Thai Journal of Mathematics
Volume 19 Number 4 (2021)
Pages 1585–1599
http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

A Study of Generalized Quasi-Hyperideals in Ordered Ternary Semihypergroups

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Abstract In this paper, we introduce and study generalized quasi-hyperideals in ordered ternary semihypergroups. Also, we define some generalized kinds of hyperideals in ordered ternary semihypergroups and study the relation between them.

MSC: 20N20; 20M17; 20N99; 06F99

Keywords: ordered ternary semihypergroup; generalized quasi-hyperideal

Submission date: 28.10.2018 / Acceptance date: 18.03.2021

1. Introduction

Firstly, Kasner's [1] gave the idea of n-ary algebras *i.e.* the sets with one n-ary operation. Dörnte [2] introduced the notion of n-ary groups. n-ary semigroup is said to ternary semigroup for n=3 [3] if it satisfies one associative ternary operation (cf. [4]). Ideal theory in ternary semigroup were studied by Sioson [5]. He also defined regular ternary semigroups and characterized them in terms of quasi-ideals. In [6, 7] Dudek et. al. studied the ideals in n-ary semigroups. Dixit and Dewan [8] studied the properties of quasi-ideals and bi-ideals in ternary semigroups.

Marty [9] proposed the notion of algebraic hyperstructures as an extension of the branch of classical algebraic structures. In algebraic structures, the composition of two elements is an element, while in algebraic hyperstructure the composition of two elements is a non-empty set. Important notions and results on hypergroups can be found in [10]. Corsini and Leoreanu [11] explored various applications in hyperstructures.

Ternary semihypergroups are algebraic structures with one ternary associative hyperoperation. A ternary semihypergroup is a particular case of an n-ary semihypergroup (n-semihypergroup) for $n = 3$ (cf. [12]). Davvaz and Leoreanu [13] studied binary relations on ternary semihypergroups and studied some basic properties of compatible relations on them. Hila and Naka [14] defined the notion of regularity in ternary semihypergroups and

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Published by The Mathematical Association of Thailand.

characterized them by using various hyperideals of ternary semihypergroups. In [15–17], Hila et al. studied some classes of hyperideals in ternary semihypergroups.

Steinfeld [18] introduced the notion of quasi-ideal for semigroups. It is a generalization of the notion of one sided ideal. The concept of the (m, n)-quasi-ideal in semigroups was given by Lajos [19]. This notion and its generalization is studied by many researchers in different algebraic structures [20–22]. Firstly, Hila et al. [23] introduced quasi-hyperideals and (m, n)-quasi-hyperideals in semihypergroups and then Hila et al. [24, 25] introduced and studied the structure of quasi-hyperideals, bi-hyperideals and their generalization in ternary semihypergroups.

The concept of ordering hypergroups was investigated by Chavlina [26] as a special class of hypergroups. Heidari and Davvaz [27] studied a semihypergroup $(S, \circ)$ besides a binary relation $\leq$, where $\leq$ is a partial order relation such that satisfies the monotone condition. In [28], polygroups which are partially ordered were introduced and some properties and related results were obtained. Yaqoob and Gulistan [29] introduced the concept of partially ordered left almost semihypergroups and studied their related properties.

The way we pass from the results on ordered semigroups based on ideals to the results on semigroups-without order-based on ideals, and conversely was shown in [30]. Iampan [31] gave the definition of an ordered ternary semigroup and characterized the minimality and maximality of ordered lateral ideals in ordered ternary semigroups (cf. [32]). Also, some works on ordered ternary semigroups in terms of different kinds of ideals and their generalizations have been done in [21, 33]. Recently, in [34–36] classes of ordered ternary semihypergroups have been characterized in terms of int-soft hyperideals and fuzzy hyperideals.

In this paper, we introduce and study generalized quasi-hyperideals in ordered ternary semihypergroups. Also, we define some generalized kinds of hyperideals in ordered ternary semihypergroups and study the relation between them. Several examples illustrate the results.

2. Preliminaries

Let $\mathcal{H}$ be a non-empty set and let $\wp^*(\mathcal{H})$ be the set of all non-empty subsets of $\mathcal{H}$. A hyperoperation on $\mathcal{H}$ is a map $\circ : \mathcal{H} \times \mathcal{H} \to \wp^*(\mathcal{H})$ and the couple $(\mathcal{H}, \circ)$ is called a hypergroupoid. A hypergroupoid $(H, \circ)$ is called a semihypergroup if for all x, y, z of H we have $(x \circ y) \circ z = x \circ (y \circ z)$, which means that

$$\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v$$

If $x \in \mathcal{H}$ and A and B are non-empty subsets of $\mathcal{H}$, then we denote

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b,\ x\ o\ A = \{x\} \circ A \text{ and } A\ o\ x = A \circ \{x\}.$$

Definition 2.1 ([3])**.** A non-empty set $\mathcal{H}$ with a ternary operation $\mathcal{H} \times \mathcal{H} \times \mathcal{H} \to \mathcal{H}$, written as $(x_1, x_2, x_3) \mapsto [x_1, x_2, x_3]$, is called a ternary semigroup if it satisfies the following identity, for any $x_1, x_2, x_3, x_4, x_5 \in \mathcal{H}$,

$$[[x_1 x_2 x_3] x_4 x_5] = [x_1 [x_2 x_3 x_4] x_5] = [[x_1 x_2 [x_3 x_4 x_5]].$$

For non-empty subsets A, B and C of a ternary semigroup $\mathcal{H}$,

$$[ABC] := \{[abc] : a \in A, b \in B \text{ and } c \in C\}.$$

If $A = \{a\}$, then we write $[\{a\}BC]$ as $[aBC]$ and similarly if $B = \{b\}$ or $C = \{c\}$, we write $[AbC]$ and $[ABc]$, respectively. For the sake of simplicity, we write $[x_1x_2x_3]$ as $x_1x_2x_3$ and $[ABC]$ as ABC.

Definition 2.2. A ternary hypergroupoid is the pair $(\mathcal{H}, f)$ where f is a ternary hyperoperation on the set $\mathcal{H}$, that is, $f : \mathcal{H} \times \mathcal{H} \times \mathcal{H} \to \wp^*(\mathcal{H})$.

If A, B, C are non-empty subsets of H, then we define

$$f(A, B, C) = \bigcup_{a \in A, b \in B, c \in C} f(a, b, c).$$

Definition 2.3. A ternary hypergroupoid $(\mathcal{H}, f)$ is called a ternary semihypergroup [13] if for all $a_1, a_2, ..., a_5 \in H$, we have

$$f(f(a_1, a_2, a_3), a_4, a_5) = f(a_1, f(a_2, a_3, a_4), a_5) = f(a_1, a_2, f(a_3, a_4, a_5)).$$

Since the set $\{x\}$ can be indentified with the element x, any ternary semigroup is a ternary semihypergroup.

It is clear that due to the associative law in ternary semihypergroup $\mathcal{H}$, for any elements $x_1, x_2, ..., x_{2n+1} \in \mathcal{H}$ and positive integers m, n with $m \leq n$, one may write,

$$\begin{aligned} f(x_1, x_2, ..., x_{2n+1}) &= f(x_1, ..., x_m, x_{m+1}, x_{m+2}, ..., x_{2n+1}) \\ &= f(x_1, ..., f(f(x_m, x_{m+1}, x_{m+2}), x_{m+3}, x_{m+4}), ..., x_{2n+1}). \end{aligned}$$

Definition 2.4 ([13])**.** Let $(\mathcal{H}, f)$ be a ternary semihypergroup. A binary relation ρ is called:

(1) compatible on the left if $a\rho b$ and $x \in f(x_1, x_2, a)$ imply that there exists $y \in f(x_1, x_2, b)$ such that $x\rho y$,
(2) compatible on the right if $a\rho b$ and $x \in f(a, x_1, x_2)$ imply that there exists $y \in f(b, x_1, x_2)$ such that $x\rho y$,
(3) compatible on the lateral if $a\rho b$ and $x \in f(x_1, a, x_2)$ imply that there exists $y \in f(x_1, b, x_2)$ such that $x\rho y$,
(4) compatible on the two-sided if $a_1\rho b_1$, $a_2\rho b_2$, and $x \in f(a_1, z, a_2)$ imply that there exists $y \in f(b_1, z, b_2)$ such that $x\rho y$,
(5) compatible if $a_1\rho b_1$, $a_2\rho b_2$, $a_3\rho b_3$ and $x \in f(a_1, a_2, a_3)$ imply that there exists $y \in f(b_1, b_2, b_3)$ such that $x\rho y$.

Definition 2.5 ([36])**.** A ternary semihypergroup $(\mathcal{H}, f)$ is called a partially ordered ternary semihypergroup if there exists a partially ordered relation $' \leq '$ on $\mathcal{H}$ such that $' \leq '$ are compatible on left, compatible on right, compatible on lateral and compatible.

Example 2.6. Let $\mathcal{H}=\left\{ \begin{pmatrix} a & 0 & 0 & b \\ c & 0 & d & e \\ f & g & 0 & h \\ i & 0 & 0 & j \end{pmatrix} : a, b, c, d, e, f, g, h, i, j \in \mathbb{N}_0 \right\}$, where $\mathbb{N}_0$ denotes the set of all non-negative integers, is an ordered ternary semigroup under the ordinary multiplication of numbers with partial ordered relation $\leq_{\mathbb{N}}$ is "less than or equal to". Now, let $A_1 = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b \end{pmatrix} : a, b \in \mathbb{N}_0 \right\}$, $A_2 = \left\{ \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & b & 0 \\ 0 & c & 0 & 0 \\ d & 0 & 0 & 0 \end{pmatrix} :\right.$

$a, b, c, d \in \mathbb{N}_0 \Big\}$. Then, $\mathcal{H}$ is a ternary semihypergroup under ternary hyperoperation $'f'$ defined as follows:

$$f(X, Y, Z) = X \cdot A_1 \cdot Y \cdot A_2 \cdot Z \text{ for all } X, Y, Z \in \mathcal{H},$$

where $' \cdot '$ is the usual multiplication of matrices over $\mathbb{N}_0$. Now we define partial ordered relation $\leq_{\mathcal{H}}$ on $\mathcal{H}$ by, for any A, $B \in \mathcal{H}$

$$A \leq_{\mathcal{H}} B \text{ if and only if } a_{ij} \leq_{\mathbb{N}} b_{ij}, \text{ where } a_{ij} \in A, b_{ij} \in B, \text{ for all } i \text{ and } j.$$

Then it is easy to verify that $\mathcal{H}$ is an ordered ternary semihypergroup with partial ordered relation $\leq_{\mathcal{H}}$.

Throughout the paper, we denote $(\mathcal{H}, f, \leq)$ as an ordered ternary semihypergroup.

Let $(\mathcal{H}, f, \leq)$ be an ordered ternary semihypergroup. Then for any subset A of $\mathcal{H}$, we denote $(A] := \{h \in \mathcal{H} | h \leq a, \text{for some } a \in A\}$. If $A = \{a\}$, we also write $(\{a\}]$ as $(a]$. If A and B are non-empty subsets of $\mathcal{H}$, then we say that $A \leq B$ if for every $a \in A$, there exists $b \in B$ such that $a \leq b$.

Lemma 2.7 ([36]). *For subsets $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ of $(\mathcal{H}, f, \leq)$, the following statements hold:*

(1) $\mathcal{A} \subseteq (\mathcal{A}]$ for every $\mathcal{A} \subseteq \mathcal{H}$,
(2) If $\mathcal{A} \subseteq \mathcal{B}$, then $(\mathcal{A}] \subseteq (\mathcal{B}]$ for every $\mathcal{A}$, $\mathcal{B} \subseteq \mathcal{H}$,
(3) $((\mathcal{A}]] = (\mathcal{A}]$ for every $\mathcal{A} \subseteq \mathcal{H}$.
(4) $f((\mathcal{A}], (\mathcal{B}], (\mathcal{C}]) \subseteq (f(\mathcal{A}, \mathcal{B}, \mathcal{C})]$,
(5) $(f((\mathcal{A}], (\mathcal{B}], (\mathcal{C}])] \subseteq (f(\mathcal{A}, \mathcal{B}, \mathcal{C})]$ for all $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C} \subseteq \mathcal{H}$.

Definition 2.8 ([36]). Let $\emptyset \neq T \subseteq \mathcal{H}$. Then T is called an ordered ternary subsemihypergroup of $\mathcal{H}$ if and only if $f(T, T, T) \subseteq T$ and $(\mathrm{T}] \subseteq T$.

Definition 2.9 ([36]). An element a of $(\mathcal{H}, f, \leq)$ is called regular if there exists an element x in $\mathcal{H}$ such that $a \in (f(a, x, a)]$. $\mathcal{H}$ is called regular ordered ternary semihypergroup if every element of $\mathcal{H}$ is regular.

Definition 2.10 ([36]). A non-empty subset I of $(\mathcal{H}, f, \leq)$ is called a right (lateral, left) hyperideal of $\mathcal{H}$ if

(1) $f(I, \mathcal{H}, \mathcal{H}) \subseteq I$ $(f(\mathcal{H}, I, \mathcal{H}) \subseteq I$, $f(\mathcal{H}, \mathcal{H}, I) \subseteq I)$,
(2) If $i \in I$ and $h \leq i$, then $h \in I$ for every $h \in \mathcal{H}$.

Example 2.11. Let $\mathcal{H}$ be the ternary semihypergroup of Example 2.6.

Let $R = \left\{ \begin{pmatrix} a & 0 & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ c & 0 & 0 & d \end{pmatrix} : a, b, c, d \in \mathbb{N}_0 \right\}$. Then R is a right hyperideal of $\mathcal{H}$.

Definition 2.12 ([36]). A non-empty subset I of $(\mathcal{H}, f, \leq)$ is called a two sided hyperideal of $\mathcal{H}$ if it is a left, right hyperideal of $\mathcal{H}$ and I is called hyperideal of $\mathcal{H}$ if it is a left, right and lateral hyperideal of $\mathcal{H}$.

Example 2.13. Let $\mathcal{H}$ be the ternary semihypergroup of Example 2.6.

Let $I = \left\{ \begin{pmatrix} a & 0 & 0 & e \\ b & 0 & 0 & f \\ c & 0 & 0 & g \\ d & 0 & 0 & h \end{pmatrix} : a, b, c, d, e, f, g, h \in \mathbb{N}_0 \right\}$. Then I is a hyperideal of $\mathcal{H}$.

Lemma 2.14 ([36]). *Let $(\mathcal{H}, f, \leq)$ be an ordered ternary semihypergroup. For any $\emptyset \neq A \subseteq \mathcal{H}$,*

(1) $(f(A, \mathcal{H}, \mathcal{H}) \cup A]$ is the smallest right hyperideal of $\mathcal{H}$ containing A,
(2) $(f(\mathcal{H}, \mathcal{H}, A, \mathcal{H}, \mathcal{H}) \cup f(\mathcal{H}, A, \mathcal{H}) \cup A]$ is the smallest lateral hyperideal of H containing A,
(3) $(f(\mathcal{H}, \mathcal{H}, A) \cup A]$ is the smallest left hyperideal of $\mathcal{H}$ containing A,
(4) $(f(A, \mathcal{H}, \mathcal{H}) \cup f(\mathcal{H}, \mathcal{H}, A, \mathcal{H}, \mathcal{H}) \cup f(\mathcal{H}, A, \mathcal{H}) \cup f(\mathcal{H}, \mathcal{H}, A) \cup A]$ is the smallest hyperideal of $\mathcal{H}$ containing A.

Lemma 2.15 ([36]). *Let $(\mathcal{H}, f, \leq)$ be an ordered ternary semihypergroup. For any $\emptyset \neq A \subseteq \mathcal{H}$,*

(1) $(f(A, \mathcal{H}, \mathcal{H})]$ is a right hyperideal of $\mathcal{H}$,
(2) $(f(\mathcal{H}, \mathcal{H}, A, \mathcal{H}, \mathcal{H}) \cup f(\mathcal{H}, A, \mathcal{H})]$ is a lateral hyperideal of $\mathcal{H}$,
(3) $(f(\mathcal{H}, \mathcal{H}, A)]$ is a left hyperideal of $\mathcal{H}$,
(4) $(f(A, \mathcal{H}, \mathcal{H}) \cup f(\mathcal{H}, \mathcal{H}, A, \mathcal{H}, \mathcal{H}) \cup f(\mathcal{H}, A, \mathcal{H}) \cup f(\mathcal{H}, \mathcal{H}, A)]$ is a hyperideal of $\mathcal{H}$.

3. Generalized Quasi-Hyperideals

In this section, we define quasi-hyperideal and $(m, (p, q), n)$-quasi-hyperideal in ordered ternary semihypergroup and establish some of their elementary properties.

Definition 3.1. A non-empty subset $\mathcal{Q}$ of an ordered ternary semihypergroup $(\mathcal{H}, f, \leq)$ is called a quasi-hyperideal of $\mathcal{H}$ if

(1) $(f(\mathcal{H}, \mathcal{H}, \mathcal{Q})] \cap (f(\mathcal{H}, \mathcal{Q}, \mathcal{H})] \cap (f(\mathcal{Q}, \mathcal{H}, \mathcal{H})] \subseteq \mathcal{Q}$,
(2) $(f(\mathcal{H}, \mathcal{H}, \mathcal{Q})] \cap (f(\mathcal{H}, \mathcal{H}, \mathcal{Q}, \mathcal{H}, \mathcal{H})] \cap (f(\mathcal{Q}, \mathcal{H}, \mathcal{H})] \subseteq \mathcal{Q}$,
(3) $(\mathcal{Q}] \subseteq \mathcal{Q}$.

Example 3.2. Let $\mathcal{H}$ be the ternary semihypergroup of Example 2.6.
Let $\mathcal{Q} = \left\{ \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : a, b \in \mathbb{N}_0 \right\}$. Then $\mathcal{Q}$ is a quasi hyperideal of $\mathcal{H}$ which is not a hyperideal of $\mathcal{H}$.

Definition 3.3. A ternary sub-semihypergroup $\mathcal{Q}$ of $\mathcal{H}$ is called a generalized quasi-hyperideal or an $(m, (p, q), n)$-quasi-hyperideal of $\mathcal{H}$ if

(1) $(f(\mathcal{Q}, \mathcal{H}^m, \mathcal{H}^m)] \cap (f(\mathcal{H}^p, \mathcal{Q}, \mathcal{H}^q)] \cap f(\mathcal{H}^n, \mathcal{H}^n, \mathcal{Q})] \subseteq \mathcal{Q}$,
(2) $(f(\mathcal{Q}, \mathcal{H}^m, \mathcal{H}^m)] \cap (f(\mathcal{H}^p, \mathcal{H}, \mathcal{Q}, \mathcal{H}, \mathcal{H}^q)] \cap (f(\mathcal{H}^n, \mathcal{H}^n, \mathcal{Q})] \subseteq \mathcal{Q}$,
(3) $(\mathcal{Q}] \subseteq \mathcal{Q}$.

where m, n, p, q are positive integers and $p + q =$ even.

Example 3.4. All the quasi-hyperideals of the Examples 2.11, 2.13 and 3.2 are $(m,(p,q),n)$ quasi-hyperideals of $\mathcal{H}$.

Remark 3.5. Every quasi-hyperideal of $\mathcal{H}$ is an $(1, (1, 1), 1)$-quasi-hyperideal of $\mathcal{H}$. But an $(m, (p, q), n)$-quasi-hyperideal need not be a quasi-hyperideal of $\mathcal{H}$.

Example 3.6. Let $\mathcal{H}$ be a set of all strictly lower triangular matrices of order 6 over $\mathbb{Z}_0^-$, the set of all non-positive integers. i.e.

$$S = \{(a_{ij})_{6\times 6} \mid a_{ij} = 0 \ if \ i \leq j \ and \ a_{ij} \in \mathbb{Z}_0^- \ if \ i > j\}.$$

Here $\mathbb{Z}_0^-$ is an ordered ternary semihypergroup under the ordinary multiplication of numbers with partial ordered relation $\geq_{\mathbb{Z}_0^-}$ is "greater than or equal to". Now we define partial ordered relation $\geq_{\mathcal{H}}$ on $\mathcal{H}$ by, for any $A,\ B \in \mathcal{H}$

$$A \geq_{\mathcal{H}} B \text{ if and only if } a_{ij} \geq_{\mathbb{Z}_0^-} b_{ij}, \text{ where } a_{ij} \in A, b_{ij} \in B, \text{ for all } i \text{ and } j.$$

Then it is easy to verify that $\mathcal{H}$ is an ordered ternary semihypergroup under usual multiplication of matrices over $\mathbb{Z}_0^-$ with partial ordered relation $\geq_{\mathcal{H}}$.

Now, let $A_1 = \left\{ \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & d & 0 & 0 \\ 0 & 0 & 0 & 0 & e & 0 \\ 0 & 0 & 0 & 0 & 0 & f \end{pmatrix} : a,b,c,d,e,f \in \mathbb{Z}_0^- \right\}$,

$A_2 = \left\{ \begin{pmatrix} a & b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} : a,b \in \mathbb{Z}_0^- \right\}$. Then, $\mathcal{H}$ is an ordered ternary semihypergroup under ternary hyperoperation $'f'$ defined as follows:

$$f(X,Y,Z) = X \cdot A_1 \cdot Y \cdot A_2 \cdot Z, \text{ for all } X,Y,Z \in \mathcal{H},$$

where $'\cdot'$ is the usual multiplication of matrices over $\mathbb{Z}_0^-$.

Let $\mathcal{Q} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & d & 0 & 0 \end{pmatrix} : a,b,c,d \in \mathbb{Z}_0^- \right\}$. Then it is easy to verify that $\mathcal{Q}$ is an ordered ternary sub-semihypergroup of $\mathcal{H}$ and $\mathcal{Q}$ is an $(2,(2,2),2)$ quasi-hyperideal of $\mathcal{H}$. Now

$(f(\mathcal{Q},\mathcal{H},\mathcal{H})] \cap (f(\mathcal{H},\mathcal{Q},\mathcal{H}) \cup f(\mathcal{H},\mathcal{H},\mathcal{Q},\mathcal{H},\mathcal{H})] \cap (f(\mathcal{H},\mathcal{H},\mathcal{Q})] =$

$$\left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 & 0 & 0 \end{pmatrix} : a,b \in \mathbb{Z}_0^- \right\} \nsubseteq \mathcal{Q}.$$

Therefore $\mathcal{Q}$ is not an $(1,(1,1),1)$-quasi-hyperideal ideal of $\mathcal{H}$.

Lemma 3.7. *Let $\{T_i \mid i \in I\}$ be the arbitrary collection of ordered ternary sub-semihypergroups of $\mathcal{H}$ such that $\bigcap_{i\in I} T_i \neq \emptyset$. Then $\bigcap_{i\in I} T_i$ is an ordered ternary sub-semihypergroup of $\mathcal{H}$.*

Proof. Let T_i be an ordered ternary sub-semihypergroup of $\mathcal{H}$ for all $i \in I$ such that $\bigcap_{i\in I} T_i \neq \emptyset$ and let $t_1,t_2,t_3 \in \bigcap_{i\in I} T_i$ for all $i \in I$. As T_i is a ternary sub-semihypergroup of

$\mathcal{H}$ for all $i \in I$, we have $f(t_1, t_2, t_3) \subseteq T_i$ for all $i \in I$. Therefore $f(t_1, t_2, t_3) \subseteq \bigcap_{i \in I} T_i$ and hence $\bigcap_{i \in I} T_i$ is a ternary sub-semihypergroup of $\mathcal{H}$.

Now suppose that $x \in (\bigcap_{i \in I} T_i]$. Then $x \leq a$, for some $a \in \bigcap_{i \in I} T_i$. Now $a \in T_i$, for all $i \in I$, it implies $x \in (T_i] = T_i$, for all $i \in I$. Thus we have $x \in \bigcap_{i \in I} T_i$, which shows that $(\bigcap_{i \in I} T_i] \subseteq \bigcap_{i \in I} T_i$. Hence $\bigcap_{i \in I} T_i$ is an ordered ternary sub-semihypergroup of $\mathcal{H}$. ∎

Theorem 3.8. *Let $(\mathcal{H}, f, \leq)$ be an ordered ternary semihypergroup and $\mathcal{Q}_i$ be an $(m,(p,q),n)$ -quasi-hyperideal of $\mathcal{H}$ such that $\bigcap_{i \in I} \mathcal{Q}_i \neq \emptyset$. Then $\bigcap_{i \in I} \mathcal{Q}_i$ is an $(m,(p,q),n)$-quasi hyperideal of $\mathcal{H}$.*

Proof. Let $\{\mathcal{Q}_i \mid i \in I\}$ be a family of $(m,(p,q),n)$-quasi-hyperideals of $\mathcal{H}$. Clearly $\mathcal{Q} = \bigcap_{i \in I} \mathcal{Q}_i$ is an ordered ternary sub-semihypergroup of $\mathcal{H}$ by the Lemma 3.7. We claim that $\mathcal{Q}$ is an $(m,(p,q),n)$-quasi-hyperideal of $\mathcal{H}$. Thus, we have

$$
\begin{aligned}
&(f(\mathcal{Q}, \mathcal{H}^m, \mathcal{H}^m)] \cap (f(\mathcal{H}^p, \mathcal{Q}, \mathcal{H}^q) \cup f(\mathcal{H}^p, \mathcal{H}, \mathcal{Q}, \mathcal{H}, \mathcal{H}^q)] \cap (f(\mathcal{H}^n, \mathcal{H}^n, \mathcal{Q})] \\
&= (f(\bigcap_{i \in I} \mathcal{Q}_i, \mathcal{H}^m, \mathcal{H}^m)] \cap (f(\mathcal{H}^p, \bigcap_{i \in I} \mathcal{Q}_i, \mathcal{H}^q) \cup f(\mathcal{H}^p, \mathcal{H}, \bigcap_{i \in I} \mathcal{Q}_i, \mathcal{H}, \mathcal{H}^q)] \cap (f(\mathcal{H}^n, \mathcal{H}^n, \bigcap_{i \in I} \mathcal{Q}_i)] \\
&\subseteq (f(\mathcal{Q}_i, \mathcal{H}^m, \mathcal{H}^m)] \cap (f(\mathcal{H}^p, \mathcal{Q}_i, \mathcal{H}^q) \cup f(\mathcal{H}^p, \mathcal{H}, \mathcal{Q}_i, \mathcal{H}, \mathcal{H}^q)] \cap (f(\mathcal{H}^n, \mathcal{H}^n, \mathcal{Q}_i)], \text{ for all } i \in I. \\
&\subseteq \mathcal{Q}_i, \text{ for all } i \in I.
\end{aligned}
$$

Therefore $(f(\mathcal{Q}, \mathcal{H}^m, \mathcal{H}^m)] \cap (f(\mathcal{H}^p, \mathcal{Q}, \mathcal{H}^q) \cup f(\mathcal{H}^p, \mathcal{H}, \mathcal{Q}, \mathcal{H}, \mathcal{H}^q)] \cap (f(\mathcal{H}^n, \mathcal{H}^n, \mathcal{Q})] \subseteq \bigcap_{i \in I} \mathcal{Q}_i$. Hence $\mathcal{Q}$ is an $(m,(p,q),n)$-quasi hyperideal of $\mathcal{H}$. ∎

Definition 3.9. Let $(\mathcal{H}, f, \leq)$ be an ordered ternary semihypergroup. Then an ordered ternary sub-semihypergroup

(1) $\mathcal{R}$ of $\mathcal{H}$ is called an m-right hyperideal of $\mathcal{H}$ if $f(\mathcal{R}, \mathcal{H}^m, \mathcal{H}^m) \subseteq \mathcal{R}$ and $(R] = R$,

(2) $\mathcal{M}$ of $\mathcal{H}$ is called an (p,q)-lateral hyperideal of $\mathcal{H}$ if

$$f(\mathcal{H}^p, \mathcal{M}, \mathcal{H}^q) \cup f(\mathcal{H}^p, \mathcal{H}, \mathcal{M}, \mathcal{H}, \mathcal{H}^q) \subseteq \mathcal{M} \text{ and } (M] = M,$$

(3) $\mathcal{L}$ of $\mathcal{H}$ is called an n-left hyperideal of $\mathcal{H}$ if $f(\mathcal{H}^n, \mathcal{H}^n, \mathcal{L}) \subseteq \mathcal{L}$ and $(L] = L$,

where m, n, p, q are positive integers and $p + q$ is an even positive integer.

Example 3.10. Let $\mathcal{H}$ be the ternary semihypergroup of Example 3.6. Consider

$$\mathcal{M}_{gen} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} : a \in \mathbb{Z}_0^- \right\}.$$

Then it is easy to see that $\mathcal{M}_{gen}$ is an ordered ternary sub-semihypergroup of $\mathcal{H}$ and $\mathcal{M}_{gen}$ is a $(3,1)$-lateral hyperideal of $\mathcal{H}$. Now

$$f(\mathcal{H},\mathcal{M}_{gen},\mathcal{H}) = \left\{ \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ a&b&0&0&0&0\\ c&d&0&0&0&0 \end{pmatrix} : a,b,c,d \in \mathbb{Z}_0^- \right\} \nsubseteq \mathcal{M}_{gen}.$$ Therefore $\mathcal{M}_{gen}$ is not a lateral hyperideal of $\mathcal{H}$.

Theorem 3.11. *Every m-right, (p,q)-lateral and n-left hyperideal of $\mathcal{H}$ is an $(m,(p,q),n)$-quasi-hyperideal of $\mathcal{H}$. But converse need not be true.*

Proof. Proof is easy. Conversely, take an ordered ternary semihypergroup $\mathcal{H}$ given in the Example 3.2.

Let $\mathcal{H} = \left\{ \begin{pmatrix} 0&0&0&a\\ 0&0&b&c\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix} : a,b,c \in \mathbb{N}_0 \right\}$. Then $\mathcal{H}$ is a $(3,(2,2),3)$-quasi hyperideal of $\mathcal{H}$. But it is not a 3-right hyperideal, a $(2,2)$-lateral hyperideal and a 3-left hyperideal of $\mathcal{H}$. ■

Theorem 3.12. *Let $(\mathcal{H},f,\leq)$ be an ordered ternary semihypergroup. Then the following statements hold:*

(1) Let $\mathcal{R}_i$ be an m-right hyperideal of $\mathcal{H}$ such that $\bigcap_{i\in I}\mathcal{R}_i \neq \emptyset$. Then $\bigcap_{i\in I}\mathcal{R}_i$ is an m-right hyperideal of $\mathcal{H}$.
(2) Let $\mathcal{M}_i$ be a (p,q)-lateral hyperideal of $\mathcal{H}$ such that $\bigcap_{i\in I}\mathcal{M}_i \neq \emptyset$. Then $\bigcap_{i\in I}\mathcal{M}_i$ is a (p,q)-lateral hyperideal of $\mathcal{H}$.
(3) Let $\mathcal{L}_i$ be an n-left hyperideal of $\mathcal{H}$ such that $\bigcap_{i\in I}\mathcal{L}i \neq \emptyset$. Then $\bigcap_{i\in I}\mathcal{L}_i$ is an n-left hyperideal of $\mathcal{H}$.

Proof. Analogous to the proof of the Theorem 3.8. ■

Theorem 3.13. *Let $\mathcal{R}$ be an m-right hyperideal, $\mathcal{M}$ be a (p,q)-lateral hyperideal and $\mathcal{L}$ be an n-left hyperideal of $\mathcal{H}$. Then $\mathcal{R}\cap\mathcal{M}\cap\mathcal{L}$ is an $(m,(p,q),n)$-quasi-hyperideal of $\mathcal{H}$.*

Proof. Suppose that $\mathcal{Q} = \mathcal{R}\cap\mathcal{M}\cap\mathcal{L}$. By the Theorem 3.11, every m-right, (p,q)-lateral and n-left hyperideal of $\mathcal{H}$ are $(m,(p,q),n)$-quasi-hyperideals of $\mathcal{H}$. Therefore $\mathcal{R}$, $\mathcal{M}$ and $\mathcal{L}$ are $(m,(p,q),n)$-quasi-hyperideals of $\mathcal{H}$. If $\mathcal{R}\cap\mathcal{M}\cap\mathcal{L}$ is non-empty. Then by the Theorem 3.8, we have $\mathcal{Q} = \mathcal{R}\cap\mathcal{M}\cap\mathcal{L}$ is an $(m,(p,q),n)$-quasi-hyperideal of $\mathcal{H}$. ■

Theorem 3.14. *Let $\mathcal{A}$ be any non-empty subset of $\mathcal{H}$. Then*

(1) $(f(\mathcal{A},\mathcal{H}^m,\mathcal{H}^m)]$ is an m-right hyperideal of $\mathcal{H}$,
(2) $f(\mathcal{H}^p,\mathcal{A},\mathcal{H}^q)\cup f(\mathcal{H}^p,\mathcal{H},\mathcal{A},\mathcal{H},\mathcal{H}^q)$ is a (p,q)-lateral hyperideal of $\mathcal{H}$,
(3) $f(\mathcal{H}^n,\mathcal{H}^n,\mathcal{A})$ is an n-left hyperideal of $\mathcal{H}$,
(4) $f(\mathcal{A},\mathcal{H}^m,\mathcal{H}^m)\cap f(\mathcal{H}^p,\mathcal{A},\mathcal{H}^q)\cup f(\mathcal{H}^p,\mathcal{H},\mathcal{A},\mathcal{H},\mathcal{H}^q)\cap f(\mathcal{H}^n,\mathcal{H}^n,\mathcal{A})$ is an $(m,(p,q),n)$ -quasi hyperideal of $\mathcal{H}$.

Proof. (1) It is easy to show that $(f(\mathcal{A},\mathcal{H}^m,\mathcal{H}^m)]$ is a ternary sub-semihypergroup and $((f(\mathcal{A},\mathcal{H}^m,\mathcal{H}^m)]] = (f(\mathcal{A},\mathcal{H}^m,\mathcal{H}^m)]$.

Now

$$\begin{aligned} f((f(\mathcal{A},\mathcal{H}^m,\mathcal{H}^m)],\mathcal{H}^m,\mathcal{H}^m) &\subseteq f((f(\mathcal{A},\mathcal{H}^m,\mathcal{H}^m)],(\mathcal{H}^m],(\mathcal{H}^m])] \\ &\subseteq (f(f(\mathcal{A},\mathcal{H}^m,\mathcal{H}^m),\mathcal{H}^m,\mathcal{H}^m)] \\ &= (f(\mathcal{A},f(\mathcal{H}^m,\mathcal{H}^m,\mathcal{H}^m),\mathcal{H}^m)] \\ &\subseteq (f(\mathcal{A},\mathcal{H}^m,\mathcal{H}^m)]. \end{aligned}$$

Therefore $(f(\mathcal{A},\mathcal{H}^m,\mathcal{H}^m)]$ is an m-right hyperideal of $\mathcal{H}$.

(2), (3) and (4) can be proved analogously to (1). ■

Theorem 3.15. *Let $\mathcal{A}$ be an ordered ternary sub-semihypergroup of $\mathcal{H}$. Then*

(1) $(\mathcal{A}\cup f(\mathcal{A},\mathcal{H}^m,\mathcal{H}^m)]$ is an m-right hyperideal of $\mathcal{H}$ containing $\mathcal{A}$,
(2) $(\mathcal{A}\cup f(\mathcal{H}^p,\mathcal{A},\mathcal{H}^q)\cup f(\mathcal{H}^p,\mathcal{H},\mathcal{A},\mathcal{H},\mathcal{H}^q)]$ is a (p,q)-lateral hyperideal of $\mathcal{H}$ containing $\mathcal{A}$,
(3) $(\mathcal{A}\cup f(\mathcal{H}^n,\mathcal{H}^n,\mathcal{A})]$ is an n-left hyperideal of $\mathcal{H}$ containing $\mathcal{A}$,
(4) $(f(\mathcal{A},\mathcal{H}^m,\mathcal{H}^m)]\cap(f(\mathcal{H}^p,\mathcal{A},\mathcal{H}^q)\cup f(\mathcal{H}^p,\mathcal{H},\mathcal{A},\mathcal{H},\mathcal{H}^q)]\cap(f(\mathcal{H}^n,\mathcal{H}^n,\mathcal{A})]\cup(\mathcal{A}]$ is an $(m,(p,q),n)$-quasi hyperideal of $\mathcal{H}$ containing $\mathcal{A}$.

Proof. Proof is analogous to the Theorem 3.14. ■

Theorem 3.16. *Let $\mathcal{Q}$ be an $(m,(p,q),n)$-quasi hyperideal of $\mathcal{H}$. Then*

(1) $\mathcal{R}=(\mathcal{Q}\cup f(\mathcal{Q},\mathcal{H}^m,\mathcal{H}^m)]$ is an m-right hyperideal of $\mathcal{H}$,
(2) $\mathcal{M}=(\mathcal{Q}\cup f(\mathcal{H}^p,\mathcal{Q},\mathcal{H}^q)\cup f(\mathcal{H}^p,\mathcal{H},\mathcal{Q},\mathcal{H},\mathcal{H}^q)]$ is a (p,q)-lateral hyperideal of $\mathcal{H}$,
(3) $\mathcal{L}=(\mathcal{Q}\cup f(\mathcal{H}^n,\mathcal{H}^n,\mathcal{Q})]$ is an n-left hyperideal of $\mathcal{H}$.

Proof. Proof is analogous to the Theorem 3.14. ■

An $(m,(p,q),n)$-quasi hyperideal $\mathcal{Q}$ has the $(m,(p,q),n)$ intersection property if $\mathcal{Q}$ is the intersection of an m-right, a (p,q)-lateral and an n-left hyperideal of $\mathcal{H}$.

Remark 3.17. Every m-right hyperideal, (p,q)-lateral hyperideal and n-left hyperideal have the intersection property.

Theorem 3.18. *Let $(\mathcal{H},f,\leq)$ be an ordered ternary semihypergroup and $\mathcal{Q}$ be an $(m,(p,q),n)$-quasi-hyperideal of $\mathcal{H}$. Then the following statements are equivalent:*

(1) $\mathcal{Q}$ has the $(m,(p,q),n)$ intersection property,
(2) $(\mathcal{Q}\cup f(\mathcal{Q},\mathcal{H}^m,\mathcal{H}^m)]\cap(\mathcal{Q}\cup f(\mathcal{H}^p,\mathcal{Q},\mathcal{H}^q)\cup f(\mathcal{H}^p,\mathcal{H},\mathcal{Q},\mathcal{H},\mathcal{H}^q)]\cap(\mathcal{Q}\cup f(\mathcal{H}^n,\mathcal{H}^n,\mathcal{Q})]=\mathcal{Q}$,
(3) $(\mathcal{Q}\cup f(\mathcal{Q},\mathcal{H}^m,\mathcal{H}^m)]\cap(\mathcal{Q}\cup f(\mathcal{H}^p,\mathcal{Q},\mathcal{H}^q)\cup f(\mathcal{H}^p,\mathcal{H},\mathcal{Q},\mathcal{H},\mathcal{H}^q)]\cap(f(\mathcal{H}^n,\mathcal{H}^n,\mathcal{Q}]\subseteq\mathcal{Q}$,
(4) $(f(\mathcal{Q},\mathcal{H}^m,\mathcal{H}^m)]\cap(\mathcal{Q}\cup f(\mathcal{H}^p,\mathcal{Q},\mathcal{H}^q)\cup f(\mathcal{H}^p,\mathcal{H},\mathcal{Q},\mathcal{H},\mathcal{H}^q)]\cap(\mathcal{Q}\cup f(\mathcal{H}^n,\mathcal{H}^n,\mathcal{Q})]\subseteq\mathcal{Q}$,
(5) $(\mathcal{Q}\cup f(\mathcal{Q},\mathcal{H}^m,\mathcal{H}^m)]\cap(f(\mathcal{H}^p,\mathcal{Q},\mathcal{H}^q)\cup f(\mathcal{H}^p,\mathcal{H},\mathcal{Q},\mathcal{H},\mathcal{H}^q)]\cap(\mathcal{Q}\cup f(\mathcal{H}^n,\mathcal{H}^n,\mathcal{Q})]\subseteq\mathcal{Q}$.

Proof. (1) $\Rightarrow$ (2). Let $\mathcal{Q}$ has the $(m,(p,q),n)$ intersection property. It is obvious that $\mathcal{Q}\subseteq(\mathcal{Q}\cup f(\mathcal{Q},\mathcal{H}^m,\mathcal{H}^m)]\cap(\mathcal{Q}\cup f(\mathcal{H}^p,\mathcal{Q},\mathcal{H}^q)\cup f(\mathcal{H}^p,\mathcal{H},\mathcal{Q},\mathcal{H},\mathcal{H}^q)]\cap(\mathcal{Q}\cup f(\mathcal{H}^n,\mathcal{H}^n,\mathcal{Q})]$....(i). To prove (2) we have to show that $(\mathcal{Q}\cup f(\mathcal{Q},\mathcal{H}^m,\mathcal{H}^m)]\cap(\mathcal{Q}\cup f(\mathcal{H}^p,\mathcal{Q},\mathcal{H}^q)\cup f(\mathcal{H}^p,\mathcal{H},\mathcal{Q},\mathcal{H},\mathcal{H}^q)]\cap(\mathcal{Q}\cup f(\mathcal{H}^n,\mathcal{H}^n,\mathcal{Q})]\subseteq\mathcal{Q}$. As it is known that $\mathcal{Q}$ has $(m,(p,q),n)$ intersection property, it implies there exist an m-right hyperideal $\mathcal{R}$, a (p,q)-lateral hyperideal $\mathcal{M}$ and an n-left hyperideal $\mathcal{L}$ of $\mathcal{H}$ s.t. $\mathcal{R}\cap\mathcal{M}\cap\mathcal{L}=\mathcal{Q}$. Then $\mathcal{Q}\subseteq\mathcal{R}$, $\mathcal{Q}\subseteq\mathcal{M}$ and

$\mathcal{Q} \subseteq \mathcal{L}$. Also we have that $(f(\mathcal{H}^n, \mathcal{H}^n, \mathcal{Q})] \subseteq (f(\mathcal{H}^n, \mathcal{H}^n, \mathcal{L})] \subseteq \mathcal{L}$ and in the similar way $(f(\mathcal{H}^p, \mathcal{Q}, \mathcal{H}^q) \cup f(\mathcal{H}^p, \mathcal{H}, \mathcal{Q}, \mathcal{H}, \mathcal{H}^q)] \subseteq \mathcal{M}$ and $(f(\mathcal{Q}, \mathcal{H}^m, \mathcal{H}^m)] \subseteq \mathcal{R}$ which implies $\mathcal{Q} \cup (f(\mathcal{H}^n, \mathcal{H}^n, \mathcal{Q})] = (\mathcal{Q} \cup f(\mathcal{H}^n, \mathcal{H}^n, \mathcal{Q})] \subseteq \mathcal{L}$, $(\mathcal{Q} \cup f(\mathcal{H}^p, \mathcal{Q}, \mathcal{H}^q) \cup f(\mathcal{H}^p, \mathcal{H}, \mathcal{Q}, \mathcal{H}, \mathcal{H}^q)] \subseteq \mathcal{M}$ and $(\mathcal{Q} \cup f(\mathcal{Q}, \mathcal{H}^m, \mathcal{H}^m)] \subseteq \mathcal{R}$. Hence we have $(\mathcal{Q} \cup f(\mathcal{Q}, \mathcal{H}^m, \mathcal{H}^m)] \cap (\mathcal{Q} \cup f(\mathcal{H}^p, \mathcal{Q}, \mathcal{H}^q) \cup f(\mathcal{H}^p, \mathcal{H}, \mathcal{Q}, \mathcal{H}, \mathcal{H}^q)] \cap (\mathcal{Q} \cup f(\mathcal{H}^n, \mathcal{H}^n, \mathcal{Q})] \subseteq \mathcal{L} \cap \mathcal{M} \cap \mathcal{R} = \mathcal{Q}$....(ii). From (i) and (ii), we have $(\mathcal{Q} \cup f(\mathcal{Q}, \mathcal{H}^m, \mathcal{H}^m)] \cap (\mathcal{Q} \cup f(\mathcal{H}^p, \mathcal{Q}, \mathcal{H}^q) \cup f(\mathcal{H}^p, \mathcal{H}, \mathcal{Q}, \mathcal{H}, \mathcal{H}^q)] \cap (\mathcal{Q} \cup f(\mathcal{H}^n, \mathcal{H}^n, \mathcal{Q})] = \mathcal{Q}$.

(2) $\Rightarrow$ (1). Consider $(\mathcal{Q} \cup f(\mathcal{Q}, \mathcal{H}^m, \mathcal{H}^m)] \cap (\mathcal{Q} \cup f(\mathcal{H}^p, \mathcal{Q}, \mathcal{H}^q) \cup f(\mathcal{H}^p, \mathcal{H}, \mathcal{Q}, \mathcal{H}, \mathcal{H}^q)] \cap (\mathcal{Q} \cup f(\mathcal{H}^n, \mathcal{H}^n, \mathcal{Q})] = \mathcal{Q}$. By the Theorem 3.16, $(\mathcal{Q} \cup f(\mathcal{Q}, \mathcal{H}^m, \mathcal{H}^m)]$ is an m-right hyperideal of $\mathcal{H}$, $(\mathcal{Q} \cup f(\mathcal{H}^p, \mathcal{Q}, \mathcal{H}^q) \cup f(\mathcal{H}^p, \mathcal{H}, \mathcal{Q}, \mathcal{H}, \mathcal{H}^q)]$ is a (p, q)-lateral hyperideal of $\mathcal{H}$ and $(\mathcal{Q} \cup f(\mathcal{H}^n, \mathcal{H}^n, \mathcal{Q})]$ is an n-left hyperideal of $\mathcal{H}$. Let $\mathcal{R} = (\mathcal{Q} \cup f(\mathcal{Q}, \mathcal{H}^m, \mathcal{H}^m)]$, $\mathcal{M} = (\mathcal{Q} \cup f(\mathcal{H}^p, \mathcal{Q}, \mathcal{H}^q) \cup f(\mathcal{H}^p, \mathcal{H}, \mathcal{Q}, \mathcal{H}, \mathcal{H}^q)]$ and $\mathcal{L} = (\mathcal{Q} \cup f(\mathcal{H}^n, \mathcal{H}^n, \mathcal{Q})]$. Now $(f(\mathcal{Q}, \mathcal{H}^m, \mathcal{H}^m)] \cap (f(\mathcal{H}^p, \mathcal{Q}, \mathcal{H}^q) \cup f(\mathcal{H}^p, \mathcal{H}, \mathcal{Q}, \mathcal{H}, \mathcal{H}^q)] \cap (f(\mathcal{H}^n, \mathcal{H}^n, \mathcal{Q})] \subseteq \mathcal{Q}$, as $\mathcal{Q}$ is an ordered $(m, (p, q), n)$-quasi hyperideal of $\mathcal{H}$. We have

$$\begin{aligned}
\mathcal{L} \cap \mathcal{M} \cap \mathcal{R} &= (\mathcal{Q} \cup f(\mathcal{Q}, \mathcal{H}^m, \mathcal{H}^m)] \cap (\mathcal{Q} \cup f(\mathcal{H}^p, \mathcal{Q}, \mathcal{H}^q) \cup f(\mathcal{H}^p, \mathcal{H}, \mathcal{Q}, \mathcal{H}, \mathcal{H}^q)] \\
&\quad \cap (\mathcal{Q} \cup f(\mathcal{H}^n, \mathcal{H}^n, \mathcal{Q})] \\
&= \mathcal{Q} \cup (f(\mathcal{Q}, \mathcal{H}^m, \mathcal{H}^m)] \cap (f(\mathcal{H}^p, \mathcal{Q}, \mathcal{H}^q) \cup f(\mathcal{H}^p, \mathcal{H}, \mathcal{Q}, \mathcal{H}, \mathcal{H}^q)] \cap (f(\mathcal{H}^n, \mathcal{H}^n, \mathcal{Q})] \\
&\subseteq \mathcal{Q} \cup \mathcal{Q} \\
&= \mathcal{Q}.
\end{aligned}$$

(2) $\Rightarrow$ (3). Consider $\mathcal{Q} = (\mathcal{Q} \cup f(\mathcal{Q}, \mathcal{H}^m, \mathcal{H}^m)] \cap (\mathcal{Q} \cup f(\mathcal{H}^p, \mathcal{Q}, \mathcal{H}^q) \cup f(\mathcal{H}^p, \mathcal{H}, \mathcal{Q}, \mathcal{H}, \mathcal{H}^q)] \cap (\mathcal{Q} \cup f(\mathcal{H}^n, \mathcal{H}^n, \mathcal{Q})]$. As we know $(f(\mathcal{H}^n, \mathcal{H}^n, \mathcal{Q})] \subseteq (\mathcal{Q} \cup f(\mathcal{H}^n, \mathcal{H}^n, \mathcal{Q})]$, we have $(\mathcal{Q} \cup f(\mathcal{Q}, \mathcal{H}^m, \mathcal{H}^m)] \cap (\mathcal{Q} \cup f(\mathcal{H}^p, \mathcal{Q}, \mathcal{H}^q) \cup f(\mathcal{H}^p, \mathcal{H}, \mathcal{Q}, \mathcal{H}, \mathcal{H}^q)] \cap (f(\mathcal{H}^n, \mathcal{H}^n, \mathcal{Q})] \subseteq (\mathcal{Q} \cup f(\mathcal{Q}, \mathcal{H}^m, \mathcal{H}^m)] \cap (\mathcal{Q} \cup f(\mathcal{H}^p, \mathcal{Q}, \mathcal{H}^q) \cup f(\mathcal{H}^p, \mathcal{H}, \mathcal{Q}, \mathcal{H}, \mathcal{H}^q)] \cap (\mathcal{Q} \cup f(\mathcal{H}^n, \mathcal{H}^n, \mathcal{Q})]$. It follows that $(\mathcal{Q} \cup f(\mathcal{Q}, \mathcal{H}^m, \mathcal{H}^m)] \cap (\mathcal{Q} \cup f(\mathcal{H}^p, \mathcal{Q}, \mathcal{H}^q) \cup f(\mathcal{H}^p, \mathcal{H}, \mathcal{Q}, \mathcal{H}, \mathcal{H}^q)] \cap (f(\mathcal{H}^n, \mathcal{H}^n, \mathcal{Q})] \subseteq \mathcal{Q}$.

(3) $\Rightarrow$ (2). Let $(\mathcal{Q} \cup f(\mathcal{Q}, \mathcal{H}^m, \mathcal{H}^m)] \cap (\mathcal{Q} \cup f(\mathcal{H}^p, \mathcal{Q}, \mathcal{H}^q) \cup f(\mathcal{H}^p, \mathcal{H}, \mathcal{Q}, \mathcal{H}, \mathcal{H}^q)] \cap (f(\mathcal{H}^n, \mathcal{H}^n, \mathcal{Q})] \subseteq \mathcal{Q}$. Then $\mathcal{Q} \subseteq (\mathcal{Q} \cup f(\mathcal{Q}, \mathcal{H}^m, \mathcal{H}^m)] \cap (\mathcal{Q} \cup f(\mathcal{H}^p, \mathcal{Q}, \mathcal{H}^q) \cup f(\mathcal{H}^p, \mathcal{H}, \mathcal{Q}, \mathcal{H}, \mathcal{H}^q))] \cap (\mathcal{Q} \cup f(\mathcal{H}^n, \mathcal{H}^n, \mathcal{Q})]$. Now we have to show that $(\mathcal{Q} \cup f(\mathcal{Q}, \mathcal{H}^m, \mathcal{H}^m)] \cap (\mathcal{Q} \cup f(\mathcal{H}^p, \mathcal{Q}, \mathcal{H}^q) \cup f(\mathcal{H}^p, \mathcal{H}, \mathcal{Q}, \mathcal{H}, \mathcal{H}^q)] \cap (f(\mathcal{H}^n, \mathcal{H}^n, \mathcal{Q})] \subseteq \mathcal{Q}$. For this suppose that $x \in (\mathcal{Q} \cup f(\mathcal{Q}, \mathcal{H}^m, \mathcal{H}^m)] \cap (\mathcal{Q} \cup f(\mathcal{H}^p, \mathcal{Q}, \mathcal{H}^q) \cup f(\mathcal{H}^p, \mathcal{H}, \mathcal{Q}, \mathcal{H}, \mathcal{H}^q)] \cap (\mathcal{Q} \cup f(\mathcal{H}^n, \mathcal{H}^n, \mathcal{Q})]$. Then we have to show that $x \in \mathcal{Q}$. Now $(\mathcal{Q} \cup f(\mathcal{Q}, \mathcal{H}^m, \mathcal{H}^m)] \cap (\mathcal{Q} \cup f(\mathcal{H}^p, \mathcal{Q}, \mathcal{H}^q) \cup f(\mathcal{H}^p, \mathcal{H}, \mathcal{Q}, \mathcal{H}, \mathcal{H}^q))] \cap (f(\mathcal{H}^n, \mathcal{H}^n, \mathcal{Q})] \subseteq \mathcal{Q}$. We have $x \in \mathcal{Q}$. Therefore $(\mathcal{Q} \cup f(\mathcal{Q}, \mathcal{H}^m, \mathcal{H}^m)] \cap (\mathcal{Q} \cup f(\mathcal{H}^p, \mathcal{Q}, \mathcal{H}^q) \cup f(\mathcal{H}^p, \mathcal{H}, \mathcal{Q}, \mathcal{H}, \mathcal{H}^q)] \cap (\mathcal{Q} \cup f(\mathcal{H}^n, \mathcal{H}^n, \mathcal{Q})] = \mathcal{Q}$.

The proofs for (2) $\Rightarrow$ (4), (2)$\Rightarrow$ (5) and (4) $\Rightarrow$(2), (5) $\Rightarrow$ (2) are analogous to the proofs of (2)$\Rightarrow$(3) and (3) $\Rightarrow$ (2), respectively. ∎

Theorem 3.19. *Every regular ordered ternary semihypergroup $(\mathcal{H}, f, \leq)$ has the intersection property of $(m, (p, q), n)$-quasi-hyperideals for any positive integers m, p, q, n and $p + q$ is even.*

Proof. Let $\mathcal{H}$ be a regular ordered ternary semihypergroup and $\mathcal{Q}$ be an $(m, (p, q), n)$-quasi hyperideal of $\mathcal{H}$. Then by the Theorem 3.16, $\mathcal{R} = (\mathcal{Q} \cup f(\mathcal{Q}, \mathcal{H}^m, \mathcal{H}^m)]$, $\mathcal{M} = (\mathcal{Q} \cup f(\mathcal{H}^p, \mathcal{Q}, \mathcal{H}^q) \cup f(\mathcal{H}^p, \mathcal{H}, \mathcal{Q}, \mathcal{H}, \mathcal{H}^q)]$ and $\mathcal{L} = (\mathcal{Q} \cup f(\mathcal{H}^n, \mathcal{H}^n, \mathcal{Q})]$ are an m-right, a (p, q)-lateral and an n-left hyperideal of $\mathcal{H}$ respectively. Clearly $\mathcal{Q} \subseteq \mathcal{R}$, $\mathcal{Q} \subseteq \mathcal{M}$ and $\mathcal{Q} \subseteq \mathcal{L}$ implies $\mathcal{Q} \subseteq \mathcal{R} \cap \mathcal{M} \cap \mathcal{L}$. As $\mathcal{H}$ is regular, we have $\mathcal{Q} \subseteq (f(\mathcal{Q}, \mathcal{H}^m, \mathcal{H}^m)]$, $\mathcal{Q} \subseteq (f(\mathcal{H}^p, \mathcal{Q}, \mathcal{H}^q) \cup f(\mathcal{H}^p, \mathcal{H}, \mathcal{Q}, \mathcal{H}, \mathcal{H}^q)]$ and $\mathcal{Q} \subseteq (f(\mathcal{H}^n, \mathcal{H}^n, \mathcal{Q})]$. Therefore $\mathcal{R} = (f(\mathcal{Q}, \mathcal{H}^m, \mathcal{H}^m)]$, $\mathcal{M} = (f(\mathcal{H}^p, \mathcal{Q}, \mathcal{H}^q) \cup f(\mathcal{H}^p, \mathcal{H}, \mathcal{Q}, \mathcal{H}, \mathcal{H}^q)]$ and $L = (f(\mathcal{H}^n, \mathcal{H}^n, \mathcal{Q})]$. It will imply that $\mathcal{R} \cap \mathcal{M} \cap \mathcal{L} = (f(\mathcal{Q}, \mathcal{H}^m, \mathcal{H}^m)] \cap (f(\mathcal{H}^p, \mathcal{Q}, \mathcal{H}^q) \cup f(\mathcal{H}^p, \mathcal{H}, \mathcal{Q}, \mathcal{H}, \mathcal{H}^q)] \cap (f(\mathcal{H}^n, \mathcal{H}^n, \mathcal{Q})] \subseteq \mathcal{Q}$. Therefore $\mathcal{Q} = \mathcal{R} \cap \mathcal{M} \cap \mathcal{L}$. Hence $\mathcal{Q}$ has the $(m, (p, q), n)$ intersection property. ∎

4. Generalized Minimal Quasi-Hyperideals and $(m,(p,q),n)$-Quasi -Simple Ordered Ternary Semihypergroups

In this section, we introduce the concept of a minimal $(m,(p,q),n)$-quasi-hyperideal, a minimal m-right hyperideal, a minimal (p,q)-lateral hyperideal and a minimal n-left hyperideal in ordered ternary semihypergroups and study the relationship between them. Also m-right simple, (p,q)-lateral simple, n-left simple and $(m,(p,q),n)$-quasi-simple ordered ternary semihypergroups are defined and some properties of them are investigated.

Definition 4.1. An $(m,(p,q),n)$-quasi hyperideal $\mathcal{Q}$ of $\mathcal{H}$ is called minimal $(m,(p,q),n)$-quasi hyperideal if it does not properly contain any $(m,(p,q),n)$-quasi hyperideal of $\mathcal{H}$.

Analogously, minimal m-right, minimal (p,q)-lateral and minimal n-left hyperideal can be defined.

Theorem 4.2. *Let $(\mathcal{H},f,\leq)$ be an ordered ternary semihypergroup and $\mathcal{Q}$ be an $(m,(p,q),n)$-quasi-hyperideal of $\mathcal{H}$. Then $\mathcal{Q}$ is minimal if and only if $\mathcal{Q}$ is the intersection of some minimal m-right hyperideal $\mathcal{R}$, minimal (p,q)-lateral hyperideal $\mathcal{M}$ and minimal n-left hyperideal $\mathcal{L}$ of $\mathcal{H}$.*

Proof. Assume that $\mathcal{Q}$ is minimal $(m,(p,q),n)$-quasi hyperideal of $\mathcal{H}$. Then

$$(f(\mathcal{Q},\mathcal{H}^m,\mathcal{H}^m)]\cap(f(\mathcal{H}^p,\mathcal{Q},\mathcal{H}^q)\cup f(\mathcal{H}^p,\mathcal{H},\mathcal{Q},\mathcal{H},\mathcal{H}^q)]\cap(f(\mathcal{H}^n,\mathcal{H}^n,\mathcal{Q})]\subseteq\mathcal{Q}.$$

By the Theorem 3.14, $(f(\mathcal{Q},\mathcal{H}^m,\mathcal{H}^m)]$, $(f(\mathcal{H}^p,\mathcal{Q},\mathcal{H}^q)\cup f(\mathcal{H}^p,\mathcal{H},\mathcal{Q},\mathcal{H},\mathcal{H}^q)]$, $(f(\mathcal{H}^n,\mathcal{H}^n,\mathcal{Q})]$ are an m-right, a (p,q)-lateral and an n-left hyperideal of $\mathcal{H}$ and by Theorem 3.13, intersection of an m-right, an (p,q)-lateral and an n-left hyperideal is an $(m,(p,q),n)$-quasi hyperideal of $\mathcal{H}$. As $\mathcal{Q}$ is minimal, we have

$$(f(\mathcal{Q},\mathcal{H}^m,\mathcal{H}^m)]\cap(f(\mathcal{H}^p,\mathcal{Q},\mathcal{H}^q)\cup f(\mathcal{H}^p,\mathcal{H},\mathcal{Q},\mathcal{H},\mathcal{H}^q)]\cap(f(\mathcal{H}^n,\mathcal{H}^n,\mathcal{Q})]=\mathcal{Q}.$$

To show that $(f(\mathcal{H}^n,\mathcal{H}^n,\mathcal{Q})]$ is a minimal n-left hyperideal of $\mathcal{H}$. Let $\mathcal{L}$ be an n-left hyperideal of $\mathcal{H}$ contained in $(f(\mathcal{H}^n,\mathcal{H}^n,\mathcal{Q})]$. Then $(f(\mathcal{H}^n,\mathcal{H}^n,\mathcal{L})]\subseteq(L]\subseteq L\subseteq(f(\mathcal{H}^n,\mathcal{H}^n,\mathcal{Q})]$. Thus, $(f(\mathcal{Q},\mathcal{H}^m,\mathcal{H}^m)]\cap(f(\mathcal{H}^p,\mathcal{Q},\mathcal{H}^q)\cup f(\mathcal{H}^p,\mathcal{H},\mathcal{Q},\mathcal{H},\mathcal{H}^q)]\cap(f(\mathcal{H}^n,\mathcal{H}^n,\mathcal{L})]$ $\subseteq(f(\mathcal{Q},\mathcal{H}^m,\mathcal{H}^m)]\cap(f(\mathcal{H}^p,\mathcal{Q},\mathcal{H}^q)\cup f(\mathcal{H}^p,\mathcal{H},\mathcal{Q},\mathcal{H},\mathcal{H}^q)]\cap(f(\mathcal{H}^n,\mathcal{H}^n,\mathcal{Q})]\subseteq\mathcal{Q}$
Now $f(\mathcal{Q},\mathcal{H}^m,\mathcal{H}^m)]\cap(f(\mathcal{H}^p,\mathcal{Q},\mathcal{H}^q)\cup f(\mathcal{H}^p,\mathcal{H},\mathcal{Q},\mathcal{H},\mathcal{H}^q)]\cap(f(\mathcal{H}^n,\mathcal{H}^n,\mathcal{L})]$ is an $(m,(p,q),n)$ -quasi hyperideal of $\mathcal{H}$ and $\mathcal{Q}$ is a minimal $(m,(p,q),n)$-quasi hyperideal of $\mathcal{H}$. We have

$$(f(\mathcal{Q},\mathcal{H}^m,\mathcal{H}^m)]\cap(f(\mathcal{H}^p,\mathcal{Q},\mathcal{H}^q)\cup f(\mathcal{H}^p,\mathcal{H},\mathcal{Q},\mathcal{H},\mathcal{H}^q)]\cap(f(\mathcal{H}^n,\mathcal{H}^n,\mathcal{L})]=\mathcal{Q}.$$

Then $\mathcal{Q}\subseteq(f(\mathcal{H}^n,\mathcal{H}^n,\mathcal{L})]$ and we have

$$\begin{aligned}(f(\mathcal{H}^n,\mathcal{H}^n,\mathcal{Q})] &\subseteq (f(\mathcal{H}^n,\mathcal{H}^n,(f(\mathcal{H}^n,\mathcal{H}^n,\mathcal{L})]]\\ &\subseteq (f(\mathcal{H}^n,\mathcal{H}^n,(f(\mathcal{H}^n,\mathcal{H}^n,\mathcal{L}))]\\ &= (f(\mathcal{H}^n,f(\mathcal{H}^n,\mathcal{H}^n,\mathcal{H}^n),\mathcal{L})]\\ &\subseteq (f(\mathcal{H}^n,\mathcal{H}^n,\mathcal{L})]\\ &\subseteq \mathcal{L}.\end{aligned}$$

It implies $\mathcal{L}=(f(\mathcal{H}^n,\mathcal{H}^n,\mathcal{Q})]$. Therefore $(f(\mathcal{H}^n,\mathcal{H}^n,\mathcal{Q})]$ is a minimal n-left hyperideal of $\mathcal{H}$. Similarly other cases can be proved.

Conversely, suppose $\mathcal{Q}=\mathcal{L}\cap\mathcal{M}\cap\mathcal{R}$, where $\mathcal{L},\mathcal{M}$ and $\mathcal{R}$ are minimal n-left, minimal (p,q)-lateral and minimal m-right hyperideals of $\mathcal{H}$, respectively. Then $\mathcal{Q}\subseteq\mathcal{L},\mathcal{Q}\subseteq\mathcal{M}$ and $\mathcal{Q}\subseteq\mathcal{R}$. By the Theorem 3.13, $\mathcal{Q}$ will be an $(m,(p,q),n)$-quasi hyperideal of $\mathcal{H}$. Now we have to show that $\mathcal{Q}$ is minimal. For this let $\mathcal{Q}'$ be an $(m,(p,q),n)$-quasi hyperideal of $\mathcal{H}$ contained in $\mathcal{Q}$. By the Theorem 3.14, $(f(\mathcal{Q}',\mathcal{H}^m,\mathcal{H}^m)]$, $(f(\mathcal{H}^p,\mathcal{Q}',\mathcal{H}^q)\cup$

$f(\mathcal{H}^p, \mathcal{H}, \mathcal{Q}', \mathcal{H}, \mathcal{H}^q)]$, $(f(\mathcal{H}^n, \mathcal{H}^n, \mathcal{Q}')]$ are an m-right, a (p, q)-lateral and an n-left hyperideal of $\mathcal{H}$, respectively. Now,

$$\begin{aligned}(f(\mathcal{H}^n, \mathcal{H}^n, \mathcal{Q}')] &\subseteq (f(\mathcal{H}^n, \mathcal{H}^n, \mathcal{Q})] \\ &\subseteq (f(\mathcal{H}^n, \mathcal{H}^n, \mathcal{L})] \\ &\subseteq \mathcal{L}.\end{aligned}$$

But $\mathcal{L}$ is minimal, it implies $(f(\mathcal{H}^n, \mathcal{H}^n, \mathcal{Q}')] = \mathcal{L}$. Similarly $(f(\mathcal{Q}', \mathcal{H}^m, \mathcal{H}^m)] = \mathcal{R}$ and $(f(\mathcal{H}^p, \mathcal{Q}', \mathcal{H}^q) \cup f(\mathcal{H}^p, \mathcal{H}, \mathcal{Q}', \mathcal{H}, \mathcal{H}^q)] = \mathcal{M}$. As $\mathcal{Q}'$ is an $(m, (p, q), n)$-quasi hyperideal of $\mathcal{H}$. We have

$$\begin{aligned}\mathcal{Q} &= \mathcal{L} \cap \mathcal{M} \cap \mathcal{R} \\ &= (f(\mathcal{H}^n, \mathcal{H}^n, \mathcal{Q}')] \cap (f(\mathcal{H}^p, \mathcal{Q}', \mathcal{H}^q) \cup f(\mathcal{H}^p, \mathcal{H}, \mathcal{Q}', \mathcal{H}, \mathcal{H}^q)] \cap (f(\mathcal{H}^n, \mathcal{H}^n, \mathcal{Q}')] \\ &\subseteq \mathcal{Q}'.\end{aligned}$$

It implies $\mathcal{Q} = \mathcal{Q}'$. Therefore $\mathcal{Q}$ is a minimal $(m, (p, q), n)$-quasi hyperideal of $\mathcal{H}$. ∎

Theorem 4.3. *Let $(\mathcal{H}, f, \leq)$ be an ordered ternary semihypergroup. Then the following holds:*

(1) An m-right hyperideal $\mathcal{R}$ is minimal if and only if $(f(a, \mathcal{H}^m, \mathcal{H}^m)] = \mathcal{R}$ for all $a \in \mathcal{R}$.

(2) A (p, q)-lateral hyperideal $\mathcal{M}$ is minimal if and only if

$$(f(\mathcal{H}^p, a, \mathcal{H}^q) \cup f(\mathcal{H}^p, \mathcal{H}, a, \mathcal{H}, \mathcal{H}^q)] = \mathcal{M} \text{ for all } a \in \mathcal{M}.$$

(3) An n-left hyperideal $\mathcal{L}$ is minimal if and only if $(f(\mathcal{H}^n, \mathcal{H}^n, a)] = \mathcal{L}$ for all $a \in \mathcal{L}$.

(4) An $(m, (p, q), n)$-quasi-hyperideal $\mathcal{Q}$ is minimal if and only if $(f(a, \mathcal{H}^m, \mathcal{H}^m)] \cap (f(\mathcal{H}^p, a, \mathcal{H}^q) \cup f(\mathcal{H}^p, \mathcal{H}, a, \mathcal{H}, \mathcal{H}^q)] \cap (f(\mathcal{H}^n, \mathcal{H}^n, a)] = \mathcal{Q}$ for all $a \in \mathcal{Q}$.

Proof. (2) Suppose that a (p, q)-lateral hyperideal $\mathcal{M}$ is minimal. Let $a \in \mathcal{M}$. Then

$$\begin{aligned}(f(\mathcal{H}^p, a, \mathcal{H}^q) \cup f(\mathcal{H}^p, \mathcal{H}, a, \mathcal{H}, \mathcal{H}^q)] &\subseteq (f(\mathcal{H}^p, M, \mathcal{H}^q) \cup f(\mathcal{H}^p, \mathcal{H}, M, \mathcal{H}, \mathcal{H}^q)] \\ &\subseteq \mathcal{M}.\end{aligned}$$

By the Theorem 3.14(2), we have $(f(\mathcal{H}^p, a, \mathcal{H}^q) \cup f(\mathcal{H}^p, \mathcal{H}, a, \mathcal{H}, \mathcal{H}^q)]$ is a (p, q)-lateral hyperideal of $\mathcal{H}$. As $\mathcal{M}$ is minimal (p, q)-lateral hyperideal of $\mathcal{H}$, we have $(f(\mathcal{H}^p, a, \mathcal{H}^q) \cup f(\mathcal{H}^p, \mathcal{H}, a, \mathcal{H}, \mathcal{H}^q)] = \mathcal{M}$.

Conversely, suppose that $(f(\mathcal{H}^p, a, \mathcal{H}^q) \cup f(\mathcal{H}^p, \mathcal{H}, a, \mathcal{H}, \mathcal{H}^q)] = \mathcal{M}$ for all $a \in \mathcal{M}$. Let $\mathcal{M}'$ be any (p, q)-lateral hyperideal of $\mathcal{H}$ contained in $\mathcal{M}$. Let $m \in \mathcal{M}'$. Then $m \in \mathcal{M}$. By assumption, we have $(f(\mathcal{H}^p, a, \mathcal{H}^q) \cup f(\mathcal{H}^p, \mathcal{H}, a, \mathcal{H}, \mathcal{H}^q)] = \mathcal{M}$ for all $m \in \mathcal{M}$. Now

$$\begin{aligned}\mathcal{M} &= (f(\mathcal{H}^p, a, \mathcal{H}^q) \cup f(\mathcal{H}^p, \mathcal{H}, a, \mathcal{H}, \mathcal{H}^q)] \\ &\subseteq (f(\mathcal{H}^p, \mathcal{M}', \mathcal{H}^q) \cup f(\mathcal{H}^p, \mathcal{H}, \mathcal{M}', \mathcal{H}, \mathcal{H}^q)] \\ &\subseteq \mathcal{M}'.\end{aligned}$$

It implies $\mathcal{M} \subseteq \mathcal{M}'$. Thus, $\mathcal{M} = \mathcal{M}'$. Hence, $\mathcal{M}$ is minimal (p, q)-lateral hyperideal of $\mathcal{H}$.

Analogously we can prove (1), (3) and (4). ∎

Definition 4.4. Let $(\mathcal{H}, f, \leq)$ be an ordered ternary semihypergroup. Then $\mathcal{H}$ is called an $(m, (p, q), n)$-quasi simple if $\mathcal{H}$ is an unique $(m, (p, q), n)$-quasi hyperideal of $\mathcal{H}$.

Analogously, m-right, (p, q)-lateral and n-left simple ordered ternary semihypergroups can be defined.

Theorem 4.5. *Let $(\mathcal{H}, f, \leq)$ be an ordered ternary semihypergroup. The following statements hold true:*

(1) $\mathcal{H}$ is an m-right simple if and only if $(f(a, \mathcal{H}^m, \mathcal{H}^m)] = \mathcal{H}$ for all $a \in \mathcal{H}$.
(2) $\mathcal{H}$ is an (p, q)-lateral simple if and only if $(f(\mathcal{H}^p, a, \mathcal{H}^q) \cup f(\mathcal{H}^p, \mathcal{H}, a, \mathcal{H}, \mathcal{H}^q)] = \mathcal{H}$ for all $a \in \mathcal{H}$.
(3) $\mathcal{H}$ is an n-left simple if and only if $(f(\mathcal{H}^n, \mathcal{H}^n, a)] = \mathcal{H}$ for all $a \in \mathcal{H}$.
(4) $\mathcal{H}$ is an $(m, (p, q), n)$-quasi simple if and only if $(f(a, \mathcal{H}^m, \mathcal{H}^m)] \cap (f(\mathcal{H}^p, a, \mathcal{H}^q) \cup f(\mathcal{H}^p, \mathcal{H}, a, \mathcal{H}, \mathcal{H}^q)] \cap (f(\mathcal{H}^n, \mathcal{H}^n, a)] = \mathcal{H}$ for all $a \in \mathcal{H}$.

Proof. (1) Assume that $\mathcal{H}$ is an m-right simple, then we have $\mathcal{H}$ is a minimal m-right hyperideal of $\mathcal{H}$. By the Theorem 4.3(1), $(f(a, \mathcal{H}^m, \mathcal{H}^m)] = \mathcal{H}$ for all $a \in \mathcal{H}$.

Conversely, suppose that $(f(a, \mathcal{H}^m, \mathcal{H}^m)] = \mathcal{H}$ for all $a \in \mathcal{H}$. By the Theorem 4.3(1), $\mathcal{H}$ is a minimal m-right hyperideal of $\mathcal{H}$, and therefore $\mathcal{H}$ is an m-right simple.

(2), (3) and (4) can be proved analogously to (1). ∎

Theorem 4.6. *Let $(\mathcal{H}, f, \leq)$ be an ordered ternary semihypergroup. The following statements hold true:*

(1) If an m-right hyperideal $\mathcal{R}$ of $\mathcal{H}$ is an m-right simple, then $\mathcal{R}$ is a minimal m-right hyperideal of $\mathcal{H}$.
(2) If a (p, q)-lateral hyperideal $\mathcal{M}$ of $\mathcal{H}$ is an (p, q)-lateral simple, then $\mathcal{M}$ is a minimal (p, q)-lateral hyperideal of $\mathcal{H}$.
(3) If an n-left hyperideal $\mathcal{L}$ of $\mathcal{H}$ is an n-left simple, then $\mathcal{L}$ is a minimal n-left hyperideal of $\mathcal{H}$.
4). If an $(m, (p, q), n)$-quasi hyperideal $\mathcal{Q}$ of $\mathcal{H}$ is an $(m, (p, q), n)$-quasi simple, then $\mathcal{Q}$ is a minimal $(m, (p, q), n)$-quasi hyperideal of $\mathcal{H}$.

Proof. (1) Let $\mathcal{R}$ be an m-right simple. By the Theorem 4.5(1), we have $(f(a, \mathcal{R}^m, \mathcal{R}^m)] = \mathcal{R}$ for all $a \in \mathcal{R}$. For every $a \in \mathcal{R}$, we have

$$\begin{aligned} \mathcal{R} &= (f(a, \mathcal{R}^m, \mathcal{R}^m)] \\ &\subseteq (f(a, \mathcal{H}^m, \mathcal{H}^m)] \\ &\subseteq (f(\mathcal{R}, \mathcal{H}^m, \mathcal{H}^m)] \\ &\subseteq \mathcal{R}. \end{aligned}$$

Then $(f(a, \mathcal{H}^m, \mathcal{H}^m)] = \mathcal{R}$ for all $a \in \mathcal{R}$. By the Theorem 4.3(1), we have $\mathcal{R}$ is minimal.

(2), (3) and (4) can be proved analogously to (1). ∎

Acknowledgements

We would like to thank the referees for their comments and suggestions on the manuscript.

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