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Some Fixed Point Theorems in Generalized Complex Valued Metric Spaces

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Abstract In this paper, we study and estoblished some fixed point theorems for general Kannan contraction mapping in a new class of generalized conplex valued metric space. The results exfend and improve some results of Y. Elkouch and E.M. Marhrani [Y. Elkouch, E.M. Morhrani, On some fixed point theorem in generalized metric space, Fixed Point Theory Appl. 2017 (2017) Article no. 23].

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1. Introduction

The axiomatic development of a metric space was essentially carried out by Frence mathematician Freehen in the year 1960. Recently, the Banach fixed point theorem in a complex valued metric space introduced by Banach [1], has been generalized in many space. In 2011 Azam [2], introduced the notrun of complex valued metric space and established sufficient conditions for the existence of common fixed point of a pair of mappings satisfying a contractive condition. In 2012, Sintunavarat and Kumam [3] introduced new spaces called the complex valued metric spaces and established the existence of fixed point theorems under the contraction condition. One year later, Sintunavarat, Cho and Kumam, [4] established the existence of fixed point theorems under the contraction condition in complex valued metric spaces, they introduce the concepts of a C-Cauchy sequence and C-complete in complex-valued metric spaces and establish the existence of common fixed point theorems in C-complete complex-valued metric spaces.

In 2015, Jleli and Samet [5] introduced a very interesting concept of a generalized metric space, which covers different well-known metric structures including classical metric spaces, b-metric spaces, dislocated metric spaces, modular spaces, and so on. In 2016, Kumam, Sarwar, and Zada [6] to establish fixed point results satisfying contractive conditions of rational type in the setting of complex valued metric spaces. In 2017, Ali [7] to prove some fixed point theorems for a pair of weakly compatible satisfying (CLRg)

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property im complex value metric spaces. In 2019, Verma and Pethak [8] to prove some common fixed point results for two pair weakly compatible mappings in complex valued metric space and applied in GV-fuzzy metric space.

In 2017, Elkouch and Marhrani [9], obtain some generalizations of fixed point results for Kannan contractive mapping, Chatterjea and Hardy-Rogers contraction mappings in a new class of generalized metric spaces.

Let us recall that a mapping T on a metric space (X, d) is a *Kannan contraction* [10] if there exists $\alpha \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \alpha d(x, Tx) + \alpha d(y, Ty), \tag{1.1}$$

for all $x, y \in X$.

They prove that, in a complete generalized metric space (X, D), and let f be a self-mapping on X satisfying for some constant $\lambda \in [0, \frac{1}{2})$ such that $C\lambda < 1$. Then under some condition the sequence $\{f^n x_0\}$ converges to some $\omega \in X$, with a unique fixed point of f.

Motivation of this paper, we introduce some results of Elkouch and Marharni [9] to generalized complex valued metric spaces and then we prved some fixed point theorems for general Kannan contraction mapping defined in section 2. The results is exfended and improve the results of Elkouch and Morharni [9].

2. Preliminaries

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order relation $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \leq Re(z_2) \text{ and } Im(z_1) \leq Im(z_2).$$

Thus $z_1 \precsim z_2$ if one of the followings holds:

(1) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$,
(2) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$,
(3) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$,
(4) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

We write $z_1 \precnsim z_2$ if $z_1 \precsim z_2$ and $z_1 \neq z_2$ i.e. one of (2),(3) and (4) is satisfied and we will write $z_1 \prec z_2$ only (4) is satisfied.

Remark 2.1. We can easily check the following:

(i) If $a, b \in \mathbb{R}, 0 \leq a \leq b$ and $z_1 \precsim z_2$ then $az_1 \precsim bz_2, \forall z_1, z_2 \in \mathbb{C}$.
(ii) $0 \precsim z_1 \precnsim z_2 \Rightarrow |z_1| < |z_2|$.
(iii) $z_1 \precsim z_2$ and $z_2 \prec z_3 \Rightarrow z_1 \prec z_3$.

In this work, we improve the Kannan contraction (1.1) into the *general Kannan contraction mapping* T on a complex valued metric space (X, d) if there exists $\alpha \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \precsim \alpha(d(x, Tx) + d(y, Ty)), \tag{2.1}$$

for all $x, y \in X$.

In this paper we defined the set $C(D, X, x)$ by let X be a nonempty set, and $D : X \times X \to \mathbb{C}$ be a given mapping. For every $x \in X$, we define the set

$$C(D, X, x) = \left\{ \{x_n\} \subseteq X : \lim_{n \to \infty} |D(x_n, x)| = 0 \right\}. \tag{2.2}$$

Definition 2.2. Let X be a nonempty set, a mapping $D : X \times X \to \mathbb{C}$ is called a *generalized complex value metric space* if it satisfies the following condition
1. For every $x, y \in X$, we have
$$0 \precsim D(x, y).$$
2. For every $x, y \in X$, we have
$$D(x, y) = 0 \Rightarrow x = y.$$
3. For every $x, y \in X$, we have
$$D(x, y) = D(y, x).$$
4. There exists a complex constant $0 \prec r$ such that for all $x, y \in X$ and $\{x_n\} \in C(D, X, x)$, we have
$$D(x, y) \precsim r \limsup_{n\to\infty} |D(x_n, y)|.$$

Definition 2.3. Let (X, D) be a generalized complex valued metric space, let $\{x_n\}$ be a sequence in X, and let $x \in X$. We say that $\{x_n\}$ is *converges* to x in X, if $\{x_n\} \in C(D, X, x)$. We denote this by $\lim_{n\to\infty} x_n = x$.

Example 2.4. Let $X = [0, 1]$ and let $D : X \times X \to \mathbb{C}$ be the mapping define by for any $x, y \in X$
$$\begin{cases} D(x, y) = (x + y)i; x \neq 0 \text{ and } y \neq 0 \\ D(x, 0) = D(0, x) = \frac{x}{2} i \end{cases}$$

Proof. (1) Let $x, y \in X$, we have $x \geq 0$ and $y \geq 0$, thus $x + y \geq 0$

If $D(x, y) = (x + y)i = 0 + (x + y)i \succsim 0 + 0i = 0$.

If $D(x, 0) = \frac{x}{2} i = 0 + \frac{x}{2} i \succsim 0 + 0i = 0$.
$$\text{Hence } D(x, y) \succsim 0.$$
(2) Let $D(x, y) = 0$, since $x \geq 0$ and $y \geq 0$, it follows that
$$0 = D(x, y) = (x + y)i.$$
Then $x = 0 = y$. If $y = 0$ with $D(x, 0) = D(0, x) = \frac{x}{2} i$, we have
$$\frac{x}{2} i = 0 \text{ and } y = 0.$$
Then $x = 0 = y$.

(3) If $x \neq 0$ and $y \neq 0$, $D(x, y) = (x + y)i = (y + x)i = D(y, x)$ and $D(x, 0) = D(0, x)$.

(4) Let $x, y \in X$ and $\{x_n\} \in C(D, X, x)$, we see that $\limsup_{n\to\infty} |D(x_n, x)| = 0$ and put $r = i$, then we have $D(0, y) = \frac{y}{2} i$ and if $x_n = 0$ for all $n \in \mathbb{N}$, we have
$$\limsup_{n\to\infty} |D(x_n, y)| = \limsup_{n\to\infty} |\frac{y}{2} i| = \frac{y}{2}.$$
Then $D(0, y) = \frac{y}{2} i \precsim r \limsup_{n\to\infty} |D(x_n, y)|$. If $x_n \neq 0$ for all $n \in \mathbb{N}$, we have
$$\lim_{n\to\infty} |D(x_n, y)| = \lim_{n\to\infty} (|(x_n + y)i|) = \lim_{n\to\infty} (x_n + y) = x + y.$$
Hence, $\limsup_{n\to\infty} |D(x_n, y)| = x + y$, and we have $D(x, y) = (x + y)i$, it follows that
$$D(x, y) = (x + y)i \precsim r \limsup_{n\to\infty} |D(x_n, y)|.$$
Hence, $([0, 1], D)$ is generalized complex valued metric space. ∎

Definition 2.5. Let (X, D) be a generalized complex valued metric space. Then a sequence $\{x_n\}$ in X is said to *Cauchy sequence* in X, if $\lim_{n\to\infty} |D(x_n, x_{n+m})| = 0$.

Definition 2.6. Let (X, D) be a generalized complex valued metric space. If every Cauchy sequence is convergent in X then (X, D) is called a *complete complex valued metric space.*

Lemma 2.7 ([11]). *Let λ be a real number such that $0 \leq \lambda < 1$, and let $\{b_n\}$ be a sequence of positives reals number such that $\lim_{n\to\infty} b_n = 0$. Then, for any sequence of positives numbers $\{a_n\}$ satisfying*

$$a_{n+1} \leq a_n + b_n, \forall n \in \mathbb{N},$$

we have $\lim_{n\to\infty} a_n = 0$.

3. Main Results

In this section we prove some proposition for use in the main theorem and prove some fixed point theorem in generalized complex valued metric space.

Proposition 3.1. *Let X be a nonempty set and $D : X \times X \to \mathbb{C}$. Then $C(D, X, x)$ is a nonempty set if and only if $D(x, x) = 0$.*

Proof. $\Rightarrow$) Suppose $C(D, X, x)$ is a nonempty set. To show that $D(x, x) = 0$.
By Definition of $C(D, X, x)$ there exists $\{x_n\} \subset X$ such that

$$\lim_{n\to\infty} |D(x_n, x)| = 0.$$

By Definition 2.2 (4), there exists $0 \prec r$ such that

$$D(x, x) \precsim r \limsup_{n\to\infty} |D(x_n, x)| = 0.$$

Thus $D(x, x) \precsim 0$. By Definition 2.2 (1), we have $0 \precsim D(x, x) \precsim 0$. Hence $D(x, x) = 0$.

$\Leftarrow$) Suppose $D(x, x) = 0$. To show that $C(D, X, x) \neq \emptyset$. Since $D(x, x) = 0 = 0 + 0i$, put $x_n = x$, for all $n \in \mathbb{N}$. We see that

$$\lim_{n\to\infty} |D(x_n, x)| = \lim_{n\to\infty} |D(x, x)| = |0 + 0i| = 0.$$

Hence $C(D, X, x) \neq \emptyset$. ∎

Lemma 3.2. *Let (X, D) be a generalized complex valued metric space, and let $f : X \to X$ be a mapping satisfying condition that for any $x, y \in X$,*

$$D(fx, fy) \precsim \lambda(D(x, fx) + D(y, fy)),$$

for some $\lambda \in [0, \frac{1}{2})$. Then any fixed point $\omega \in X$ of f satisfies

$$|D(\omega, \omega)| < \infty \Rightarrow D(\omega, \omega) = 0.$$

Proof. Let $\omega \in X$ be a fixed point of f such that $|D(\omega, \omega)| < \infty$. Since ω is a fixed point of f then $f\omega = \omega$. To show that $D(\omega, \omega) = 0$, we consider

$$D(\omega, \omega) = D(f\omega, f\omega) \precsim \lambda[D(\omega, f\omega) + D(\omega, f\omega)].$$

By Remark 2.1 (ii), we have

$$\begin{aligned} |D(\omega,\omega)| &\leq |\lambda[D(\omega,f\omega)+D(\omega,f\omega)]| \\ &\leq |\lambda D(\omega,f\omega)|+|\lambda D(\omega,f\omega)| \\ &= \lambda|D(\omega,f\omega)|+\lambda|D(\omega,f\omega)| \\ &= 2\lambda|D(\omega,f\omega)|, \end{aligned}$$

It follows that,

$$(1-2\lambda)|D(\omega,\omega)| \leq 0.$$

Since $\lambda \in [0,\frac{1}{2})$, we obtain that $0 \leq |D(\omega,\omega)| \leq 0$, then $|D(\omega,\omega)| = 0$. Hence, $D(\omega,\omega) = 0$. ∎

Next, we can some sets that, for a mapping $f : X \to X$, and for every $x \in X$, we define

$$\delta(D,f,x) = sup\{|D(f^i x, f^j x)| : i,j \in \mathbb{N}\}. \tag{3.1}$$

Theorem 3.3. *Let (X,D) be a complete generalized complex valued metric space, and let f be a self-mapping on X satisfying (2.1) for some constant $\lambda \in [0,\frac{1}{2})$ such that $|r\lambda| < 1$.*

(i) If there exists an element $x_0 \in X$ such that $\delta(D,f,x_0) < \infty$, then the sequence $\{f^n x_0\}$ converges to some $\omega \in X$.

(ii) If $|D(\omega,f\omega)| < \infty$, then ω is a fixed point of f.
Moreover, for each fixed point $\omega^{'}$ of f in X such that $|D(\omega^{'},\omega^{'})| < \infty$, we have $\omega = \omega^{'}$.

Proof. Let $n \in \mathbb{N}(n \geq 1)$. For all $i,j \in \mathbb{N}$, from (2.1), we have

$$D(f^{n+i}x_0, f^{n+j}x_0) \precsim \lambda[D(f^{n+i-1}x_0, f^{n+i}x_0) + D(f^{n+j-1}x_0, f^{n+j}x_0)].$$

By Remark 2.1 (ii), we have

$$\begin{aligned} |D(f^{n+i}x_0, f^{n+j}x_0)| &\leq \lambda|[D(f^{n+i-1}x_0, f^{n+i}x_0) + D(f^{n+j-1}x_0, f^{n+j}x_0)]| \\ &\leq \lambda|D(f^{n+i-1}x_0, f^{n+i}x_0)| + \lambda|D(f^{n+j-1}x_0, f^{n+j}x_0)|, \end{aligned}$$

and then

$$\begin{aligned} |D(f^{n+i}x_0, f^{n+j}x_0)| &\leq \lambda\delta(D,f,f^{n-1}x_0) + \lambda\delta(D,f,f^{n-1}x_0) \\ &\leq 2\lambda\delta(D,f,f^{n-1}x_0), \end{aligned}$$

we have,

$$|D(f^{n+i}x_0, f^{n+j}, x_0)| \leq 2\lambda\delta(D,f,f^{n-1}x_0). \tag{3.2}$$

By (3.2) we have $2\lambda\delta(D,f,f^{n-1}x_0)$ is upper bound of the set $\{|D(f^{n+i}x_0, f^{n+j}x_0)| : i,j \in \mathbb{N}\}$ since $\delta(D,f,f^n x_0)$ is least upper bound of $\{|D(f^{n+i}x_0, f^{n+j}x_0)|\}$ it follows that

$$\delta(D,f,f^n x_0) \leq (2\lambda)\delta(D,f,f^{n-1}x_0).$$

Consequently, we obtain

$$\delta(D,f,f^n x_0) \leq (2\lambda)^n \delta(D,f,x_0).$$

Since, $|D(f^{2n}x_0, f^{m+n}x_0)| = |D(f^n(f^n x_0), f^m(f^n x_0))|$, we have

$$|D(f^{2n}x_0, f^{m+n}x_0)| \leq \delta(D,f,f^n x_0) \leq (2\lambda)^n \delta(D,f,x_0) \tag{3.3}$$

for all integer m. Since $\delta(D, f, x_0) < \infty$ and $2\lambda \in [0, 1)$, we have $\lim_{n\to\infty}(2\lambda)^n = 0$, it follows that

$$\lim_{n\to\infty} |D(f^n x_0, f^{m+n} x_0)| = \lim_{n\to\infty} |D(f^{2n} x_0, f^{m+n} x_0)| = 0.$$

Then we have $\{f^n x_0\}$ is a cauchy sequence, and thus there exists $\omega \in X$ such that

$$\lim_{n\to\infty} |D(f^n x_0, \omega)| = 0.$$

By Definition of generalized complex valued metric space (4), we have

$$D(f\omega, \omega) \precsim r \limsup_{n\to\infty} |D(f\omega, f^{n+1} x_0)|. \tag{3.4}$$

By Remark 2.1 (ii), we have

$$|D(f\omega, \omega)| \leq |r| \limsup_{n\to\infty} |D(f\omega, f^{n+1} x_0)|. \tag{3.5}$$

By (2.1) we have

$$D(f^{n+1} x_0, f\omega) \precsim \lambda(D(f^{n+1} x_0, f^n x_0) + D(\omega, f\omega)). \tag{3.6}$$

By Remark 2.1 (ii), we have

$$|D(f^{n+1} x_0, f\omega)| \leq |\lambda(D(f^{n+1} x_0, f^n x_0) + D(\omega, f\omega))|. \tag{3.7}$$

By (3.3) and (3.7) we obtain

$$\limsup_{n\to\infty} |D(f\omega, f^{n+1} x_0)| \leq |\lambda D(\omega, f\omega)|. \tag{3.8}$$

By (3.5) and (3.8), we obtain

$$|D(\omega, f\omega)| \leq |r\lambda||D(\omega, f\omega)|$$

or

$$(1 - |r\lambda|)|D(\omega, f\omega)| \leq 0.$$

Since $|r\lambda| < 1$ and $|D(\omega, f\omega)| < \infty$, we deduce that $|D(\omega, f\omega)| = 0$, and then $D(\omega, f\omega) = 0$. Which implies that $f\omega = \omega$. If $\omega^{'}$ is any fixed point of f such that $|D(\omega^{'}, \omega^{'})| < \infty$, we obtain

$$\begin{aligned} |D(\omega, \omega^{'})| &= |D(f\omega, f\omega^{'})| \\ &\leq |\lambda(D(f\omega, \omega) + D(f\omega^{'}, \omega^{'}))| \\ &\leq |\lambda(D(\omega, \omega) + D(\omega^{'}, \omega^{'}))| \\ &= 0. \end{aligned}$$

Which implies that $\omega^{'} = \omega$. ∎

If $D : X \times X \to [0, +\infty)$ we can reduce the equation (2.2) to the set

$$C(D, X, x) = \left\{ \{x_n\} \subseteq X : \lim_{n\to\infty} D(x_n, x) = 0 \right\},$$

and the Definition 2.2 can be reduced to next.

Definition 3.4 ([9])**.** D is called a *generalized metric space on* X if it satisfies the following condition,

1. For every $x, y \in X$, we have

$$D(x,y) = 0 \Rightarrow x = y,$$

2. For every $x, y \in X$, we have

$$D(x,y) = D(y,x).$$

3. There exists a real constant $C > 0$ such that, for all $(x,y) \in X \times X$ and $\{x_n\} \in C(D,X,x)$, we have

$$D(x,y) \leq C \limsup_{n\to\infty} D(x_n,y).$$

Next, equation 3.1 can be reduced to for every $x \in X$,

$$\delta(D,f,x) = sup\{D(f^i x, f^j x) : i,j \in \mathbb{N}\}.$$

From above condition we have a result of Elkouch and Marhrani [9].

Corollary 3.5 ([9])**.** *Let (X,D) be a complete generalized metric space, and let f be a self-mapping on X satisfying (1.1) for some constant $\lambda \in [0, \frac{1}{2})$ such that $C\lambda < 1$.*

If there exists an element $x_0 \in X$ such that $\delta(D,f,x_0) < \infty$, then the sequence $\{f^n x_0\}$ converges to some $\omega \in X$. Moreover, if $D(\omega, f\omega) < \infty$, then ω is a fixed point of f. Moreover, for each fixed point $\omega^{'}$ of f in X such that $D(\omega^{'}, \omega^{'}) < \infty$, we have $\omega = \omega^{'}$.

Next theorem we prove a fixed point theorem for some contractive condition with different from (2.1).

Theorem 3.6. *Let (X,D) be a complete generalized complex valued metric space and f be a self-mapping on X satisfying that for some $\lambda \in [0, \frac{1}{2})$ such that*

$$D(fx,fy) \precsim \lambda(D(y,fx) + D(x,fy)) \tag{3.9}$$

for any $x, y \in X$.

(i) If there exists an element $x_0 \in X$ such that $\delta(D,f,x_0) < \infty$, then the sequence $\{f^n x_0\}$ converges to some $\omega \in X$.

(ii) If $|D(\omega, f\omega)| < \infty$, then ω is a fixed point of f.
Moreover, for each fixed point $\omega^{'}$ of f in X such that $|D(\omega, \omega^{'})| < \infty$, we have $\omega = \omega^{'}$.

Proof. Let $n \in \mathbb{N}(n \geq 1)$ for each $i, j \in \mathbb{N}$ we have

$$D(f^{n+i}x_0, f^{n+j}x_0) = D(f(f^{n+i-1}x_0), f(f^{n+j-1}x_0))$$

from (3.9), then

$$D(f^{n+i}x_0, f^{n+j}x_0) \precsim \lambda[D(f^{n+j-1}x_0, f^{n+i}x_0) + D(f^{n+i-1}x_0, f^{n+j}x_0)].$$

By Definition 2.2 (3), we have

$$D(f^{n+i}x_0, f^{n+j}x_0) \precsim \lambda[D(f^{n+i}x_0, f^{n+j-1}x_0) + D(f^{n+i-1}x_0, f^{n+j}x_0)].$$

By Remark 2.1 (ii), we have

$$\begin{aligned} |D(f^{n+i}x_0, f^{n+j}x_0)| &\leq \lambda|[D(f^{n+i}x_0, f^{n+j-1}x_0) + D(f^{n+i-1}x_0, f^{n+j}x_0)]| \\ &\leq \lambda|[D(f^{n+i}x_0, f^{n+j-1}x_0)| + \lambda|D(f^{n+i-1}x_0, f^{n+j}x_0)]|. \end{aligned}$$

From definition of δ in (3.1), we have

$$\begin{aligned} |D(f^{n+i}x_0, f^{n+j}x_0)| &\leq \lambda(\delta(D,f,f^{n-1}x_0)) + \lambda(\delta(D,f,f^{n-1}x_0)) \\ &\leq 2\lambda\delta(D,f,f^{n-1}x_0), \end{aligned}$$

it follows that

$$|D(f^{n+i}x_0, f^{n+j}, x_0)| \leq 2\lambda\delta(D,f,f^{n-1}x_0). \tag{3.10}$$

From (3.10), we see that $2\lambda\delta(D,f,f^{n-1}x_0)$ is a bounded above of $\{|D(f^{n+i}x_0, f^{n+j}x_0)| : i,j \in \mathbb{N}\}$. Since, $\delta(D,f,f^n x_0)$ is a least upper bound of $\{|D(f^{n+i}x_0, f^{n+j}x_0)|\}$ it follows that

$$\delta(D,f,f^n x_0) \leq (2\lambda)\delta(D,f,f^{n-1}x_0).$$

Similary, we obtain

$$\begin{aligned} \delta(D,f,f^n x_0) &\leq (2\lambda)\delta(D,f,f^{n-1}x_0) \\ &\leq (2\lambda)^2\delta(D,f,f^{n-2}x_0) \\ &\cdot \\ &\cdot \\ &\cdot \\ &\leq (2\lambda)^n\delta(D,f,x_0). \end{aligned}$$

From $|D(f^n x_0, f^{m+n}x_0)| = |D(f(f^{n-1}x_0), f^{m+1}(f^{n-1}x_0))|$ we have

$$|D(f^n x_0, f^{m+n}x_0)| \leq \delta(D,f,f^{n-1}x_0) \leq (2\lambda)^n\delta(D,f,x_0),$$

for any $m \in \mathbb{N}$. Since, $\delta(D,f,x_0) < \infty$ and $2\lambda \in [0,1)$ thus $\lim_{n\to\infty}(2\lambda)^n = 0$, it follows that

$$\lim_{n\to\infty} |D(f^n x_0, f^{m+n}x_0)| = 0. \tag{3.11}$$

Then, $\{f^n x_0\}$ is a cauchy sequence then there exists $\omega \in X$ such that

$$\lim_{n\to\infty} |D(f^n x_0, \omega)| = 0. \tag{3.12}$$

By Definition 2.2 (4), we have

$$D(f\omega, \omega) \precsim r \limsup_{n\to\infty} |D(f\omega, f^{n+1}x_0)|. \tag{3.13}$$

By Remark 2.1 (ii), we have

$$|D(f\omega, \omega)| \leq |r| \limsup_{n\to\infty} |D(f\omega, f^{n+1}x_0)|. \tag{3.14}$$

From (3.9) again,

$$D(f^{n+1}x_0, f\omega) \precsim \lambda(D(\omega, f^{n+1}x_0) + D(f^n x_0, f\omega)), \tag{3.15}$$

and from Remark 2.1 (ii), we have

$$|D(f^{n+1}x_0, f\omega)| \leq |\lambda(D(\omega, f^{n+1}x_0) + D(f^n x_0, f\omega))|.$$

Let $a_n := \lambda|D(f^n x_0, f\omega)|$ and $b_n := \lambda|(D(\omega, f^{n+1}x_0)|$. From (3.12), we have $\lim_{n\to\infty} b_n = 0$. By Lemma 2.7, we have $\lim_{n\to\infty} |D(f^n x_0, f\omega)| = 0$. Thus,

$$\limsup_{n\to\infty} |D(f\omega, f^{n+1}x_0)| = 0. \tag{3.16}$$

From (3.14) and (3.16), we have

$$|D(f\omega,\omega)| \leq |r| \limsup_{n\to\infty} |D(f\omega, f^{n+1}x_0)| = |r| \cdot 0 = 0.$$

Thus, $|D(\omega, f\omega)| = 0$, that is $D(\omega, f\omega) = 0$. It follows that $f\omega = \omega$. If $\omega^{'}$ is a fixed point of f with $|D(\omega^{'}, f\omega^{'})| < \infty$, then

$$\begin{aligned} |D(\omega,\omega^{'})| &= |D(f\omega, f\omega^{'})| \\ &\leq |\lambda(D(\omega^{'}, f\omega) + D(\omega, f\omega^{'}))| \\ &= \lambda|D(\omega^{'},\omega) + D(\omega,\omega^{'})| \\ &= 2\lambda|(D(\omega,\omega^{'})|. \end{aligned}$$

It follows that

$$(1-2\lambda)|D(\omega,\omega^{'})| \leq 0.$$

Since $2\lambda < 1$ and $|D(\omega,\omega^{'})| < \infty$, we have $|D(\omega,\omega^{'})| \leq 0$. Thus, $|D(\omega,\omega^{'})| = 0$ and $D(\omega,\omega^{'}) = 0$. Hence, $\omega^{'} = \omega$. ∎

Corollary 3.7 ([9]). *Let (X, D) be a complete generalized metric space and f be a self-mapping on X satisfying that for some $\lambda \in [0, \frac{1}{2})$ such that*

$$D(fx, fy) \leq \lambda(D(y, fx) + D(x, fy)) \tag{3.17}$$

for any $x, y \in X$. If there exists an element $x_0 \in X$ such that $\delta(D, f, x_0) < \infty$, then the sequence $\{f^n x_0\}$ converges to some $\omega \in X$. Moreover, if $D(\omega, f\omega) < \infty$, then ω is a fixed point of f, for any fixed point $\omega^{'}$ of f in X such that $D(\omega,\omega^{'}) < \infty$, we have $\omega = \omega^{'}$.

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References

[1] S. Banach, Sur operations dams les ensembles abstraits et leur application aus equation integral, Fund. Math. 3 (1922) 133–181.

[2] A. Azam, F. Brain, M. Khan, Common fixed point theorems in complex valued metric space, Numer. Funct. Anal. Optim. 32 (3) (2011) 243–253.

[3] W. Sintunavarat, P. Kumam, Generalized common fixed point theorems in complex valued metric spaces and applications, J. Inequal. Appl. 2012 (2012) Article no. 84.

[4] W. Sintunavarat, Y.J. Cho, P. Kumam, Urysohn integral equations approach by common fixed points in complex valued metric spaces, Adv. Difference Equ. 2013 (2013) Article no. 49.

[5] M. Jleli, B. Samet, A generalized metric space and related fixed point theorems, Fixed Point Theory Appl. 2015 (2015) Article no. 33.

[6] P. Kumam, M. Sarwar, M.B. Zada, Fixed point results satisfying rational type contractive conditions in complex valued metric spaces, Annales Mathematicae Silesianae 30 (1) (2016) 89–110.

[7] S. Ali, Some common fixed point theorems for two weakly compatible mappings in complex valued metric spaces, Thai J. Math. 15 (3) (2017) 797–806.

[8] R.K. Verma, H.K. Pathak, Common fixed point theorems in complex valued metric space and application, Thai J. Math. 17 (1) (2019) 75–88.

[9] Y. Elkouch, E.M. Morhrani, On some fixed point theorem in generalized metric space, Fixed Point Theory Appl. 2017 (2017) Article no. 23.

[10] R. Kannan, Some resulth on fixed points, Bull. Calcutta Math. Soc. 60 (1968) 71–76.

[11] V. Berinde, Iterative Approximation of Fixed points, Lecture Notes in Mathematics (2nd Edition), Springer, Berlin, 2007.