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Fixed Point Results on *b*-Metric Space via *b*-Simulation Functions

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Abstract In this paper, we present a common fixed point result for two mappings satisfying generalized contractive condition in b-metric space via b-simulation functions. Our results extend and improve several previous results.

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1. Introduction

Czerwik in [1] introduced the concept of b-metric space. Since then, several papers deal with fixed point theory for single-valued and multivalued operators in b-metric spaces have been established (see also [2–5]). Pacurar [6] obtained some results on sequences of almost contractions and about their fixed points in b-metric spaces. Recently, Hussain and Shah [7] presented new results on KKM mappings in cone b-metric spaces.

Very recently Aghajani and et al. in [8] proved some common fixed point theorems in b-meric space and presented some basic property of this spaces. Also in [9] the authors generalized the concept of G-metric space and introduced the concept of G_b-metric space. Furthermore they have proved some fixed point result in such spaces.

The aim of this paper is to present some common fixed point result for two mappings considering b-simulation functions in b-metric space. The results obtained in this paper generalize and extend several ones obtained earlier in a lot of papers concerning metric space such as [10–14].

Consistent with [1] and [5, p. 264], the following definition and results will be needed in the sequel.

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2. Preliminaries

Definition 2.1 ([1]). Let X be a nonempty set and $b \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is said to be a b-metric on X if, for all $x, y, z \in X$, the following conditions are satisfied:

(b1) $d(x, y) = 0$ iff $x = y$,
(b2) $d(x, y) = d(y, x)$,
(b3) $d(x, z) \leq b[d(x, y) + d(y, z)]$.

In this case, the triplet (X, d, b) is called a b-metric space.

It should be noted that, the class of b-metric spaces is effectively larger than that of metric spaces, since a b-metric is a metric when $b = 1$.

Singh and et al. [5, p. 264] presented an example shows that a b-metric on a nonempty set X need not be a metric on X.

Example 2.2 ([8]). Let (X, d) be a metric space, and $\rho(x, y) = (d(x, y))^p$, where $p > 1$ is a real number. Then ρ is a b-metric with $b = 2^{p-1}$. Obviously conditions (b1) and (b2) of Definition 2.1 are satisfied. If $1 < p < \infty$, then the convexity of the function $f(x) = x^p$ $(x > 0)$ implies

$$\left(\frac{a+b}{2}\right)^p \leq \frac{1}{2}\left(a^p + b^p\right),$$

and hence, $(a + b)^p \leq 2^{p-1}(a^p + b^p)$ holds. Thus for each $x, y, z \in X$ we obtain

$$\begin{aligned} \rho(x, y) &= (d(x, y))^p \\ &\leq [d(x, z) + d(z, y)]^p \\ &\leq 2^{p-1}[(d(x, z))^p + (d(z, y))^p] \\ &= 2^{p-1}[\rho(x, z) + \rho(z, y)]. \end{aligned}$$

So condition (b3) of Definition 2.1 is hold and so ρ is a b-metric.

It should be noted that in preceding example, if (X, d) is a metric space, then (X, ρ) is not necessarily a metric space.

For example, if $X = \mathbb{R}$ be the set of real numbers and $d(x, y) = |x - y|$ be the usual Euclidean metric, then $\rho(x, y) = (x - y)^2$ is a b-metric on $\mathbb{R}$ with $b = 2$, but is not a metric on $\mathbb{R}$, because the triangle inequality does not hold.

Example 2.3 ([15]). Let X be a nonempty set, $C_b(X) = \{f : X \to \mathbb{R} : ||f||_\infty = \sup_{x\in X} |f(x)| < \infty\}$ and let $||f|| = \sqrt[3]{||f^3||_\infty}$. Then the function $d : C_b(X) \times C_b(X) \to [0, \infty)$ defined by

$$d(f, g) = ||f - g|| \text{ for all } f, g \in C_b(X)$$

is a b-metric with constant $b = \sqrt[3]{4}$ and so $(C_b(X), d, \sqrt[3]{4})$ is a b-metric space.

Before stating and proving our results, we present some definition and proposition in b-metric space. We recall first the notions of convergence, closedness and completeness in a b-metric space.

Definition 2.4 ([4]). Let (X, d, b) be a b-metric space. Then a sequence $\{x_n\}$ in X is called:

(a) convergent if and only if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$. In this case, we write $\lim_{n\to\infty} x_n = x$.

(b) Cauchy if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

A b-metric space (X, d, b) is complete if every Cauchy sequence in X is convergent.

Proposition 2.5 ([4], Remark 2.1)**.** *In a b-metric space (X, d, b) the following assertions hold:*

(i) a convergent sequence has a unique limit,

(ii) each convergent sequence is Cauchy,

(iii) in general, a b-metric is not continuous.

Definition 2.6 ([16])**.** Let (X, d, b) be a b-metric space and f, g be two self mappings of X. Then the pair $\{f, g\}$ is said to be compatible if and only if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

Remark 2.7. Let (X, d, b) be a b-metric space. If there exists two sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n\to\infty} d(x_n, y_n) = 0$, then we can not necessarily conclude that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$, because in general, a b-metric function may not be continuous. Even it is possible that there is no limit. For example, let $X = \mathbb{R}$ and $d(x, y) = (x - y)^2$ and $x_n = (-1)^n$ and $y_n = (-1)^n + \dfrac{1}{n}$.

Lemma 2.8. *Let (X, d, b) be a b-metric space. If there exists two sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n\to\infty} d(x_n, y_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} x_n = t$ for some $t \in X$, then $\lim_{n\to\infty} y_n = t$.*

Demmaa and et al. [15] gave the definition of b-simulation function in the setting of b-metric space as follows:

Definition 2.9. Let (X, d, b) be a b-metric space. A b-simulation function is a function $\xi : [0, \infty) \times [0, \infty) \to \mathbb{R}$ satisfying the following conditions:

(ξ_1) $\xi(t, s) \le s - t$, for all $t, s \ge 0$,

(ξ_2) if $\{t_n\}$, $\{s_n\}$ are sequences in $(0, \infty)$ such that

$$0 < \lim_{n\to\infty} t_n \le \liminf_{n\to\infty} s_n \le \limsup_{n\to\infty} s_n \le b \lim_{n\to\infty} t_n < \infty,$$

then

$$\limsup_{n\to\infty} \xi(bt_n, s_n) < 0.$$

Following are some examples of b-simulation functions (see [15]).

Example 2.10. Let $\xi : [0, \infty) \times [0, \infty) \to \mathbb{R}$, be defined by

- $\xi(t, s) = \lambda s - t$ for all $t, s \in [0, \infty)$, where $\lambda \in [0, 1)$.
- $\xi(t, s) = \psi(s) - \varphi(t)$ for all $t, s \in [0, \infty)$, where $\varphi, \psi : [0, \infty) \to [0, \infty)$ are two continuous functions such that $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$ and $\psi(t) < t \le \varphi(t)$ for all $t > 0$.
- $\xi(t, s) = s\frac{f(t,s)}{g(t,s)}t$ for all $t, s \in [0, \infty)$, where $f, g : [0, \infty) \times [0, \infty) \to (0, \infty)$ are two continuous functions with respect to each variable such that $f(t, s) > g(t, s)$ for all $t, s > 0$.

- $\xi(t,s) = s - \varphi(s) - t$ for all $t, s \in [0,\infty)$, where $\varphi : [0,\infty) \to [0,\infty)$ is a lower semi-continuous function such that $\varphi(t) = 0$ if and only if $t = 0$.
- $\xi(t,s) = s\varphi(s) - t$ for all $t, s \in [0,\infty)$, where $\varphi : [0,\infty) \to [0,\infty)$ is such that $\lim_{t\to r^+} \varphi(t) < 1$ for all $r > 0$.

Definition 2.11. The self-mapping f of a b-metric space (X, d, b) is said to be b-continuous at $x \in X$ if and only if it is b-sequentially continuous at x, that is, whenever $\{x_n\}$ is b-convergent to x, $\{f(x_n)\}$ is b-convergent to $f(x)$.

3. Fixed Points via b-Simulation Functions

The following lemmas, are needed to establish the main result.

Lemma 3.1. *Let (X, d, b) be a b-metric space and let $f, g : X \to X$ be two mappings. Suppose that $f(X) \subseteq g(X)$ and there exists a b-simulation function ξ such that*

$$\xi(bd(fx, fy), d(gx, gy)) \geq 0 \text{ for all } x, y \in X. \tag{3.1}$$

Then there exists a sequence $\{y_n\}$ in X such that $\lim_{n\to\infty} d(y_{n-1}, y_n) = 0$.

Proof. Let $x_0 \in X$ be arbitrary. Since $f(X) \subseteq g(X)$, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ such that $y_n = f(x_n) = g(x_{n+1})$ for every $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $y_{n_0} = y_{n_0+1}$, then it follows from (3.1) and (ξ_1) that for all $n \in \mathbb{N}$

$$\begin{aligned} 0 &\leq \xi(bd(fx_{n_0+1}, fx_{n_0+2}), d(gx_{n_0+1}, gx_{n_0+2})) \\ &= \xi(bd(y_{n_0+1}, y_{n_0+2}), d(y_{n_0}, y_{n_{0+1}})) \\ &\leq d(y_{n_0}, y_{n_0+1}) - bd(y_{n_0+1}, y_{n_0+2}). \end{aligned}$$

Since $d(y_{n_0}, y_{n_0+1}) = 0$, the above inequality shows that $d(y_{n_0+1}, y_{n_0+2}) = 0$, therefore $y_{n_0+1} = y_{n_0+2}$. Thus, $y_{n_0} = y_{n_0+1} = y_{n_0+2} = \cdots$, which implies that $\lim_{n\to\infty} d(y_{n-1}, y_n) = 0$. Now, suppose that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}$. Then, it follows from (3.1) and (ξ_1) that for all $n \in \mathbb{N}$, we have

$$\begin{aligned} 0 &\leq \xi(bd(fx_n, fx_{n+1}), d(gx_n, gx_{n+1})) \\ &= \xi(bd(y_n, y_{n+1}), d(y_{n-1}, y_n)) \\ &\leq d(y_{n-1}, y_n) - bd(y_n, y_{n+1}). \end{aligned}$$

The above inequality shows that

$$bd(y_n, y_{n+1}) \leq d(y_{n-1}, y_n), \text{ for all } n \in \mathbb{N},$$

which implies that $\{d(y_{n-1}, y_n)\}$ is a decreasing sequence of positive real numbers. So there is some $r \geq 0$ such that $\lim_{n\to\infty} d(y_{n-1}, y_n) = r$. Suppose that $r > 0$. It follows from the condition (ξ_2), with $t_n = d(y_n, y_{n+1})$ and $s_n = d(y_{n-1}, y_n)$, that

$$0 \leq \limsup_{n\to\infty} \xi(bd(y_n, y_{n+1}), d(y_{n-1}, y_n)) < 0,$$

which is a contradiction. Then we conclude that $r = 0$, which ends the proof. ∎

Remark 3.2. Let (X, d, b) be a b-metric space and let $f, g : X \to X$ be two mappings. Suppose that $f(X) \subseteq g(X)$ and there exists a b-simulation function ξ such that (3.1) holds. Then there exists a sequence $\{y_n\}$ in X, such that $bd(y_m, y_n) \leq d(y_{m-1}, y_{n-1})$ for all $m, n \in \mathbb{N}$.

Proof. By a similar argument of Lemma 3.1 for every $n \in \mathbb{N}$ we have $y_n = f(x_n) = g(x_{n+1})$. Hence, it follows from (3.1) and (ξ_1) that for all $m, n \in \mathbb{N}$, we have

$$\begin{aligned} 0 &\leq \xi(bd(fx_m, fx_n), d(gx_m, gx_n)) \\ &= \xi(bd(y_m, y_n), d(y_{m-1}, y_{n-1})) \\ &\leq d(y_{m-1}, y_{n-1}) - bd(y_m, y_n). \end{aligned}$$

The above inequality shows that

$$bd(y_m, y_n) \leq d(y_{m-1}, y_{n-1}), \text{ for all } m, n \in \mathbb{N}.$$

■

Lemma 3.3. *Let (X, d, b) be a b-metric space and let $f, g : X \to X$ be two mappings. Suppose that $f(X) \subseteq g(X)$ and there exists a b-simulation function ξ such that (3.1) holds. Then there exists a sequence $\{y_n\}$ in X, such that $\{y_n\}$ is bounded sequence.*

Proof. By a similar argument of Lemma 3.1 for every $n \in \mathbb{N}$ we have $y_n = f(x_n) = g(x_{n+1})$. If there exists $n_0 \in \mathbb{N}$ such that $y_{n_0} = y_{n_0+1}$, we have $d(y_i, y_j) \leq M$ for all $i, j = 0, 1, 2, \cdots$, where

$$M = \max\{d(y_i, y_j) : i, j \leq n_0\}.$$

Let us assume that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}$ and suppose $\{y_n\}$ is not a bounded sequence. Then, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that for $n_1 = 1$ and for each $k \in \mathbb{N}$, n_{k+1} is the minimum integer such that $d(y_{n_k+1}, y_{n_k}) > 1$ and

$$d(y_m, y_{n_k}) \leq 1 \text{for } n_k \leq m \leq n_{k+1} - 1.$$

By the triangle inequality, we obtain

$$\begin{aligned} 1 &< d(y_{n_{k+1}}, x_{n_k}) \\ &\leq bd(y_{n_{k+1}}, y_{n_{k+1}-1}) + bd(y_{n_{k+1}-1}, y_{n_k}) \\ &\leq bd(y_{n_{k+1}}, y_{n_{k+1}-1}) + b. \end{aligned}$$

Letting $k \to \infty$ in the above inequality and using Lemma 3.1, we get

$$1 \leq \liminf_{k\to\infty} d(y_{n_{k+1}}, y_{n_k}) \leq \limsup_{k\to\infty} d(y_{n_{k+1}}, y_{n_k}) \leq b. \tag{3.2}$$

Again, from Remark 3.2, we have

$$\begin{aligned} bd(y_{n_{k+1}}, y_{n_k}) &\leq d(y_{n_{k+1}-1}, y_{n_k-1}) \\ &\leq bd(y_{n_{k+1}-1}, y_{n_k}) + bd(y_{n_k}, y_{n_k-1}) \\ &\leq b + bd(y_{n_k}, y_{n_k-1}) \end{aligned}$$

Letting $k \to \infty$ in the above inequality and using (3.2), we deduce that

$$\lim_{k\to\infty} d(y_{n_{k+1}}, y_{n_k}) = 1 \text{and} \lim_{k\to\infty} d(y_{n_{k+1}-1}, y_{n_k-1}) = b.$$

Then by condition (ξ_2), with $t_k = d(y_{n_{k+1}}, y_{n_k})$ and $s_k = d(y_{n_{k+1}-1}, y_{n_k-1})$, we obtain

$$0 \leq \limsup_{k\to\infty} \xi(bd(y_{n_{k+1}}, y_{n_k}), d(y_{n_{k+1}-1}, y_{n_k-1}) < 0,$$

which is a contradiction. This ends the proof. ■

Lemma 3.4. *Let (X, d, b) be a b-metric space and let $f, g : X \to X$ be two mappings. Suppose that $f(X) \subseteq g(X)$ and there exists a b-simulation function ξ such that (3.1) holds. Then there exists a sequence $\{y_n\}$ in X, such that $\{y_n\}$ is a Cauchy sequence.*

Proof. By a similar argument of Lemma 3.1 for every $n \in \mathbb{N}$ we have $y_n = f(x_n) = g(x_{n+1})$. If there exists $n_0 \in \mathbb{N}$ such that $y_{n_0} = y_{n_0+1}$, then we have $\{y_n\}$ is a Cauchy sequence. Let us assume that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}$ and let

$$C_n = \sup\{d(y_i, y_j) : i, j \geq n\}.$$

From Lemma 3.3, we know that $C_n < \infty$ for every $n \in \mathbb{N}$. Since $\{C_n\}$ is a positive decreasing sequence, there is some $C \geq 0$ such that

$$\lim_{n\to\infty} C_n = C. \tag{3.3}$$

Let us suppose that $C > 0$. By the definition of $\{C_n\}$, for every $k \in \mathbb{N}$, there exists $n_k, m_k \in \mathbb{N}$ such that $m_k > n_k \geq k$ and

$$C_k - \frac{1}{k} < d(y_{m_k}, y_{n_k}) \leq C_k.$$

Letting $k \to \infty$ in the above inequality, we get

$$\lim_{k\to\infty} d(y_{m_k}, y_{n_k}) = C. \tag{3.4}$$

Again, from Remark 3.2 and the definition of $\{C_n\}$, we deduce

$$bd(y_{m_k}, y_{n_k}) \leq d(y_{m_k-1}, y_{n_k-1}) \leq C_{k-1}.$$

Letting $k \to \infty$ in the above inequality, using (3.3) and (3.4), we get

$$bC \leq \liminf_{k\to\infty} d(y_{m_{k-1}} y_{n_{k-1}}) \leq \limsup_{k\to\infty} d(y_{m_{k-1}}, y_{n_{k-1}}) \leq C. \tag{3.5}$$

Now, if $b > 1$, the previous inequality implies a contradiction since $C > 0$. If $b = 1$, by the condition (ξ_2), with $t_k = d(y_{m_k}, y_{n_k})$ and $s_k = d(y_{m_{k-1}}, y_{n_{k-1}})$, we get

$$0 \leq \limsup_{k\to\infty} \xi(bd(y_{m_k}, y_{n_k}), d(y_{m_k-1}, y_{n_k-1})) < 0,$$

which is a contradiction. Thus we have $C = 0$, that is,

$$\lim_{n\to\infty} C_n = 0 \text{ for all } b \geq 1.$$

This proves that $\{y_n\}$ is a Cauchy sequence. ∎

Now, we present our main result.

Theorem 3.5. *Let (X, d, b) be a complete b-metric space, $f, g : X \to X$ be two mappings with $f(X) \subseteq g(X)$ and the pair $\{f, g\}$ is compatible. Suppose that there exists a b-simulation function ξ such that (3.1) holds, that is,*

$$\xi(bd(fx, fy), d(gx, gy)) \geq 0, \text{ for all } x, y \in X.$$

If g is continuous, then f and g have a coincidence point, that is, there exists $y \in X$ such that $f(y) = g(y)$. Moreover, if g is one to one, then f and g have unique common fixed point.

Proof. Let $x_0 \in X$, since $f(X) \subseteq g(X)$, hence for every $n \in \mathbb{N}$ we have $y_n = f(x_n) = g(x_{n+1})$. Now, by Lemma 3.4, the sequence $\{y_n\}$ is Cauchy and since (X, d, b) is complete, then there exists some $y \in X$ such that $\lim_{n\to\infty} y_n = y$. That is,

$$y = \lim_{n\to\infty} y_n = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n). \tag{3.6}$$

We claim that y is a coincidence point of f, g. Since, g is continuous, hence we have

$$\lim_{n\to\infty} gf(x_n) = \lim_{n\to\infty} gg(x_n) = g(y).$$

Also, since $\{f, g\}$ is compatible, we have $\lim_{n\to\infty} d(fg(x_n), gf(x_n)) = 0$. Hence, by Lemma 2.8 we deduce

$$\lim_{n\to\infty} fg(x_n) = g(y).$$

From (3.1) we have,

$$\begin{aligned} 0 &\leq \xi(bd(fy, fgx_n), d(gy, ggx_n)) \\ &\leq d(gy, ggx_n) - bd(fy, fgx_n)). \end{aligned}$$

Letting $n \to \infty$ in the above inequality, we get

$$\begin{aligned} 0 &\leq \liminf_{n\to\infty} d(gy, ggx_n) - b\limsup_{n\to\infty} d(fy, fgx_n)) \\ &= -b\limsup_{n\to\infty} d(fy, fgx_n) \\ &\leq 0. \end{aligned}$$

Thus,

$$\limsup_{n\to\infty} d(fy, fgx_n) = 0.$$

That is

$$\lim_{n\to\infty} fg(x_n) = f(y),$$

therefore, $f(y) = g(y)$.

Now, assume there exists $u \in X$ such that $f(u) = g(u)$ then the (ξ_2) inequality implies

$$\begin{aligned} 0 &\leq \xi(bd(fy, fu), d(gy, gu)) \\ &\leq d(gy, gu) - bd(fy, fu) \\ &\leq 0, \end{aligned}$$

hence $bd(fy, fu) \leq d(fy, fu)$, if $b > 1$, then $f(y) = f(u)$. If $b = 1$, by the condition (ξ_2), with $t_k = d(fy, fu)$ and $s_k = d(gy, gu)$, we get

$$0 \leq \limsup_{k\to\infty} \xi(bd(fy, fu), d(gy, gu) < 0,$$

which is a contradiction. Thus we have $f(u) = f(y) = g(u) = g(y)$.

Now, suppose the map g is one to one. If y, u are two coincidence points of f and g, in this case by the above argument we have $f(y) = g(y) = f(u) = g(u)$. Since g is one to one it follows that $y = u$. Also, since $g(y) = f(y)$ and the pair $\{f, g\}$ is compatible we have $fg(y) = gf(y)$. Therefore, $gf(y) = fg(y) = ff(y)$. That is $f(y)$ is a coincidence point of f and g. Therefore, $f(y) = y$ hence $f(y) = g(y) = y$. That is f and g have unique common fixed point $y \in X$. ∎

Now we give an example to support our main result.

Example 3.6. Let $X = [0, 1]$ be endowed with the b-metric $d(x, y) = (x - y)^2$, where $b = 2$. Define f and g on X by

$$f(x) = (\frac{x}{2})^4 \text{ and } g(x) = (\frac{x}{2})^2$$

Obviously $f(X) \subseteq g(X)$ and furthermore the pair $\{f, g\}$ is compatible mappings. Consider the b-simulation function as

$$\xi(t,s) = \frac{1}{2}s - t,$$

for all $t, s \geq 0$. Then for each $x, y \in X$ we have

$$\begin{aligned} d(fx, fy) &= (fx - fy)^2 = ((\frac{x}{2})^4 - (\frac{y}{2})^4)^2 \\ &= ((\frac{x}{2})^2 + (\frac{y}{2})^2)^2((\frac{x}{2})^2 - (\frac{y}{2})^2)^2 \\ &\leq (\frac{1}{4} + \frac{1}{4})^2 d(gx, gy) = \frac{1}{4}d(gx, gy). \end{aligned}$$

Thus f and g satisfy all conditions given in Theorem 3.5 and so they have a unique common fixed point.

We show the unifying power of b-simulation functions by applying Theorem 3.5 to deduce different kinds of contractive conditions in the existing literature.

Compatible mapping bring a standard fixed point results. See [17–24]. Hence, if we take $g = I$ (the identity map) in Theorem 3.5, we obtain Theorem 3.4 of [15].

Corollary 3.7. *Let (X, d, b) be a complete b-metric space, $f, g : X \to X$ be two mappings with $f(X) \subseteq g(X)$ and the pair $\{f, g\}$ is compatible. Suppose that there exists $\lambda \in (0, 1)$ such that*

$$bd(fx, fy) \leq \lambda d(gx, gy) \text{ for all } x, y \in X.$$

If g is continuous, then f and g have a coincidence point. Moreover, if g is one to one, then f and g have unique common fixed point.

Proof. The result follows from Theorem 3.5, by taking b-simulation function as

$$\xi(t,s) = \lambda s - t,$$

for all $t, s \geq 0$. ∎

4. An Application to the Integral Equation

Let $C^k[a, b] = \{f : [a, b] \to \mathbb{R} : f$ is continuous and has derivative of order$k\}$. For every $x \in [0, 1]$, consider the integral equation

$$f(x) = h(x) + \lambda \int_0^x k(x,t)f(t)dt,$$

where $f, h \in C^k[0, 1]$, $\lambda \neq 0$ and $k(x, t)$ is continuous on the squared region $[0, 1]\times[0, 1] \longrightarrow [-M, M]$ with $|M| < \frac{1}{|\lambda|}$. Then there exists a unique $f_0 \in C^k[0, 1]$ such that

$$f_0(x) - h(x) = \lambda \int_0^x k(x,t)f_0(t)dt.$$

In the following we can show this fact: for every $f \in C^k[0, 1]$, define $T : C^k[0, 1] \to C^k[0, 1]$ by $T(f) = T_f$, where, for every $x \in [0, 1]$,

$$T_f(x) = h(x) + \lambda \int_0^x k(x,t)f(t)dt.$$

If we consider $d(f,g) = ||f-g||_\infty$, for every $f, g \in C^k[0,1]$, then it is easy to see that d is a complete metric on $C^k[0,1]$. Therefore, for all $f, g \in C^k[0,1]$, we have,

$$\begin{aligned} d(T(f),T(g)) &= \sup_{x\in[0,1]} |T_f(x) - T_g(x)| \\ &\leq \sup_{x\in[0,1]} |\lambda| \int_0^x |k(x,t)|(|f(t)-g(t)|)dt \\ &\leq |\lambda M| \int_0^x |f(t)-g(t)|dt \\ &\leq |\lambda M| \sup_{x\in[0,1]} |f(x)-g(x)| \int_0^x dt \\ &\leq |\lambda M| ||f-g||_\infty \\ &= |\lambda M| d(f,g). \end{aligned}$$

Hence, the assertion follows from using Corollary 3.7, there exists a unique $f_0 \in C^k[0,1]$ such that $T(f_0) = f_0$. That is

$$f_0(x) - h(x) = \lambda \int_0^x k(x,t) f_0(t) dt,$$

for every $x \in [0,1]$.

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