0 \qquad (\forall x_n \in S).$$

By taking the limit inferior of the both sides of the last inequality when $n \to \infty$ we get

$$\langle x^*, x - \bar{x}\rangle - c \parallel x - \bar{x} \parallel \leq 0 \quad (\forall x \in clS).$$

This means that $(x^*, c) \in N_{clS}{}^a(\bar{x})$ and so the proof is completed. ∎

Proposition 3.20. *Let S be a cone, then*

$$N_S{}^a(\lambda\bar{x}) = N_S{}^a(\bar{x}) \quad (\forall \lambda > 0).$$

Proof. It follows from the hypothesis that

$$\begin{aligned}
(x^*, c) \in N_S{}^a(\lambda\bar{x}) \quad &\Longleftrightarrow \quad \langle x^*, \lambda x - \lambda\bar{x}\rangle - c\|\lambda x - \lambda\bar{x}\| \leq 0 \quad (\forall\ x \in S) \\
&\Longleftrightarrow \quad \lambda(\langle x^*, x - \bar{x}\rangle - c\|x - \bar{x}\|) \leq 0 \quad (\forall\ x \in S) \\
&\Longleftrightarrow \quad (x^*, c) \in N_S{}^a(\bar{x}).
\end{aligned}$$

This completes the proof. ∎

Proposition 3.21. *Let $S_1, S_2 \subset X$, $S_1 \cap S_2 \neq \emptyset$. Then*

$$N_{S_1 \cup S_2}{}^a(\bar{x}) = N_{S_1}{}^a(\bar{x}) \cap N_{S_2}{}^a(\bar{x}) \subset N_{S_1 \cap S_2}{}^a(\bar{x}).$$

Proof. Suppose that $(x^*, c) \in N_{S_1 \cup S_2}{}^a(\bar{x})$, therefore

$$(x^*, x - \bar{x}) - c \parallel x - \bar{x} \parallel \leq 0 \quad \forall x \in S_1 \cup S_2$$

so that we have :

$$(x^*, x - \bar{x}) - c \parallel x - \bar{x} \parallel \leq 0 \quad \forall x \in S_1$$

and

$$(x^*, x - \bar{x}) - c \parallel x - \bar{x} \parallel \leq 0 \quad \forall x \in S_2.$$

This means that $(x^*, c) \in N_{S_1}{}^a(\bar{x}) \cap N_{S_2}{}^a(\bar{x})$. Also we obtain

$$(x^*, x - \bar{x}) - c \parallel x - \bar{x} \parallel \leq 0 \quad \forall x \in S_1 \cap S_2$$

and the last inclusion obtained. Conversely, if $(x^*, c) \in N_{S_1}{}^a(\bar{x}) \cap N_{S_2}{}^a(\bar{x})$, then

$$(x^*, x - \bar{x}) - c \parallel x - \bar{x} \parallel \leq 0 \quad \forall x \in S_1$$

and

$$(x^*, x - \bar{x}) - c \parallel x - \bar{x} \parallel \leq 0 \quad \forall x \in S_2.$$

Hence

$$(x^*, x - \bar{x}) - c \parallel x - \bar{x} \parallel \leq 0 \quad \forall x \in S_1 \cup S_2,$$

so that $(x^*, c) \in N_{S_1 \cup S_2}{}^a(\bar{x})$, and this completes the proof. ∎

The next example indicates that the converse of the last inclusion may fail.

Example 3.22. If $X = \mathbb{R}, S_1 = \{0,1\}, S_2 = \{0,2\}, \bar{x} = 0$, then we get

$$N_{S_1}{}^a(\bar{x}) = N_{S_2}{}^a(\bar{x}) = \{(\alpha, c) \in \mathbb{R} \times \mathbb{R} : \alpha \leq c\},$$

while $N_S{}^a(\bar{x}) = \mathbb{R}^2$.

Remark 3.23. Since $S_1 \cap S_2 \subset S_1, S_2$, then by Proposition 3.11 , we have

$$N_{S_1}{}^c(\bar{x}), N_{S_2}{}^a(\bar{x}) \subset N_{S_1}{}^a(\bar{x}) \cap N_{S_2}{}^a(\bar{x}),$$

and so that

$$N_{S_1}{}^a(\bar{x}) \cap N_{S_2}{}^a(\bar{x}) \subset N_{S_1 \cap S_2}{}^a(\bar{x}),$$

and similarly

$$N_{S_1}{}^a(\bar{x}) \cup N_{S_2}{}^a(\bar{x}) \subset N_{S_1 \cap S_2}{}^a(\bar{x}).$$

Proposition 3.24. *Let $S = S_1 \cap S_2 \neq \emptyset$. Then*

$$N_{S_1}{}^a(\bar{x}) + N_{S_2}{}^a(\bar{x}) \subset N_S{}^a(\bar{x}).$$

Proof. Assume that $(x_1{}^*, c_1) \in N_{S_1}{}^a(\bar{x})$ and $(x_2{}^*, c_2) \in N_{S_2}{}^a(\bar{x})$, therefore

$$\langle x_1{}^*, x - \bar{x}\rangle - c_1\|x - \bar{x}\| \leq 0 \quad (\forall\ x \in S_1),$$

$$\langle x_2{}^*, x - \bar{x}\rangle - c_2\|x - \bar{x}\| \leq 0 \quad (\forall\ x \in S_2),$$

for any $x \in S = S_1 \cap S_2$, we obtain

$$\langle x_1{}^* + x_2{}^*, x - \bar{x}\rangle - (c_1 + c_2)\|x - \bar{x}\| \leq 0 \quad (\forall\ x \in S),$$

i.e, $(x_1{}^* + x_2{}^*, c_1 + c_2) \in N_S{}^a(\bar{x})$ and the proof is completed. ∎

The next example shows that the inclusion in the result of Proposition 3.24 may be strict.

Example 3.25. Let $X = \mathbb{R}, S_1 = \{0,1\}, S_2 = \{0,2\}, \bar{x} = 0$, then we have

$$N_{S_1}{}^a(\bar{x}) = N_{S_2}{}^a(\bar{x}) = \{(\alpha, c) \in \mathbb{R} \times \mathbb{R} : \alpha \leq c\}$$

while $N_{S_1 \cap S_2}{}^a(\bar{x}) = \mathbb{R}^2$.

Proposition 3.26. *Let $S = S_1 + S_2$, $\bar{x} = \bar{x}_1 + \bar{x}_2$, $\bar{x}_i \in S_i$, $i = 1, 2$. Then*

$$N_S{}^a(\bar{x}) = N_{S_1}{}^a(\bar{x_1}) \cap N_{S_2}{}^a(\bar{x_2}).$$

Proof. Assume $(x^*, c) \in N_S{}^a(\bar{x})$, then we have

$$\langle x^*, x - \bar{x}\rangle - c \parallel x - \bar{x} \parallel \leq 0 \quad (\forall x \in S),$$

therefore

$$\langle x^*, (x_1 + x_2) - (\bar{x}_1 + \bar{x}_2)\rangle - c \parallel (x_1 + x_2) - (\bar{x}_1 + \bar{x}_2) \parallel \leq 0 \quad (\forall x = x_1 + x_2 \in S = S_1 + S_2),$$

from the last inequality, with $x_2 = \bar{x}_2$ and $x_1 = \bar{x}_1$,respectively , we obtain

$$\langle x^*, x_1 - \bar{x}_1\rangle - c \parallel x_1 - \bar{x}_1 \parallel \leq 0 \quad (\forall x_1 \in S_1) \Longrightarrow (x^*, c) \in N_{S_1}{}^a(\bar{x}_1)$$

$$\langle x^*, x_2 - \bar{x}_2\rangle - c \parallel x_2 - \bar{x}_2 \parallel \leq 0 \quad (\forall x_2 \in S_2) \Longrightarrow (x^*, c) \in N_{S_2}{}^a(\bar{x}_2)$$

and so

$$(x^*, c) \in N_{S_1}{}^a(\bar{x}_1) \cap N_{S_2}{}^a(\bar{x}_2).$$

The converse of the inclusion can be proved by a similar way. ∎

Proposition 3.27. *Let* $^{S} = \{(x,x) : x \in S\}$, $^{x} = (\bar{x},\bar{x})$. *Then*

$$N_{S}{}^{a}({}^{x}) = \{((x^*,y^*),c) \in X^* \times X^* \times \mathbb{R}^+ : ((x^*+y^*),2c) \in N_S{}^a(\bar{x})\}.$$

Note that $\| (x,y) \| = \| x \| + \| y \|, \forall x,y \in X$.

Proof. It follows from the hypothesis that

$$\begin{aligned}((x^*,y^*),c) \in N_{S}{}^{a}({}^{x}) \quad &\Longleftrightarrow \quad \langle (x^*,y^*),(x,x) - {}^{x}\rangle - c \parallel (x,x) - {}^{x} \parallel \le 0 \quad (\forall (x,x) \in {}^{S}),\\ &\Longleftrightarrow \quad \langle x^*+y^*, x-\bar{x}\rangle - 2c \parallel x-\bar{x} \parallel \le 0 \quad (\forall\ x \in S)\\ &\Longleftrightarrow \quad (x^*+y^*,2c) \in N_S{}^a(\bar{x}).\end{aligned}$$

This completes the proof. ■

Proposition 3.28. *Let* $X = X_1 \times X_2$, $S = S_1 \times S_2$, $\bar{x} = (\bar{x}_1,\bar{x}_2), \bar{x}_i \in S_i \subset X_i, i = 1,2$. *Then*

$$\pi(N_S{}^a(\bar{x})) = \pi(N_S{}^a(\bar{x}_1)) \times \pi(N_S{}^a(\bar{x}_2)).$$

Proof. It is easy to verify the following relations:

$$\begin{aligned}((x^*,y^*),c) \in N_S{}^a(\bar{x}) \quad &\Longleftrightarrow \quad \langle (x^*,y^*),(x_1,x_2) - (\bar{x}_1,\bar{x}_2)\rangle\\ &\qquad\quad -c \parallel (x_1,x_2) - (\bar{x}_1,\bar{x}_2) \parallel \le 0 \quad (\forall (x_1,x_2) \in {}^{S}),\\ &\Longleftrightarrow \quad \langle x^*, x_1 - \bar{x}_1\rangle - c \parallel x_1 - \bar{x}_1 \parallel \le 0 \quad (\forall\ x_1 \in S_1),\\ &\qquad\quad \langle y^*, x_2 - \bar{x}_2\rangle - c \parallel x_2 - \bar{x}_2 \parallel \le 0 \quad (\forall\ x_2 \in S_2)\\ &\Longleftrightarrow \quad ((x^*,c),(y^*,c)) \in N_{S_1}{}^a(\bar{x}_1) \times N_{S_2}{}^a(\bar{x}_2).\end{aligned}$$

■

4. Augmented Normal Cones and Weak Subdifferentials

Kruger in [13] introduced new approach in order to define the normal cone by using the Fréchet subdifferential of the distance function. Recall that the distance function to the set S is defined by the formula

$$d_S(x) = inf_{y \in S} \parallel x - y \parallel .$$

We are going to generalize this approach for augmented normal cones related by weak subdifferential in what follows. Contrary to the indicator function whose weak subdifferential can be used for defining the augmented normal cone, the distance function is Lipschitz continuous. This makes it more convenient in some situations.

Proposition 4.1.

$$\partial^w d_S(\bar{x}) \subset \{(x^*,c) \in N_S{}^c(\bar{x}) : \parallel x^* \parallel \le c+1\}.$$

Proof. Suppose that $(x^*,c) \in \partial^w d_S(\bar{x})$, then we have

$$d_S(x) - d_S(\bar{x}) \ge \langle x^*, x - \bar{x}\rangle - c \parallel x - \bar{x} \parallel \quad \forall x \in X$$

Hence if $x \in S$, we obtain

$$\langle x^*, x - \bar{x}\rangle - c \parallel x - \bar{x} \parallel \le 0,$$

and so $(x^*, c) \in N_S{}^c(\bar{x})$. Also if $x \notin S$, we get, note $\overline{x} \in S$,

$$\| x - \bar{x} \| \geq d_S(x) = \inf_{y \in S} \| x - y \| = d_S(x) \geq \langle x^*, x - \bar{x} \rangle - c \| x - \bar{x} \| \quad \forall x \notin S$$

therefore

$$\langle x^*, x - \bar{x} \rangle \leq (c+1) \| x - \bar{x} \| .$$

Consequently, it follows from the above inequalities that

$$\langle x^*, x - \bar{x} \rangle \leq (c+1) \| x - \bar{x} \| \quad \forall x \in X.$$

Then

$$\| x^* \| \leq c + 1.$$

∎

Remark 4.2. In Proposition 4.1 if we take $c = 0$ then we obtain:

$$\partial d_S(\bar{x}) \subset \{x^* \in N_S(\bar{x}) : \| x \| \leq 1\},$$

that is the result found by Kruger in [13] for Fréchet subdifferential.

The following example shows that the inclusion of Proposition 4.1 may be strict.

Example 4.3. Consider $S = [0, 1], \bar{x} = 0$, then we have

$$\partial^w d_S(0) = \emptyset, \quad \{(x^*, c) \in N_S{}^a(0) : \| x \| \leq c + 1\} \neq \emptyset.$$

It follows from Proposition 3.8 that an augmented normal cone is a particular case of a weak subdifferential. In the following we establish a link between the weak subdifferential of an arbitrary function and the augmented normal cone of its epigraph. Recall that the epigraph of f is the set

$$epif = \{(u, \mu) \in X \times \mathbb{R} : f(u) \leq \mu\}.$$

The following result shows that the relationship between weak subdifferential of f and Augmented normal cone related by $epif$.

Proposition 4.4. *1) If $(x^*, c) \in \partial^w f(\bar{x})$, then $((x^*, -1), c) \in N^c_{epif}(\bar{x}, f(\bar{x}))$,*
2) If $\mu \geq f(\bar{x})$ and $((x^*, \lambda), c) \in N^c{}_{epif}(\bar{x}, \mu)$, then $| \lambda | \leq c$.

Proof. 1) If $(x^*, c) \in \partial^w f(\bar{x})$,then we have

$$f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x} \rangle - c \| x - \bar{x} \| \quad \forall x \in X.$$

Then

$$\langle (x^*, -1), (x - \bar{x}, f(x) - f(\bar{x}) \rangle \leq c \| x - \bar{x} \| .$$

It is obvious that

$$c \| x - \bar{x} \| \leq c \| x - \bar{x} \| + c \mid f(x) - f(\bar{x}) \mid$$

and

$$c \| x - \bar{x} \| + c \mid f(x) - f(\bar{x}) \mid = c \| (x - \bar{x}, f(x) - f(\bar{x}) \| \quad \forall x \in X.$$

Thus the above inequalities imply

$$((x^*, -1), (x - \bar{x}, f(x) - f(\bar{x})) \leq c \| (x - \bar{x}, f(x) - f(\bar{x})) \| \quad \forall x \in X.$$

This means that $((x^*, -1), c) \in N^c_{epif}(\bar{x}, f(\bar{x}))$.

2) Suppose that $((x^*, \lambda), c) \in N^c{}_{epif}(\bar{x}, \mu)$, then we have:

$$\langle (x^*, \lambda), (x - \bar{x}, u - \mu) \rangle \leq c \| (x - \bar{x}, u - \mu) \| \quad \forall (x, u) \in epif.$$

If we take $x = \bar{x}, u = f(\bar{x})$, then

$$\lambda(f(\bar{x}) - \mu) \leq c \mid f(\bar{x}) - \mu \mid .$$

Therefore

$$(\lambda + c)(f(\bar{x}) - \mu) \leq 0,$$

and by taking $\mu \geq f(\bar{x})$, we get $\lambda \geq -c$.

Similarly from

$$\langle (x^*, \lambda), (x - \bar{x}, u - \mu) \rangle \leq c \parallel (x - \bar{x}, u - \mu) \parallel \quad \forall (x, u) \in epif$$

and $\mu = f(\bar{x})$ we have:

$$\langle (x^*, \lambda), (x - \bar{x}, u - f(\bar{x})) \rangle \leq c \parallel (x - \bar{x}, u - f(\bar{x})) \parallel \quad \forall (x, u) \in epif$$

for arbitrary $\epsilon > 0$, by taking $x = \bar{x}$ and $u = f(\bar{x}) + \epsilon$ in the last inequality, we deduce that

$$\lambda \epsilon \leq c \mid \epsilon \mid,$$

and so that, $\lambda \leq c$. This completes the proof. ∎

Acknowledgement

S. Suantai would like to thank Thailand Science Research and Innovation under the project IRN62W0007 for the financial support.

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