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On Measurability of $C[0,1]$

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Abstract In classical stochastic calculus, the measurability of the space $C[0,1]$ of all continuous functions on $[0,1]$ is handled by continuous modification. In this note, we shall prove that $C[0,1]$ is measurable in a setting of a dyadic Henstock integral.

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In stochastic analysis, the measure of the space $C[0,1]$ of all continuous functions defined on $[0,1]$ is handled by continuous modification. In this paper, we shall prove that $C[0,1]$ is measurable and the measure of $C[0,1]$ is one by integrals using the Henstock approach.

1. McShane–Wiener Integral

In this section, we shall define two types of McShane–Wiener integrals.

Let $\mathbb{R}$ denotes the set of real numbers. We define the set

$$\begin{aligned}\mathbb{R}^{[0,1]} &= \prod_{t\in[0,1]} \mathbb{R}_t,\\ &= \{\xi : t \mapsto \xi(t), t \in [0,1], \xi(t) \in \mathbb{R} \text{ with } \xi(0) = 0\}.\end{aligned}$$

where $\mathbb{R}_t = \mathbb{R}$ for each t, i.e., $\mathbb{R}^{[0,1]}$ can also be viewed as a set of real-valued function ξ defined on $[0,1]$ with $\xi(0) = 0$. Let $\mathcal{Q}_2 = \{m2^{-n} \in [0,1] : m, n \text{ are positive integers}\}$ be the *dyadic rational*. Clearly that $\mathcal{Q}_2$ is a countable dense subset of $[0,1]$. Let $\mathcal{N}(\mathcal{Q}_2)$ be the class of all finite subsets $N = \{t_1, t_2, \ldots, t_n\}$ of $\mathcal{Q}_2$ with $t_1 < t_2 < \cdots < t_n$.

The following notation shall be used: $\xi_i = \xi(t_i)$ and $I_i = I_{t_i}$ for all $t_i \in N$; and $\xi(N) = (\xi_1, \xi_2, \ldots, \xi_n)$.

The *cylindrical intervals* (or simply *intervals*) in $\mathbb{R}^{[0,1]}$, denoted by $I[N]$, are of the form

$$I[N] = I(N) \times \mathbb{R}^{[0,1]\setminus N},$$

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where $N = \{t_1, t_2, \ldots, t_n\} \in \mathcal{N}(\mathcal{Q}_2)$ and $I(N) = I_1 \times I_2 \times \cdots \times I_n$ is the n Cartesian product of one-dimensional, compact or unbounded closed, intervals I_i in $\mathbb{R}$. Let $\mathcal{I}(\mathcal{Q}_2)$ be the class of all interval in $\mathbb{R}^{[0,1]}$ with $N \in \mathcal{N}(\mathcal{Q}_2)$.

Let $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ denote the set of extended real numbers. Denote $\prod_{t\in[0,1]} \bar{\mathbb{R}}_t$ by $\bar{\mathbb{R}}^{[0,1]}$, the class of all extended real-valued functions ξ defined on $[0,1]$ with $\xi(0) = 0$.

Let $\delta(\xi, N)$ be a positive function defined on $\bar{\mathbb{R}}^{[0,1]} \times \mathcal{N}(\mathcal{Q}_2)$. A point-interval pair $(\xi, I[N])$, where $\xi \in \bar{\mathbb{R}}^{[0,1]}$ and $N \in \mathcal{N}(\mathcal{Q}_2)$, is said to be δ-*fine* if for each $t_i \in N$, we have (i) $I_i \subset (\xi_i - \delta(\xi, N), \xi_i + \delta(\xi, N))$ whenever $\xi_i \neq \pm\infty$; (ii) $I_i \subseteq ((\delta(\xi, N))^{-1}, \infty)$ whenever $\xi_i = \infty$; or (iii) $I_i \subseteq (-\infty, -(\delta(\xi, N))^{-1})$ whenever $\xi_i = -\infty$. Let $L(\xi)$ be a set-valued function defined on $\bar{\mathbb{R}}^{[0,1]}$ with values in $\mathcal{N}(\mathcal{Q}_2)$.

Let γ be a pair of functions (δ, L), where $\delta : \bar{\mathbb{R}}^{[0,1]} \times \mathcal{N}(\mathcal{Q}_2) \to (0, \infty)$ and $L : \bar{\mathbb{R}}^{[0,1]} \to \mathcal{N}(\mathcal{Q}_2)$. A point-interval pair $(\xi, I[N])$ is said to be γ-*fine* with respect to $\mathcal{Q}_2$ if $N \supseteq L(\xi)$ and $(\xi, I(N))$ is δ-fine, where $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$. A finite collection of point-interval pairs $D = \{(\xi, I[N])\}$ is said to be a γ-*fine partial division* of $\mathbb{R}^{[0,1]}$ with respect to $\mathcal{Q}_2$ if $\{I[N]\}$ is a partial partition of $\mathbb{R}^{[0,1]}$ and each $(\xi, I[N])$ is γ-fine. In addition, if $\{I[N]\}$ is a partition of $\mathbb{R}^{[0,1]}$, then D is said to be a γ-*fine division* of $\mathbb{R}^{[0,1]}$ with respect to $\mathcal{Q}_2$. Given a function γ, a γ-fine division of $\mathbb{R}^{[0,1]}$ with respect to $\mathcal{Q}_2$ exists, see [1, p.121].

Given $N = \{t_1, t_2, \ldots, t_n\} \in \mathcal{N}(\mathcal{Q}_2)$, let

$$G(I[N]) = \int_{I(N)} h_N(u)\, du,$$

where $u = (u_1, u_2, \ldots, u_n)$ and

$$h_N(u) = \left((2\pi)^n \prod_{j=1}^{n} (t_j - t_{j-1})\right)^{-1/2} \exp\left(-\frac{1}{2}\sum_{j=1}^{n} \frac{(u_j - u_{j-1})^2}{t_j - t_{j-1}}\right)$$

with $t_0 = 0$ and $u_0 = 0$, mentioned in [1–3]. The n-dimensional integral above is a Riemann or improper Riemann integral. Hence, it is a Henstock integral, see [1–3].

Definition 1.1 (McShane–Wiener Integral on $\mathbb{R}^{[0,1]}$ with respect to $\mathcal{Q}_2$)**.** The functional $f : \bar{\mathbb{R}}^{[0,1]} \to \mathbb{R}$ is said to be McShane–Wiener integrable (or simply Wiener integrable) to $A \in \mathbb{R}$ on $\mathbb{R}^{[0,1]}$ with respect to $\mathcal{Q}_2$ if for each $\epsilon > 0$, there exists a pair of functions $\gamma = (\delta, L)$, where $\delta : \bar{\mathbb{R}}^{[0,1]} \times \mathcal{N}(\mathcal{Q}_2) \to (0, \infty)$ and $L : \bar{\mathbb{R}}^{[0,1]} \to \mathcal{N}(\mathcal{Q}_2)$, such that whenever $D = \{(\xi, I[N])\}$ is a γ-fine division of $\mathbb{R}^{[0,1]}$ with respect to $\mathcal{Q}_2$, we have

$$\left|(D)\sum f(\xi)G(I[N]) - A\right| \le \epsilon,$$

where we assume that $f(\xi) = 0$ if one of the components of ξ is $\pm\infty$. The number A is called the McShane–Wiener integral (or simply Wiener integral) of f on $\mathbb{R}^{[0,1]}$ with respect to $\mathcal{Q}_2$ and is denoted by $\int_{\mathbb{R}^{[0,1]}(\mathcal{Q}_2)} f$.

We remark that the above integrals are mentioned in [1, p.316 – 320]. In his book, Muldowney remarks that this integral is a version of continuous modification of the integral. We use the above integral to show that $C[0,1]$ is integrable. It is known that $C[0,1]$ is not integrable if we use tag points in $[0,1]$ instead of $\mathcal{Q}_2$.

Definition 1.2. In the Definition 1.1, if we replace the interval $[0,1]$ by $\mathcal{Q}_2$, i.e., $f : \bar{\mathbb{R}}^{\mathcal{Q}_2} \to \mathbb{R}$; $\gamma = (\delta, L)$ is defined on $\mathbb{R}^{\mathcal{Q}_2}$ and $\mathcal{N}(\mathcal{Q}_2)$; and $D = \{(\xi, I[N])\}$ is a γ-fine division of $\mathbb{R}^{\mathcal{Q}_2}$, then f is said to be *McShane–Wiener integrable* (or simply *Wiener integrable*) to $A \in \mathbb{R}$ on $\mathbb{R}^{\mathcal{Q}_2}$ and is denoted by $\int_{\mathbb{R}^{\mathcal{Q}_2}} f$.

We note that the basic properties of integrals, such as linear property and the integrability over subinterval hold for the McShane–Wiener Integral on $\mathbb{R}^{[0,1]}$ with respect to $\mathcal{Q}_2$ and the McShane–Wiener Integral on $\mathbb{R}^{\mathcal{Q}_2}$. The integrals $\int_{\mathbb{R}^{[0,1]}(\mathcal{Q}_2)} f$ and $\int_{\mathbb{R}^{\mathcal{Q}_2}} f$ are not equivalent. The first integral is an integration over $\mathbb{R}^{[0,1]}$, while the second is an integration over $\mathbb{R}^{\mathcal{Q}_2}$.

2. Measures on $\mathbb{R}^{[0,1]}$

Now we shall discuss the measurability of $C[0,1]$ and $C(\mathcal{Q}_2)$.

Definition 2.1. Let $\mathcal{M}$ be the collection of all subsets M of $\overline{\mathbb{R}}^{[0,1]}$ such that χ_M is McShane–Wiener integrable on $\mathbb{R}^{[0,1]}$ with respect to $\mathcal{Q}_2$. If $M \in \mathcal{M}$, then M is said to be a *measurable set*. Let $\mathcal{M}^{\mathcal{Q}_2}$ be the collection of all subsets M of $\overline{\mathbb{R}}^{\mathcal{Q}_2}$ such that χ_M is McShane–Wiener integrable on $\mathbb{R}^{\mathcal{Q}_2}$. If $M \in \mathcal{M}^{\mathcal{Q}_2}$, then M is said to be a $\mathcal{Q}_2$*-measurable set.*

Definition 2.2. Let $\mathcal{P} : \mathcal{M} \to \mathbb{R}$ and $M \in \mathcal{M}$, define

$$\mathcal{P}(M) = \int_{\mathbb{R}^{[0,1]}(\mathcal{Q}_2)} \chi_M;$$

and $\mathcal{P}^{\mathcal{Q}_2} : \mathcal{M}^{\mathcal{Q}_2} \to \mathbb{R}$ and $M \in \mathcal{M}^{\mathcal{Q}_2}$, define

$$\mathcal{P}^{\mathcal{Q}_2}(M) = \int_{\mathbb{R}^{\mathcal{Q}_2}} \chi_M.$$

Lemma 2.3. $\mathbb{R}^{\mathcal{Q}_2}$ *is* $\mathcal{Q}_2$*-measurable and* $\mathcal{P}^{\mathcal{Q}_2}(\mathbb{R}^{\mathcal{Q}_2})=1$*;* $\mathbb{R}^{[0,1]}$ *is measurable and* $\mathcal{P}(\mathbb{R}^{[0,1]})=1$.

Proof. Let $\epsilon > 0$. Let $L : \overline{\mathbb{R}}^{\mathcal{Q}_2} \to \mathcal{N}(\mathcal{Q}_2)$ and $\delta : \overline{\mathbb{R}}^{\mathcal{Q}_2} \times \mathcal{N}(\mathcal{Q}_2) \to (0,\infty)$ be any functions, and $\gamma = (\delta, L)$.

Let $D = \{(\xi, I[N])\}$ be a γ-fine division of $\mathbb{R}^{\mathcal{Q}_2}$. Note that for any $(\xi, I[N]) \in D$, we have $N = \{t_1, t_2, \ldots, t_n\} \subset \mathcal{Q}_2$. Hence

$$(D)\sum \chi_{\mathbb{R}^{\mathcal{Q}_2}}(\xi)G(I[N]) = (D)\sum 1 \cdot G(I[N]) = (D)\sum G(I[N]) = 1.$$

Therefore, $\mathcal{P}^{\mathcal{Q}_2}(\mathbb{R}^{\mathcal{Q}_2}) = 1$. Similarly, $\mathcal{P}(\mathbb{R}^{[0,1]}) = 1$. ∎

Theorem 2.4. $\mathcal{P}^{\mathcal{Q}_2}$ *is a probability measure on* $(\mathbb{R}^{\mathcal{Q}_2}, \mathcal{M}^{\mathcal{Q}_2})$*, that is,* $(\mathbb{R}^{\mathcal{Q}_2}, \mathcal{M}^{\mathcal{Q}_2}, \mathcal{P}^{\mathcal{Q}_2})$ *is a probability measure space;* $(\mathbb{R}^{[0,1]}, \mathcal{M}, \mathcal{P})$ *is a probability measure space.*

Proof. The proof is a consequence of the standard properties of Henstock-Wiener integral and Monotone Convergence Theorem, see [1, 2, 4–7]. ∎

Theorem 2.5. *The probability measure spaces* $(\mathbb{R}^{\mathcal{Q}_2}, \mathcal{M}^{\mathcal{Q}_2}, \mathcal{P}^{\mathcal{Q}_2})$ *and* $(\mathbb{R}^{[0,1]}, \mathcal{M}, \mathcal{P})$ *are complete, i.e., every subset of a set of measure zero is of measure zero.*

Proof. Let M be a set of $\mathcal{Q}_2$-measure zero. So, $M \in \mathcal{M}^{\mathcal{Q}_2}$ and $\mathcal{P}^{\mathcal{Q}_2}(M) = 0$. Let $\epsilon > 0$, there exists a pair of functions $\gamma = (\delta, L)$ such that whenever $D = \{(\xi, I[N])\}$ is a γ-fine division of $\mathbb{R}^{\mathcal{Q}_2}$, we have

$$\left|(D)\sum \chi_M(\xi)G(I[N])\right| \le \epsilon.$$

Let $M' \subseteq M$. Then $\chi_{M'}(\xi) \leq \chi_M(\xi)$ for all $\xi \in \overline{\mathbb{R}}^{\mathcal{Q}_2}$. Thus for every γ-fine division $D = \{(\xi, I[N])\}$ of $\mathbb{R}^{\mathcal{Q}_2}$, we have

$$\left|(D)\sum \chi_{M'}(\xi)G(I[N])\right| \leq \left|(D)\sum \chi_M(\xi)G(I[N])\right| \leq \epsilon.$$

Therefore, $\chi_{M'}$ is integrable to zero on $\mathbb{R}^{\mathcal{Q}_2}$, that is, M' is a set of $\mathcal{Q}_2$-measure zero. Similarly for $(\mathbb{R}^{[0,1]}, \mathcal{M}, \mathcal{P})$. ∎

A function ξ is said to be *uniformly continuous* on X, where X is a metric space with metric $|\cdot|$, if for any $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X$ with $|x - y| < \delta$ we have

$$|\xi(x) - \xi(y)| \leq \epsilon.$$

Let $C(X)$ be the set of all uniform continuous function on X and $D(X) = \mathbb{R}^X \setminus C(X)$. We note that if X is compact then the set of all uniform continuous function on X and the set of all continuous function on X are coincide.

The following is a well-known result, see [8].

Lemma 2.6. *Suppose ξ^* is a uniform continuous function on $\mathcal{Q}_2$. Then there exists unique continuous function ξ on $[0,1]$ such that*

$$\xi(x) = \xi^*(x)$$

for all $x \in \mathcal{Q}_2$.

In this note, $C[0,1]$ and $C(\mathcal{Q}_2)$ are the sets of all uniform continuous function ξ on $[0,1]$ and $\mathcal{Q}_2$, respectively, with $\xi(0) = 0$. Hence $C[0,1] \subseteq \mathbb{R}^{[0,1]}$ and $C(\mathcal{Q}_2) \subseteq \mathbb{R}^{\mathcal{Q}_2}$.

Notice that, the corresponding function $\xi^* \in C(\mathcal{Q}_2)$ of $\xi \in C[0,1]$ is the restriction function of ξ on $\mathcal{Q}_2$.

Let $\wp : \mathbb{R}^{[0,1]} \to \mathbb{R}^{\mathcal{Q}_2}$ be a projection of $\mathbb{R}^{[0,1]}$ onto $\mathbb{R}^{\mathcal{Q}_2}$. We note that

$$\wp^{-1}(C(\mathcal{Q}_2)) \supset C[0,1];$$

but, by Lemma 2.6, if $\xi^* \in C(\mathcal{Q}_2)$, then there exists unique $\xi \in C[0,1]$, such that

$$\wp(\xi) = \xi^*.$$

Theorem 2.7. *Suppose $\chi_{C(\mathcal{Q}_2)}$ is McShane-Wiener integrable on $\mathbb{R}^{\mathcal{Q}_2}$. 2Then the McShane-Wiener integral of $\chi_{C[0,1]}$ on $\mathbb{R}^{[0,1]}$ with respect to $\mathcal{Q}_2$ exists and*

$$\int_{\mathbb{R}^{[0,1]}(\mathcal{Q}_2)} \chi_{C[0,1]} = \int_{\mathbb{R}^{\mathcal{Q}_2}} \chi_{C(\mathcal{Q}_2)}.$$

Proof. Let $\epsilon > 0$ be given. There exists $\gamma^* = (\delta^*, L^*)$ defined on $\mathbb{R}^{\mathcal{Q}_2}$ and $\mathcal{N}(\mathcal{Q}_2)$ such that whenever $D^* = \{(\xi^*, I^*[N])\}$ is a γ^*-fine division of $\mathbb{R}^{\mathcal{Q}_2}$, we have

$$\left|(D^*)\sum \chi_{C(\mathcal{Q}_2)}(\xi^*)G(I^*[N]) - \int_{\mathbb{R}^{\mathcal{Q}_2}} \chi_{C(\mathcal{Q}_2)}\right| \leq \epsilon.$$

For each $\xi \in \mathbb{R}^{[0,1]}$, let ξ^* be the restriction function of ξ on $\mathcal{Q}_2$, i.e., $\xi^* = \wp(\xi)$. Now, we shall choose $\gamma = (\delta, L)$ defined on $\mathbb{R}^{[0,1]}$ so that for every γ-fine division D of $\mathbb{R}^{[0,1]}$ with respect to $\mathcal{Q}_2$ the corresponding division D^* of $\mathbb{R}^{\mathcal{Q}_2}$ is γ^*-fine.

Case I. If $\xi \in C[0,1]$, then $\xi^* = \wp(\xi) \in C(\mathcal{Q}_2)$. Choose

$$\delta(\xi, N) = \delta^*(\xi^*, N) \text{ and } L(\xi) = L^*(\xi^*).$$

For this case, obviously, $\chi_{C[0,1]}(\xi) = 1 = \chi_{C(\mathcal{Q}_2)}(\xi^*)$.

Case II. If $\xi \in D[0,1]$ and $\xi^* = \wp(\xi) \in D(\mathcal{Q}_2)$, then we choose

$$\delta(\xi, N) = \delta^*(\xi^*, N) \text{ and } L(\xi) = L^*(\xi^*).$$

For this trivial case, we have $\chi_{C[0,1]}(\xi) = 0 = \chi_{C(\mathcal{Q}_2)}(\xi^*)$.

Case III. If $\xi \in D[0,1]$ but $\xi^* = \wp(\xi) \in C(\mathcal{Q}_2)$. We choose a fixed $\eta^* \in D(\mathcal{Q}_2)$. We replace the ξ^* with η^* for this case. Thus η^* become the corresponding tag point of all ξ for this case and

$$\delta(\xi, N) = \delta^*(\eta^*, N) \text{ and } L(\xi) = L^*(\eta^*).$$

Thus, we have $\chi_{C[0,1]}(\xi) = 0 = \chi_{C(\mathcal{Q}_2)}(\eta^*)$.

We note that the above replacement in this case can be done because the divisions we use in the definition of the integrals are McShane, not Henstock.

Let $D = \{(\xi, I[N])\}$ be a γ-fine division of $\mathbb{R}^{[0,1]}$ with respect to $\mathcal{Q}_2$. The corresponding division $D^* = \{(\xi^*, I^*[N])\}$ to the division D form a γ^*-fine division of $\mathbb{R}^{\mathcal{Q}_2}$ (recall in case III, $\xi^* = \eta^*$) because $N \subseteq \mathcal{Q}_2$,

$$\cup_{(\xi^*, I^*[N]) \in D^*} I^*[N] \times \mathbb{R}^{[0,1] \setminus \mathcal{Q}_2} = \cup_{(\xi, I[N]) \in D} I[N] = \mathbb{R}^{[0,1]} = \mathbb{R}^{\mathcal{Q}_2} \times \mathbb{R}^{[0,1] \setminus \mathcal{Q}_2},$$

that is, $\cup_{(\xi^*, I^*[N]) \in D^*} I^*[N] = \mathbb{R}^{\mathcal{Q}_2}$. Notice that the value of $G(I[N])$ only depends on $I[N]$. Hence $G(I[N]) = G(I^*[N])$. By the choice of function γ chosen as above, we have

$$\chi_{C[0,1]}(\xi) = \chi_{C(\mathcal{Q}_2)}(\xi^*)$$

for all $\xi \in \mathbb{R}^{[0,1]}$. Thus

$$(D) \sum \chi_{C[0,1]}(\xi) G(I[N]) = (D^*) \sum \chi_{C(\mathcal{Q}_2)}(\xi^*) G(I^*[N]).$$

Therefore, we have

$$\begin{aligned} &\left| (D) \sum \chi_{C[0,1]}(\xi) G(I[N]) - \int_{\mathbb{R}^{\mathcal{Q}_2}} \chi_{C(\mathcal{Q}_2)} \right| \\ &\qquad = \left| (D^*) \sum \chi_{C(\mathcal{Q}_2)}(\xi^*) G(I^*[N]) - \int_{\mathbb{R}^{\mathcal{Q}_2}} \chi_{C(\mathcal{Q}_2)} \right| \leq \epsilon. \end{aligned}$$

That is, the McShane-Wiener integral of $\chi_{C[0,1]}$ on $\mathbb{R}^{[0,1]}$ with respect to $\mathcal{Q}_2$ exists and

$$\int_{\mathbb{R}^{[0,1]}(\mathcal{Q}_2)} \chi_{C[0,1]} = \int_{\mathbb{R}^{\mathcal{Q}_2}} \chi_{C(\mathcal{Q}_2)}.$$

■

We shall prove that $\mathcal{P}^{\mathcal{Q}_2}(C(\mathcal{Q}_2)) = 1$ in Section 3, Theorem 3.11. By Theorem 2.7, $\mathcal{P}(C[0,1]) = 1$. Hence, we have the following theorem:

Theorem 2.8. *$\mathcal{P}^{\mathcal{Q}_2}(C(\mathcal{Q}_2)) = 1$ and $\mathcal{P}(C[0,1]) = 1$.*

We remark that using the same ideas, Theorem 2.7 is true for $\mathcal{H}_\alpha(\mathcal{Q}_2)$ and $\mathcal{H}_\alpha[0,1]$, where $0 < \alpha < \frac{1}{2}$. For the definition of $\mathcal{H}_\alpha(\mathcal{Q}_2)$ see Definition 3.1.

3. Hölder Continuous of Exponential α over $\mathcal{Q}_2$

In this section, we follow the ideas in [9, Section 8.1] of the proofs of Lemmas 3.2, 3.3 and Theorem 3.8. Similar results can be found in [1, Section 6.9]. We prove the results in the setting of the McShane–Wiener integral.

Definition 3.1. For every real number $\alpha > 0$, a function $\xi \in \mathbb{R}^{\mathcal{Q}_2}$ is said to be *Hölder continuous with exponent α in the set $\mathcal{Q}_2$* if there exists a constant C such that for all $q, r \in \mathcal{Q}_2$,

$$|\xi(q) - \xi(r)| \leq C|q - r|^{\alpha}.$$

The space of all Hölder continuous with exponent α in the set $\mathcal{Q}_2$ is denoted by $\mathcal{H}_\alpha(\mathcal{Q}_2)$.

We remark that if $0 < \alpha < \beta$, then $\mathcal{H}_\beta(\mathcal{Q}_2) \subseteq \mathcal{H}_\alpha(\mathcal{Q}_2)$. Hence, if $0 < \alpha < \frac{1}{2}$, we have $\mathcal{H}_{1/2}(\mathcal{Q}_2) \subseteq \mathcal{H}_\alpha(\mathcal{Q}_2)$.

Lemma 3.2. *Let $\xi \in \mathbb{R}^{\mathcal{Q}_2}$ and $0 < \alpha < 1$. Suppose there exists $N = N(\xi)$ such that*

$$|\xi(q) - \xi(r)| \leq C|q - r|^{\alpha}$$

for all $q, r \in \mathcal{Q}_2$ with $|q - r| < 2^{-N}$. Then $\xi \in \mathcal{H}_\alpha(\mathcal{Q}_2)$.

Proof. Let $q, r \in \mathcal{Q}_2$. If $|q - r| < 2^{-N}$, then we get the required result. Suppose $|q - r| \geq 2^{-N}$, let $0 \leq q = s_0 < s_1 < \ldots < s_n = r \leq 1$ with $s_0, s_1, \ldots, s_n \in \mathcal{Q}_2$, $s_i - s_{i-1} < 2^{-N}$ and $n \leq 2^{N+1}$. Hence,

$$|\xi(q) - \xi(r)| \leq \sum_{i=1}^{n} |\xi(s_i) - \xi(s_{i-1})| \leq C \sum_{i=1}^{n} (s_i - s_{i-1})^{\alpha} \leq C \cdot 2^{N+1}|q - r|^{\alpha}.$$

Therefore,

$$|\xi(q) - \xi(r)| \leq C \cdot 2^{N+1}|q - r|^{\alpha},$$

i.e., $\xi \in \mathcal{H}_\alpha(\mathcal{Q}_2)$. ∎

To show that $\int_{\mathbb{R}^{\mathcal{Q}_2}} \chi_{\mathcal{H}_{1/2}(\mathcal{Q}_2)} = 0$. We define the following sets.

Let β be any fixed positive number. For fixed positive integer n and $\alpha > 0$, let

$$G_{\alpha,n} = \left\{ \xi \in \overline{\mathbb{R}}^{\mathcal{Q}_2} : \left| \xi\left(\frac{m}{2^n}\right) - \xi\left(\frac{m-1}{2^n}\right) \right|^{\beta} \leq 2^{-\alpha\beta n} \text{ for all } m = 1, 2, \ldots, 2^n \right\}.$$

Hence

$$G^c_{\alpha,n} = \left\{ \xi \in \overline{\mathbb{R}}^{\mathcal{Q}_2} : \left| \xi\left(\frac{m}{2^n}\right) - \xi\left(\frac{m-1}{2^n}\right) \right|^{\beta} > 2^{-\alpha\beta n} \text{ for some } m \text{ with } 0 < m \leq 2^n \right\},$$

i.e., if $\xi \in G^c_{\alpha,n}$, then there exists m such that

$$1 < 2^{\alpha\beta n} \left| \xi\left(\frac{m}{2^n}\right) - \xi\left(\frac{m-1}{2^n}\right) \right|^{\beta}.$$

Let $H_{\alpha,N} = \cap_{n=N}^{\infty} G_{\alpha,n}$. If $\xi \in H_{\alpha,N}$, then $\xi \in G_{\alpha,n}$ for all $n \geq N$, i.e.,

$$\left| \xi\left(\frac{m}{2^n}\right) - \xi\left(\frac{m-1}{2^n}\right) \right|^{\beta} \leq 2^{-\alpha\beta n} \text{ for all } m = 1, 2, \ldots, 2^n$$

for all $n \geq N$. Note that $H_{\alpha,N}$ is increasing and $H^c_{\alpha,N}$ decreasing as $N \to \infty$.

Next we shall show that $H_{\alpha,N} \subseteq \mathcal{H}_\alpha(\mathcal{Q}_2)$.

Lemma 3.3. *Let $\alpha > 0$ be fixed. For any $\xi \in H_{\alpha,N}$, we have*

$$|\xi(q) - \xi(r)| \leq \frac{3}{1-2^{-\alpha}}|q-r|^{\alpha}$$

for $q, r \in \mathcal{Q}_2$ with $|q-r| < 2^{-N}$.

Proof. Let $\xi \in H_{\alpha,N}$ and $q, r \in \mathcal{Q}_2$ with $0 < q - r < 2^{-N}$. Choose some $n \geq N$ such that $2^{-n} \leq q - r < 2^{-n+1}$. We can write $q = \frac{m}{2^n} + \frac{1}{2^{s_1}} + \frac{1}{2^{s_2}} + \ldots + \frac{1}{2^{s_k}}$ and $r = \frac{m-1}{2^n} - \frac{1}{2^{t_1}} - \frac{1}{2^{t_2}} - \ldots - \frac{1}{2^{t_l}}$, where $n < s_1 < s_2 < \cdots < s_k$ and $n < t_1 < t_2 < \cdots < t_l$. Observe that $\frac{m}{2^n}$ and $\frac{m}{2^n} + \frac{1}{2^{s_1}} + \frac{1}{2^{s_2}} + \ldots + \frac{1}{2^{s_j}}$, $j = 1, 2, \ldots, k$ are all in $\mathcal{Q}_2$ and the distance between any two consecutive points is $\frac{1}{2^{s_j}}$. Hence

$$\left|\xi(q) - \xi\left(\frac{m}{2^n}\right)\right| \leq \sum_{j=1}^{k} 2^{-\alpha s_j} \leq \sum_{j=1}^{\infty} (2^{-\alpha})^{s_j} \leq \sum_{u=n}^{\infty} (2^{-\alpha})^{u} = \frac{2^{-\alpha n}}{1-2^{-\alpha}}.$$

Similarly, we have

$$\left|\xi\left(\frac{m-1}{2^n}\right) - \xi(r)\right| \leq \frac{2^{-\alpha n}}{1-2^{-\alpha}}.$$

Hence

$$\begin{aligned}
|\xi(q) - \xi(r)| &\leq \left|\xi(q) - \xi\left(\frac{m}{2^n}\right)\right| + \left|\xi\left(\frac{m}{2^n}\right) - \xi\left(\frac{m-1}{2^n}\right)\right| + \left|\xi\left(\frac{m-1}{2^n}\right) - \xi(r)\right| \\
&\leq \frac{2^{-\alpha n}}{1-2^{-\alpha}} + 2^{-\alpha n} + \frac{2^{-\alpha n}}{1-2^{-\alpha}} \\
&\leq \frac{3}{1-2^{-\alpha}}\, 2^{-\alpha n} \leq \frac{3}{1-2^{-\alpha}}|q-r|^{\alpha}.
\end{aligned}$$

■

Lemma 3.4. *If $\xi \in H_{\alpha,N}$, then $\xi \in \mathcal{H}_\alpha(\mathcal{Q}_2)$, i.e.,*

$$|\xi(q) - \xi(r)| \leq \frac{3 \cdot 2^{N+1}}{1-2^{-\alpha}}|q-r|^{\alpha},$$

for $q, r \in \mathcal{Q}_2$.

Proof. It follows from Lemmas 3.2 and 3.3. ■

Next, we shall estimate $\int_{\mathbb{R}^{\mathcal{Q}_2}} \chi_{G^c_{\alpha,n}}$ using the MsShane–Wiener integral.

Lemma 3.5. [4, p. 56] *For fixed $p, n \in \mathbb{N}$, for each $0 < m < 2^n$, let $f : \overline{\mathbb{R}}^{\mathcal{Q}_2} \to \mathbb{R}$ be defined by $f(\xi) := \left|\xi\left(\frac{m}{2^n}\right) - \xi\left(\frac{m-1}{2^n}\right)\right|^{2p}$. Then f is McShane–Wiener integrable on $\mathbb{R}^{\mathcal{Q}_2}$ and*

$$\int_{\mathbb{R}^{\mathcal{Q}_2}} \left|\xi\left(\frac{m}{2^n}\right) - \xi\left(\frac{m-1}{2^n}\right)\right|^{2p} = K_p\, 2^{-pn},$$

where $K_p = \left(\frac{2}{\pi}\right)^p \Gamma\left(p + \frac{1}{2}\right)$.

Lemma 3.6. *$G_{\alpha,n}$ is $\mathcal{Q}_2$-measurable, i.e., $\chi_{G_{\alpha,n}}$ is McShane-Wiener integrable on $\mathbb{R}^{\mathcal{Q}_2}$.*

Proof. For $m = 1, 2, \ldots, 2^n$, let

$$G_{\alpha,n,m} = \left\{ \xi \in \overline{\mathbb{R}}^{\mathcal{Q}_2} : \left| \xi\left(\frac{m}{2^n}\right) - \xi\left(\frac{m-1}{2^n}\right) \right|^{\beta} \leq 2^{-\alpha\beta n} \right\}.$$

Then

$$\int_{\mathbb{R}^{\mathcal{Q}_2}} \chi_{G_{\alpha,n,m}} = \int_{\mathbb{R}^2} \chi_W,$$

where $W = \{(u, v) : |u - v| \leq 2^{-\alpha n}\}$. Note that $G_{\alpha,n} = \cap_{m=1}^{2^n} G_{\alpha,n,m}$. Hence $G_{\alpha,n}$ is $\mathcal{Q}_2$-measurable. ∎

Let $p > 0$ be fixed and $0 < \alpha < \frac{1}{2} - \frac{1}{p}$. Then $\lambda = p - 1 - 2\alpha p > 0$. We shall use this properties in Lemma 3.7 and Theorem 3.8.

Lemma 3.7. *Let $p > 0$ be fixed and $0 < \alpha < \frac{1}{2} - \frac{1}{p}$. Then*

$$\int_{\mathbb{R}^{\mathcal{Q}_2}} \chi_{G^c_{\alpha,n}} \leq K_p 2^{-\lambda n},$$

where $K_p = \left(\frac{2}{\pi}\right)^p \Gamma\left(p + \frac{1}{2}\right)$, where $\lambda = p - 1 - 2\alpha p$.

Proof. By Lemma 3.5, we have

$$\begin{aligned}
\int_{\mathbb{R}^{\mathcal{Q}_2}} \chi_{G^c_{\alpha,n}} &\leq \sum_{i=1}^{2^n} 2^{\alpha(2p)n} \int_{\mathbb{R}^{\mathcal{Q}_2}} \left| \xi\left(\frac{i}{2^n}\right) - \xi\left(\frac{i-1}{2^n}\right) \right|^{2p} \\
&= \sum_{i=1}^{2^n} 2^{2\alpha p n} K_p 2^{-pn} \\
&= K_p 2^n 2^{2\alpha p n} 2^{-pn} \\
&= K_p 2^{(1+2\alpha p - p)n} \\
&= K_p 2^{-\lambda n}.
\end{aligned}$$

Notice that $\lambda = p - 1 - 2\alpha p > 0$. ∎

Theorem 3.8. *Let $p > 0$ be fixed and $0 < \alpha < \frac{1}{2} - \frac{1}{p}$. Then*

$$\sum_{N=1}^{\infty} \int_{\mathbb{R}^{\mathcal{Q}_2}} \chi_{H^c_{\alpha,N}} < \infty.$$

Proof. By Lemma 3.7, we have

$$\int_{\mathbb{R}^{\mathcal{Q}_2}} \chi_{H^c_{\alpha,N}} \leq \sum_{n=N}^{\infty} \int_{\mathbb{R}^{\mathcal{Q}_2}} \chi_{G^c_{\alpha,n}} \leq K_p \sum_{n=N}^{\infty} 2^{-\lambda n} = \frac{K_p 2^{-\lambda N}}{1 - 2^{-\lambda}},$$

where $K_p = \left(\frac{2}{\pi}\right)^p \Gamma\left(p + \frac{1}{2}\right)$ and $\lambda = p - 1 - 2\alpha p$. The above equality holds because $\lambda = p - 1 - 2\alpha p > 0$. Then

$$\sum_{N=1}^{\infty} \int_{\mathbb{R}^{\mathcal{Q}_2}} \chi_{H^c_{\alpha,N}} \leq \sum_{N=1}^{\infty} \frac{K_p 2^{-N\lambda}}{1 - 2^{-\lambda}} = \frac{K_p 2^{-\lambda}}{(1 - 2^{-\lambda})^2} < \infty.$$

∎

Let

$$E = \bigcap_{n=1}^{\infty} \bigcup_{N=n}^{\infty} H_{\alpha,N}^c.$$

Then

$$E^c = \left(\bigcap_{n=1}^{\infty} \bigcup_{N=n}^{\infty} H_{\alpha,N}^c \right)^c = \bigcup_{n=1}^{\infty} \bigcap_{N=n}^{\infty} H_{\alpha,N}.$$

Theorem 3.9. *Let $p > 0$ be fixed and $0 < \alpha < \frac{1}{2} - \frac{1}{p}$, i.e., $0 < \alpha < \frac{1}{2}$. Then $\int_{\mathbb{R}^{\mathcal{Q}_2}} \chi_E = 0$ and $\int_{\mathbb{R}^{\mathcal{Q}_2}} \chi_{E^c} = 1$.*

Proof. First

$$0 \leq \int_{\mathbb{R}^{\mathcal{Q}_2}} \chi_E \leq \sum_{N=n}^{\infty} \int_{\mathbb{R}^{\mathcal{Q}_2}} \chi_{H_{\alpha,N}^c}$$

for all n. By Theorem 3.8, $\int_{\mathbb{R}^{\mathcal{Q}_2}} \chi_E = 0$. Therefore, $\int_{\mathbb{R}^{\mathcal{Q}_2}} \chi_{E^c} = 1$. ■

Theorem 3.10. *Let $p > 0$ be fixed and $0 < \alpha < \frac{1}{2} - \frac{1}{p}$. Then $\int_{R^{\mathcal{Q}_2}} \chi_{\mathcal{H}_\alpha(\mathcal{Q}_2)} = 1$.*

Proof. First we note that $E^c = \cup_{n=1}^{\infty} \cap_{N=n}^{\infty} H_{\alpha,N}$. By Lemma 3.4, $H_{\alpha,N} \subseteq \mathcal{H}_\alpha(\mathcal{Q}_2)$ for all N. Hence $E^c \subseteq \mathcal{H}_\alpha(\mathcal{Q}_2)$. By the completeness of the probability measure space, Theorem 2.5, we get $\int_{R^{\mathcal{Q}_2}} \chi_{\mathcal{H}_\alpha(\mathcal{Q}_2)} = 1$. ■

By the completeness of the probability measure space $(\mathbb{R}^{\mathcal{Q}_2}, \mathcal{M}^{\mathcal{Q}_2}, \mathcal{P}^{\mathcal{Q}_2})$, Theorem 2.5, and $\mathcal{H}_\alpha(\mathcal{Q}_2) \subseteq C(\mathcal{Q}_2)$, for $0 < \alpha < \frac{1}{2}$, we get the following theorem:

Theorem 3.11. *$C(\mathcal{Q}_2)$ is $\mathcal{Q}_2$-measurable and*

$$\mathcal{P}^{\mathcal{Q}_2}(C(\mathcal{Q}_2)) = \int_{R^{\mathcal{Q}_2}} \chi_{C(\mathcal{Q}_2)} = 1.$$

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References

[1] P. Muldowney, A Modern Theory of Random Variation, Wiley Inc., New Jersey, 2012.

[2] R. Henstock, Lectures on the Theory of Integration, World Scientific, Singapore, 1988.

[3] L.P. Yee, R. Výborný, The Integral: An Easy Approach after Kurzweil and Henstock, Cambridge University Press, Cambridge, 2000.

[4] P. Muldowney, A General Theory of Integration in Function Spaces, Pitman Research Notes in Mathematics Series, Longman Scientific, 1987.

[5] C.T. Seng, L.P. Yee, The Henstock Wiener integral, Proceeding of Symposium on Real Analysis (Xiamen 1993), J. Math. Study 27 (1) (1994) 60–65.

[6] Y.C. Hwa, Measure Theory and the Henstock–Wiener Integral, M.Sc. Thesis, NUS, 1998.

[7] Y.C. Hwa, C.T. Seng, On McShane-Wiener integral, Proceeding of International Mathematics Conference (Manila 1998), Matimyas Math. 22 (2) (1999) 39–46.

[8] H.L. Royden, Real Analysis, Maxwell-Macmillan, New York, 1988.

[9] R. Durrett, Probability: Theory and Examples (4th edition), Cambridge University Press, Cambridge, 2010.