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Existence and Behavior of Solution of Some Nonlinear Equation from Theory of Flows on Networks

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Abstract In this article the mixed problem for the class of the nonlinear hyperbolic equations that have the nonlinear main parts is studied. The considered equations are of the type of equations from theory of flows on networks. Here for the considered problem is proved the solvability theorem and also is investigated the behaviour of his solutions.

MSC: 39A10

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1. Introduction

In this article we study a class of nonlinear hyperbolic equations that one can formulate in the form (in the case of 1-dimension space)

$$u_{tt} - (f(u)_x)_x = g(u), \quad (t,x) \in R_+ \times (0,l), \quad l > 0, \tag{1.1}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad u(t,0) = u(t,l), \tag{1.2}$$

where $u_0(x), u_1(x)$ are known functions, $f(\cdot), g(\cdot) : R \to R$ are continuous functions and $l > 0$ is a number. The equation of type (1.1) describes mathematical model of the problem from theory of the flow in networks as is affirmed in articles [1–9] (e.g. Aw-Rascle equations, Antman-Cosserat model, etc.). As in the survey [4] is noted such a study can find application in accelerating missiles and space crafts, components of high-speed machinery, manipulator arm, microelectronic mechanical structures, components of bridges and other structural elements. Balance laws are hyperbolic partial differential equations that are commonly used to express the fundamental dynamics of open conservative systems (e.g. [5]). As the survey [4] possess of the sufficiently exact explanations of the significance of equations of such type therefore we not stop on this theme. It need to note that most often in these articles in which the being investigated problem descrebe the hyperbolic equation of second order as mathematical model, then for investigation the

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authors reduce it to the system of equations of first order. As it is explained in the cited above survey on the mathematical properties of the AntmanCosserat model are similar to those of the first-order system associated with the nonlinear wave equation.

In this work we use different approach for study of the solvability of the posed problem. We would like to note that by use of this approach one can investigate of the solvability of problems for such class of the hyperbolic equations that have of the nonlinear main parts. Moreover it need to note that in this approach is used the Faedo-Galerkin approximation method.

This article is organized as follows. In Section 2 we consider the class of the nonlinear hyperbolic equations of second order of such type that are arisen in the theory of flows on networks. In Section 3 we investigate the solvability of the considered problems and in Section 4 the behavior of their solutions.

2. Formulation of Problem and Main Theorem

Consider the following problem

$$u_{tt} - div(f(u)\nabla u) = g(u), \quad (t,x) \in (0,T) \times \Omega, \quad T \in (0,\infty), \tag{2.1}$$

$$u(0,x) = u_0(x), \quad u_t = (0,x) = u_1(x), \quad u\mid_{(0,T)\times\partial\Omega} = 0, \tag{2.2}$$

where $\Omega \subset R^n, n \geq 1$ is a bounded domain with sufficiently smooth boundary $\partial\Omega, T > 0$ is arbitrary fixed number, $\nabla \equiv grad, o_t = \frac{\partial o}{\partial t}, f, g : R \to R$ are a continuous functions and $u_0(x), u_1(x)$ are known functions. It is necessary to note the problem (2.1)-(2.2) is a generalization of the problem (1.1)-(1.2) that studied in the case $n \geq 1$, i.e. in the many-dimensional case. We denote by H of the Lebesgue space $L^2(\Omega)$ with usual norm and by H_0^1 of Sobolev space $W_0^{1,2}(\Omega)$ with norm $\|v\|_{H_0^1} \equiv \|\nabla v\|_{L^2} \equiv \|\nabla v\|_H$, see, e. g. [10, 11] that are Hilbert spaces. As it is well known Laplace operator $-\Delta \equiv -divgrad$ is a self-adjoint, positive operator densely defined in a Hilbert space H and on H_0^1 moreover, $\Delta : H_0^1 \to H^{-1}$, where H^{-1} is the dual to H_0^1. Here we will use some properties of Laplace operator for study of the posed problem.

Assume that in what follows the following condition is fulfilled:

(i) $f(\cdot)$ is such function that the following function

$$F(r) = \int_0^r f(s)ds, \quad r > 0$$

is a monotone function, r be a number.

In the other words we will understud the solution of this problem in the following sense

Definition 2.1. A function

$$u \in C^0(0,T; L^p(\Omega)), \quad u_t \in L^\infty(0,T; H)$$

is called a very weak solution of the problem (2.1)-(2.2) if u satisfies the following equation

$$\langle u_{tt} - \Delta F(u), v\rangle = \langle g(u), v\rangle \tag{2.3}$$

locally by a. e. $t \in (0,T)$ for any $v \in W^{2,p}(\Omega) \bigcap H_0^1(\Omega)$, $u(t)$ and $u_t(t)$ are weakly continuous on $[0,T]$ in the sense of the appropriate spaces.

It need to note the expression $\langle o, o\rangle$ denotes the following:

$$\langle v, w\rangle = \int_\Omega v(x)\cdot w(x)dx$$

for the appropriate functions $v \in X, w \in X^*$, where X is Banach space and X^* is the dual space of X.

Consider the following conditions

1) Let $f, g : R \to R$ are a continuous functions and there are a numbers $a_0, b_0, d > 0, a_1, b_1 \geq 0$ and $p > 2, 0 \leq 2p_0 \leq p$ such that the following inequations

$$\mid F(r) \mid\leq a_0 \mid r \mid^{p-1} + a_1 \mid r \mid; \quad F(r)\cdot r \geq b_0 \mid r \mid^p + b_1 r^2; \quad \mid g(r) \mid\leq d \mid r \mid^{P_0},$$

hold for any $r \in R$, moreover g is continuous function (for example, $f(r) = k_0 \mid r \mid^{p-2} -k_1 \mid r \mid^{p_1} + k_2, k_2 > 0, k_1, k_2 \geq 0, 0 \leq p_1 < p-2$, moreover $k_1 = k_1(k_0, k_2)$ and $g(r) = d \mid r \mid^{p_0}$.

2) Let the function $g : R \to R$ is the Lipschitz function, i.e. there exists such number $d_0 > 0$ that the following inequality

$$\mid g(r) - g(s) \mid\leq d_0 \mid r - s \mid \tag{2.4}$$

holds for any $r, s \in R$.

We can formulate the main theorem on the solvability of the considered problem in the following form.

Theorem 2.2. (Main Theorem) *Let functions F and g satisfy Condition 1 (F is defined in (i) by f), function g also satisfies Condition 2. Then if $u_0 \in H_0^1 \bigcap W^{1,p}(\Omega), u_1 \in L^p(\Omega)$ then the problem* (2.1)–(2.2) *possess a very weak solution $u(t,x)$ in the sense of Definition* 2.1.

3. Preliminary Results And Approach

It is well known ([10, 12–14]) that under the conditions of this problem the following problem

$$-\Delta v = w, \quad x \in \Omega \subset R^n, \quad v\mid_{(0,T)\times\partial\Omega} = 0 \tag{3.1}$$

is solvable for any $w \in L^p(\Omega), p > 1$ and has unique solution in $W^{2,p}(\Omega)\bigcap W_0^{1,p}(\Omega)$, i.e. the operator $-\Delta : W^{2,p}(\Omega)\bigcap W_0^{1,p}(\Omega) \to L^p(\Omega)$ is the isomorphism. Consequently if to set the denotation $u \equiv -\Delta v$ then of the posed problem one can rewrite in the form

$$-\Delta v_{tt} - \nabla\cdot(f(-\Delta v)\nabla(-\Delta v)) = g(-\Delta v), \quad (t,x) \in (0,T)\times\Omega,$$

or

$$-\Delta v_{tt} - \Delta F(-\Delta v) = g(-\Delta v), \quad (t,x) \in (0,T)\times\Omega,$$
$$-\Delta v(0,x) = u_0(x), \quad -\Delta v_t(0,x) = u_1(x), \quad u\mid_{(0,T)\times\partial\Omega} = 0.$$

We will study the solvability of the posed problem in very weak sense therefore for this aim we will use the following approach. Consider the following dual form for the equation (2.1)

$$\left[u_{tt} - \Delta F(u), (-\Delta)^{-1}u_t\right] = \left[g(u), (-\Delta)^{-1}u_t\right], \tag{3.2}$$

where the expression $[o, o]$ denotes the following:

$$[o, o] = \int_0^t \langle o, o\rangle ds$$

for functions $v(t,x), w(t,x)$ from the appropriate spaces. It is need to note that here we will use the approach, which we used in the article [15] for study of the differential-operator problem in the Banach space.

So from the equation (3.2) we get

$$\frac{1}{2}\frac{d}{dt}\|\nabla v_t\|_2^2 + \frac{d}{dt}\Phi(-\Delta v) = \langle g(-\Delta v), v_t\rangle$$

here Φ is a nonnegative functional and $\Phi(u) = \int_0^1 \langle F(su), u\rangle ds$.

If to bear in mind of these conditions and Condition 1 we get

$$\frac{1}{2}\frac{d}{dt}\|\nabla v_t\|_2^2 + \frac{d}{dt}\Phi(-\Delta v) \leq \frac{1}{2}\|v_t\|_2^2 + \frac{1}{2}\|g(-\Delta v)\|_2^2(t) \leq c\left[\frac{1}{2}\|\nabla v_{tt}\|_2^2 + \Phi(-\Delta v)\right] + \hat{d},$$

here $c > 0, \hat{d} \geq 0$ are constants that independent of v. Hence follows

$$\|\nabla v_t\|_2^2(t) + 2\Phi(-\Delta v)(t) \leq e^{ct}\left[\|\nabla v_1\|_2^2 + 2\Phi(u_0\right] + \frac{\hat{d}}{c}(e^{ct} - 1).$$

by virtue of the Gronwalls lemma. Thus we obtain

Lemma 3.1. *Let the condition 1 is fulfilled then each solution of the problem* (2.1)–(2.2) *satisfy the following inequalities*

$$\|\nabla v_t\|_2^2(t) \leq e^{cT}\left[\|\nabla v_1\|_2^2 + 2\Phi(u_0)\right] + \frac{\hat{d}}{c}(e^{cT} - 1) \tag{3.3}$$

$$\Phi(-\Delta v)(t) \leq e^{cT}\left[\|\nabla v_1\|_2^2 + 2\Phi(u_0)\right] + \frac{\hat{d}}{c}(e^{cT} - 1)$$

for every fixed $T \in (0, \infty)$.

Consequently, from the estimates (3.3) implies that the following a priori estima

$$v_t \in L^\infty(0,T;H_0^1), \quad u \in L^\infty(0,T;L^p(\Omega)) \Rightarrow v \in L^\infty(0,T;W^{2,p}(\Omega))$$

hold, moreover v belong to bounded subset of these spaces, by virtue of Condition 1.

Remark 3.2. It not is difficult to see that if $\hat{d} = 0$ then occurs the inequation

$$\|\nabla v_t\|_2^2(t) + 2\Phi(u)(t) \leq e^{cT}\left[\|\nabla v_1\|_2^2 + 2\Phi(u_0)\right]$$

for a. e. $t \in (0, T]$.

4. Proof of Main Theorem

Proof. (of Main Theorem) For the proof of the solvability of the problem (2.1)-(2.2) we will use the Faedo-Galerkin approximation method.

Let the system $U \equiv \{w_j(x)\}_{j=1}^\infty$ be a total system of the space

$$W^{2,p}(\Omega)\bigcap H_0^1(\Omega)$$

where $w_j(x)$ be the sufficiently smooth functions. Let functions $v_0 \in H_0^1 \bigcap W^{2,p}(\Omega), v_1 \in H_0^1 \bigcap W^{1,p}(\Omega)$ that satisfy equations: $-\Delta v_k = u_k, k = 0, 1$.

We will seek out of the approximative solutions $u_m(t,x)$ in the form

$$(-\Delta)^{-1}u_m(t,x) = v_m(t,x) = \sum_{i=1}^m c_i(t)w_i(x) \text{ or } u_m(t) \in span\{w_1, \ldots, w_m\}$$

as the solutions of the considered problem, where $c_i(t)$ are as the unknown functions that will be defined as solutions of the following Cauchy problem for system of ODE

$$\frac{d^2}{dt^2}\langle u_m, w_j\rangle - \langle F(u_m), \Delta w_j\rangle = \langle g(u_m), w_j\rangle, \quad j = 1, 2, \ldots, m \tag{4.1}$$

$$u_m(0, x) = u_{0m}(x), u_{tm}(0, x) = u_{1m}(x),$$

where u_{0m} and u_{1m} are contained in $span\{w_1, \ldots, w_m\}$, $m = 1, 2, \ldots$, moreover $u_{0m} \to u_0$ in $H_0^1 \bigcap W^{1,p}(\Omega)$; $\quad u_{1m} \to u_1$ in $H \bigcap L^p(\Omega)$ at $m \nearrow \infty$.

Remark 4.1. We would like to note that if to take into account of the inequation (2.4) and carry out above mentioned calculations then for approximative solutions $u_m(t, x)$ we obtain a priori estimates of same type as the estimates in (3.3) (with appropriate coefficients).

Thus we obtain the following problem

$$\frac{d^2}{dt^2}\langle u_m, w_j\rangle = \langle F(u_m), \Delta w_j\rangle + \langle g(u_m), wj\rangle, \quad j = 1, 2, \ldots, m \tag{4.2}$$

$$\langle u_m(t, x), w_j\rangle \mid_{t=0} = \langle u_{0m}(x), w_j\rangle, \quad \frac{d}{dt}\langle u_m(t, x), w_j\rangle \mid_{t=0} = \langle u_{1m}(x), w_j\rangle$$

that, as it is well-known, is solvable locally with respect to t.

Moreover, this problem is solvable on $(0, T]$ for any $m = 1, 2, \ldots$ and $T > 0$ by virtue of the estimates (3.3). Consequently, with use of the known procedure ([14, 16–18]) we obtain, $\nabla v_{mt} \in L^\infty(0, T; H), \nabla v_m \in L^\infty(0, T; L^p(\Omega))$. Hence follows that $u_m \in L^\infty(0, T; L^p(\Omega)), u_{mt} \in L^\infty(0, T; H^{-1}(\Omega))$, moreover they are contained in a bounded subset of these spaces by definition of $v_m(t)$. In the other hand, since $u_m(t), (v_m(t))$ is the solution of the system of ODEs and F, g are continuous, therefore the following inclusions

$$\nabla v_{mt} \in C^0(0, T; H(\Omega)), \Delta v_m \in C^0(0, T; L^p(\Omega))$$

and

$$u_{mt} \in C^0(0, T; H^{-1}(\Omega)), u_m \in C^0(0, T; L^p(\Omega))$$

hold[1].

Thus from (4.2) we get

$$u_{mtt} \in C^0\left(0, T; \left(W^{2,p}(\Omega) \bigcap W_0^{1,p}(\Omega)\right)^* + H^{-1}(\Omega)\right).$$

So, for the sequence of the approximate solutions we have: $\{u_m\}_{m=1}^\infty$ is contained in a bounded subset of the space[2]

$$C^0(0, T; L^p(\Omega)) \bigcap C^2(0, T; W^{-2,q}(\Omega) + H^{-1}(\Omega))$$

and the sequence $\{v_m\}_{m=1}^\infty$ is contained in a bounded subset of the space

$$C^0\left(0, T; W^{2,p}(\Omega) \bigcap H_0^1(\Omega)\right) \bigcap C^1(0, T; H_0^1(\Omega)) \bigcap C^2(0, T; L^q(\Omega))$$

[1]Hence follows that u_m and Δu_m are weak continuous over t with respect to the appropriate spaces, e.g. $\langle u_m(t), \omega\rangle$ is continuous for any $w \in L^p(\Omega)$.

[2]Since one can easily see that all arguments which were reduced here for $n \geq 3$ are correct for cases $n = 1, 2$ therefore we will not consider here of these cases separately.

Then the sequence $\{v_m\}_{m=1}^{\infty}$ has a precompact subset in the space

$$C^1\left(0,t;\left[W^{2,p}(\Omega),L^q(\Omega)\right]_{\frac{1}{2}}\right),$$

by virtue of the known interpolation theorems (see, [11]), and consequently, in the space $C^1(0,T;H_0^1(\Omega))$ since the imbedding $\left[W^{2,p}(\Omega),L^q(\Omega)\right]_{\frac{1}{2}}\subseteq H^1(\Omega)$ holds.

Thus for us is remained to show the following: if the sequence

$$\{u_m\}_{m=1}^{\infty}\subset C^0\left(0,T;L^p(\Omega)\right)\bigcap C^2(0,T;W^{-2,q}(\Omega)+H^{-1}(\Omega)$$

is weakly converging to u in this space and $\{F(u_m)\}_{m=1}^{\infty}$ and $\{g(u_m)\}_{m=1}^{\infty}$ have an weakly converging subsequence to η in H and to θ in $L^q(\Omega)$ $\left(q=\frac{p}{p-1}\right)$ respectively, for a. e. $t\in(0,T)$ then $\eta=F(u)$ and $\theta=g(u)$. (Here and in what follows for brevity we dont changing of indexes of subsequences.)

In the beginning we will show the equation $\theta=g(u)$. Let the sequence $\{u_m\}_{m=1}^{\infty}$ is such as above mentioned and $-\Delta v_m=u_m$. The for the operator

$$g:C^0\left(0,T;L^p(\Omega)\right)\subset C^0(0,T;H)\to C^0(0,T;H)$$

we have

$$\langle g(u_m),w_j)\rangle\to\langle\theta,w_j\rangle\quad for\forall j:j=1,2,\ldots$$

and also

$$\langle g(u_m),z)\rangle\to\langle\theta,z\rangle\quad for\forall z\in W^{2,p}(\Omega)\subset H(\Omega),$$

according to the condition (2.4).

Therefore, we consider the expression $\langle g(u_m),v_m)\rangle$ under the assumption that $u_m\rightharpoonup u$ in $L^p(\Omega)\subset H$ and $v_m\to v$ in $H^1(\Omega)$ and $g(u_m)\rightharpoonup\theta$ in the corresponding spaces. In order to prove that $\langle g(u_m),v_m)\rangle$ is the Cauchy sequence we carry out the following estimations

$$\begin{aligned}\|\langle g(u_m),v_m\rangle-\langle g(u_{m+k}),v_{m+k}\rangle\|\leq\|\langle g(u_m)-g(u_{m+k}),v_m\rangle\|+\\ \|\langle g(u_{m+k}),v_m-v_{m+k}\rangle\|\leq\langle\|g(u_m)-g(u_{m+k})\|,\|v_m\|\rangle+\\ \|\langle g(u_{m+k}),v_m-v_{m+k}\rangle\|\leq d_0\langle\|u_m-u_{m+k}\|,\|v_m\|\rangle+\|\langle g(u_{m+k}),v_m-v_{m+k}\rangle\|\end{aligned}\tag{4.3}$$

that shows the correctness of this statement since the right side converge to zero with respect to $m\nearrow\infty$. If to take account of the above assumpsion we can conduct the estimation of such type (4.3) for the expression $\|\langle g(u_m),v_m\rangle-\langle g(u),v\rangle\|$, as $g(u)$ is defined, then we obtain that equation $\theta=g(u)$ holds, i.e. $g(u_m)\rightharpoonup g(u)$ in H. In order to show the equation $\eta=F(u)$ we will use the monotonicity condition of F, i. e. for any $v,w\in C^0\left(0,T;W^{2,p}(\Omega)\right)\bigcap C^2\left(0,T;L^p(\Omega)\right)$ occurs the following inequation

$$\langle-\Delta F(-\Delta v)+\Delta F(-\Delta\tilde{v}),v-\tilde{v}\rangle\geq 0$$

and if rewrite it for $u_m=-\Delta v_m$ and $\tilde{u}=-\Delta\tilde{v}$ then we have

$$\langle\left(F(u_m-F(\tilde{u})\right),u_m-\tilde{u}\rangle\geq 0.$$

It is not difficult to see that the following convergence takes

$$\frac{d}{dt}\langle u_{mt},w_j\rangle-\langle\Delta F(u_m),w_j\rangle-\langle g(u_m),w_j\rangle\to\frac{d}{dt}\langle u_t,w_j\rangle-\langle\Delta\eta,w_j\rangle-\langle\theta,w_j\rangle,\quad\forall w_j$$

then

$$\frac{d^2}{dt^2}\langle u, w\rangle - \langle \Delta\eta, w\rangle = \langle g(u), w\rangle, \quad \forall w \in H_0^1 \bigcap W^{2,p}(\Omega) \tag{4.4}$$

for a. e. $t \in (0, T)$ by virtue of the obtained above equation $\theta = g(u)$. Consequently,

$$u_{tt} - \Delta\eta = g(u), \quad \text{in the sense of } H^{-1} + W^{-2,q}(\Omega)$$

for a. e. $t \in (0, T)$.

Let us apply monotonicity of F

$$0 \leq \langle F(u_m) - F(\tilde{u}), u_m - \tilde{u}\rangle = -\langle \Delta F(u_m) + \Delta F(\tilde{u}), v_m - \tilde{v}\rangle =$$

$$-\langle \Delta F(u_m), v_m\rangle + \langle \Delta F(u_m), \tilde{v}\rangle + \langle \Delta F(\tilde{u}), v_m - \tilde{v}\rangle =$$

(where $\tilde{u} = -\Delta\tilde{v}, \tilde{v} \in H_0^1 \bigcap W^{2,p}(\Omega)$) by use here the equation (4.2) we get

$$\langle -g(u_m + \frac{\partial^2}{\partial t^2} u_m, \tilde{v}\rangle - \langle \Delta F(u_m), v_m\rangle + \langle \Delta F(\tilde{u}, v_m - v)\rangle =$$

$$-\langle g(u_m), \tilde{v}\rangle + \frac{\partial^2}{\partial t^2}\langle u_m, \tilde{v}\rangle + \langle F(u_m), u_m\rangle + \langle \Delta F(\tilde{u}, v_m - \tilde{v})\rangle \Rightarrow$$

whence we obtain

$$0 \leq \langle g(u), \tilde{v}\rangle + \frac{d^2}{dt^2}\langle u, \tilde{v}\rangle + \langle F(u_m), \tilde{u_m}\rangle + \langle \Delta F(\tilde{u}), v - \tilde{v}\rangle \tag{4.5}$$

If pass to the limit with respect to $m : m \nearrow \infty$ in the inequation (4.5) and to take into account the following known inequation

$$\int_\Omega \liminf \left(F(u_m)u_m\right) dx \leq \langle \eta, u\rangle$$

(by the Fatous lemma, more exactly

$$\int_\Omega \liminf \left(F(-\Delta v_m)(-\Delta v_m)\right) dx \leq \langle \eta, u\rangle,$$

since $\langle -\Delta F(u_m), v_m\rangle = \langle F(-\Delta v_m), -\Delta v_m\rangle$) then with use of the equation (4.4) we get

$$0 \leq -\langle g(u), \tilde{v}\rangle + \frac{d^2}{ds^2}\langle \Delta v, \tilde{v}\rangle - \langle \Delta\eta, v\rangle + \langle \Delta F(\tilde{u}, v - \tilde{v}\rangle =$$

$$\langle -\Delta\eta, v - \tilde{v}\rangle - \langle -\Delta F(\tilde{u}), v - \tilde{v}\rangle = \langle \eta - F(\tilde{u}), u - \tilde{u}\rangle.$$

Hence we obtain the correctness of the equation $\eta = F(u)$ by virtue of arbitrariness of $\tilde{u} = -\Delta\tilde{v}$. So, we proved that limiting function $u(t, x)$ satisfies of the equation (2.3) in the sense of Definition 2.1 from the section 2. Now we will show that the function $u(t, x)$ satisfies of the initial conditions and for this we will consider the following equation

$$\langle u_{mt}, v_m\rangle(t) = \int_0^t \langle u_{mss}, v_m\rangle ds + \int_0^t \langle u_{ms}, v_{ms}\rangle ds + \langle u_{1m}, v_{0m}\rangle$$

for $t \in (0, T]$ and $u_m = -\Delta v_m$, that is equivalent to the equation

$$\langle \nabla v_{mt}, \nabla v_m\rangle(t) = \int_0^t \langle v_{mss}, u_m\rangle ds + \int_0^t \langle \nabla v_{ms}, \nabla v_{ms}\rangle ds + \langle \nabla v_{1m}, \nabla v_{0m}\rangle \tag{4.6}$$

which takes place for each $m = 1, 2, \ldots$. From obtained a priory estimations follow the boundedness of the right side of (4.6), consequently we get the boundedness of the left side of (4.6) any $t \in (0, T]$. Therefore, one can pass to the limit by $t \to 0$ by virtue of the a priory estimations. Really since

$$\{v_m\}_{m=1}^{\infty} \in C^0\left(0, T; W^{2,p}(\Omega)\right) \bigcap C^2\left(0, T; L^q(\Omega)\right)$$

and is bounded in this space we get: the right side is bounded as all terms in the left side are bounded in respective spaces, therefore one can pass to limit with respect to m as here v_{mt} are continous with respect to t for any m then v_{mt} strongly converges to v_t in H and Δv_m weakly converges to Δv in $L^p(\Omega)$. Consequently, we obtain the following equation

$$\langle u, v\rangle(t) = \int_0^t \langle v_{ss}, u\rangle ds + \int_0^t \langle \nabla v_s, \nabla v_s\rangle ds + \langle v_1, v_0\rangle$$

for, at least, a. e. $t \in (0, T)$, by virtue of the previous part of theis proof and the our assumption on the sequences $\{v_{0m}\}_{m=1}^{\infty}, \{v_{1m}\}_{m=1}^{\infty}$.

Thus the main theorem completely proved. ∎

Remark 4.2. In particular, from here follows, the solvability of the following nonlinear equation with the mixed condition such as above, which not were studied earlier

$$u_{tt} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\mid u \mid^{p-2} \frac{\partial u}{\partial x_i}\right) = h(t, x), \quad p > 2.$$

Remark 4.3. It should be noted that by using (3.1) one can reformulate of the considered problem in the following form: let $g(u) \equiv h(t, x)$ is given function

$$-\Delta\left(v_{tt} + F(-\Delta v)\right) = g(t, x) \equiv -\Delta \tilde{g}, \quad (t, x) \in (0, T) \times \Omega,$$

$$-\Delta v(0, x) = u_0(x) = -\Delta v_0(x),$$

$$-\Delta v_t(0, x) = u_1(x) = -\Delta v_1(x), \Delta v \mid_{(0,T)\times\partial\Omega} = 0.$$

In the other words we get

$$-\Delta\left(v_{tt} + F(-\Delta v) - \tilde{g}\right) = 0, \quad (t, x) \in (0, T) \times \Omega \tag{4.7}$$

hence one can obtain the following equivalent problem if F is the homogeneous operator

$$v_{tt} + F(-\Delta v) = \tilde{g}, \quad (t, x) \in (0, T) \times \Omega,$$

$$v(0, x) = v_0(x), v_t(0, x) = v_1(x), v \mid (0, T) \times \partial\Omega = 0$$

since if the equation (4.7) possess a solution then the expression $v_{tt} + F(-\Delta v) - \tilde{g}$ is a harmonic function for each t and also satisfies the homogeneous boundary condition. In this case we get, that the considered problem is equivalent to the problem

$$v_{tt} + F(-\Delta v) = \tilde{g}(t, x)$$

with the mixed conditions of such type as above.

5. Behavior of Solution of Problem (2.1)-(2.2)

Now we will investigate of the behavior of the solutions of the problem (2.1)-(2.2) under the following assumption:

g satisfies inequation $\mid g(r) \mid^2 \leq d_1\Phi(r)$ for any $r \in R$, where $d_1 > 0$. So, we will study the behavior of solution under $t \nearrow \infty$ of the problem

$$u_{tt} - \Delta F(u) = g(u), \quad (t,x) \in R_+ \times \Omega,$$

$$u(0,x) = u_0(x), u_t(0,x) = u_1(x), u \mid_{R_+\times\partial\Omega} = 0$$

for which behaving as above we get the equation

$$\|\nabla v_t\|_2^2 (t) + 2\Phi(-\Delta v)(t) = \|\nabla v_1\|_2^2 + 2\Phi(-\Delta v_0) + 2\langle g(u), v_t\rangle. \tag{5.1}$$

Remark 5.1. It not is difficult to see that if $g(u) \equiv 0$ then the equation (5.1) give we the energy functional that remain constant for $\forall t > 0$, i.e. the energy functional is independent of $t > 0$.

From (5.1) ensue the following inequality

$$\|\nabla v_t\|_2^2 (t) + 2\Phi(u)(t) \leq \|\nabla v_1\|_2^2 + 2\Phi(u_0) + \int_0^t \left[\|\nabla v_s\|_2^2 + \|g(u)\|_2^2\right](s)ds$$

then using the condition on $g(u)$ we have

$$\|\nabla v_t\|_2^2 (t) + 2\Phi(u)(t) \leq \|\nabla v_1\|_2^2 + 2\Phi(u_0 + \tilde{d}\int_0^t \left[\|\nabla v_s\|_2^2 + 2\Phi(u)\right](s)ds$$

Hence follows

$$\|\nabla v_t\|_2^2 (t) \leq \frac{1}{\tilde{d}}\left[e^{\tilde{d}t}\left(1+\tilde{d}\right) - 1\right]\left(\|\nabla v_1\|_2^2 + 2\Phi(u_0)\right) - 2\Phi(u)(t) \tag{5.2}$$

Introduce the function $E(t) = \|\nabla w\|_H^2 (t)$ and consider this function on the solution of the problem (2.1)-(2.2).

For the derivative of functional $E(t) = \|\nabla v\|_2^2 (t)$ we get

$$E_t(t) = 2\langle\nabla v_t, \nabla v\rangle \leq \|\nabla v_t\|_2^2 (t) + \|\nabla v\|_2^2 (t)$$

using here the inequation (5.2)

$$E_t(t) \leq E(t) - 2\Phi(u)(t) + \frac{1}{\tilde{d}}\left[e^{\tilde{d}t(1+\tilde{d})} - 1\right]\left(\|\nabla v_1\|_2^2 + 2\Phi(u_0)\right)$$

Hence using the condition on F (consequently, on Φ)

$$E_t(t) \leq E(t) - c\,\|-\Delta v\|^p (t) + \frac{1}{\tilde{d}}\left[e^{\tilde{d}t(1+\tilde{d})} - 1\right]\left(\|\nabla v_1\|_2^2 + 2\Phi(u_0)\right) \leq$$

$$E(t) - cE^{\frac{p}{2}}(t) + \frac{1}{\tilde{d}}\left[e^{\tilde{d}t(1+\tilde{d})} - 1\right]\left(\|\nabla v_1\|_2^2 (0) + 2\Phi(-\Delta v_0)\right) \Rightarrow$$

and at last we get

$$E_t(t) \leq E(t) - cE^{\frac{p}{2}}(t) + C_1(v_0,v_1)e^{\tilde{d}t} - C_2(v_0,v_1),$$

by virtue of the condition $\Phi(r) \geq c_0 \mid r \mid^p$ and of the continuity of embeddings $L^p(\Omega) \subset L^2(\Omega)$,$W^{2,p}(\Omega) \subset W^{1,p}(\Omega)$, where $C_j(v_0,v_1) > 0 \quad (j = 1,2)$ are constants.

So, we obtain the Cauchy problem for differential inequality

$$y_t(t) \le y(t) - cy^r(t) + C_1 e^{\tilde{d}t} - C_2, \quad y(0) = \|\nabla v_0\|_2^2 \tag{5.3}$$

where $r = p/2$. One can replace the problem (5.3) with the following problem in order to investigate of the behaviour of the solution of considered problem

$$y_t(t) \le y(t) - cy^r(t) + C_1 e^{\tilde{d}T} - C_2, \quad y(0) = \|\nabla v_0\|_2^2$$

since $\tilde{d} > 0$. The inequation (5.3) one can rewrite in the form

$$(y(t) + lC(v_0, v_1))_t \le y(t) + lC(v_0, v_1) - \varepsilon \left[y(t) + lC(v_0, v_1)\right]^r,$$

where $l > 1$ is a number, $\varepsilon = \varepsilon(c, C, l, r) > 0$ is sufficiently small number and $C = C(\tilde{d}, T, C_1, C_2)$ is a constant.

Then solving this problem we get

$$y(t) + lC(v_0, v_1) \le \left[e^{(1-r)t} \left(\nabla y_0 + lC(v_0, v_1)\right)^{1-r} + \varepsilon \left(1 - e^{(1-r)t}\right)\right]^{\frac{1}{1-r}}$$

or

$$E(t) \le \left[e^{(1-r)t} \left(\langle \nabla v_0 \rangle_H^2 + lC(v_0, v_1)\right)^{1-r} + \varepsilon \left(1 - e^{(1-r)t}\right)\right]^{\frac{1}{1-r}} - lC(v_0, v_1)$$

$$\langle \nabla v_0 \rangle_H^2(t) \le \frac{e^t \left(\langle \nabla v_0 \rangle_H^2 + lC(v_0, v_1)\right)}{\left[1 + \varepsilon \left(\langle \nabla v_0 \rangle_H^2 + lC(v_0, v_1)\right)^{r-1} \left(e^{(r-1)t} - 1\right)\right]^{\frac{1}{r-1}}} - lC(v_0, v_1) \tag{5.4}$$

here the right side is greater than zero, because $\varepsilon \le \frac{l-1}{l^r C^r}$ and $2r = p > 2$. It is necessary to note the dependence of the behavior of the solution at T is essentially, this follows from the last inequation. It should be noted that if we (roughly) simplify of the inequation (5.4) then it one can rewrite in the following form

$$\|\nabla v\|_H^2(t) \le e^T \left(\|\nabla v_0\|_H^2 + lC(v_0, v_1)\right) - lC(v_0, v_1).$$

Thus is proved the result

Theorem 5.2. *Let $u_0 \in H_0^1 \bigcap W^{1,p}(\Omega)$, $u_1 \in L^p(\Omega)$ and the appropriate functions $v_0 \in H_0^1 \bigcap W^{2,p}(\Omega)$, $v_1 \in H_0^1 \bigcap W^{1,p}(\Omega)$ satisfy equations $-\Delta v_k = u_k, k = 0, 1$. Then function $v(t, x)$, defined by the solution $u(t, x)$ of the problem* (2.1)–(2.2)*, for any $t \in (0, T)$ belong to ball $B_{R_T}^{H_0^1 \bigcap W^{1,p}(\Omega)}(0) \subset H_0^1 \bigcap W^{1,p}(\Omega)$ depending from (v_0, v_1), consequently from the inital values $u_0, u_1 \in H_0^1 \bigcap W^{1,p}(\Omega) \times L^p(\Omega)$, here $R_T = R_T(u_0, u_1, p, T) > 0$*

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